

ON THE LIFTING OF HERMITIAN MODULAR FORMS

TAMOTSU IKEDA

Notation

Let K be an imaginary quadratic field with discriminant $-D = -D_K$. We denote by $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K . The non-trivial automorphism of K is denoted by $x \mapsto \bar{x}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by $\chi = \chi_D$. We denote by $\mathcal{O}^\# = (\sqrt{-D})^{-1}\mathcal{O}$ the inverse different ideal of $K/\mathbb{Q}$.

The special unitary group $G = \mathrm{SU}(m, m)$ is an algebraic group defined over $\mathbb{Q}$ such that

$$G(R) = \left\{ g \in \mathrm{SL}_{2m}(R \otimes K) \mid g \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} {}^t \bar{g} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \right\}$$

for any $\mathbb{Q}$ -algebra R . We put $\Gamma_K^{(m)} = G(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O})$.

The hermitian upper half space $\mathcal{H}_m$ is defined by

$$\mathcal{H}_m = \{ Z \in M_m(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - {}^t \bar{Z}) > 0 \}.$$

Then $G(\mathbb{R})$ acts on $\mathcal{H}_m$ by

$$g \langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathcal{H}_m, g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We put

$$\begin{aligned} \Lambda_m(\mathcal{O}) &= \{ h = (h_{ij}) \in M_m(K) \mid h_{ii} \in \mathbb{Z}, h_{ij} = \bar{h}_{ji} \in \mathcal{O}^\#, i \neq j \}, \\ \Lambda_m(\mathcal{O})^+ &= \{ h \in \Lambda_m(\mathcal{O}) \mid h > 0 \}. \end{aligned}$$

We set $e(T) = \exp(2\pi\sqrt{-1}\mathrm{tr}(T))$ if T is a square matrix with entries in $\mathbb{C}$. For each prime p , the unique additive character of $\mathbb{Q}_p$ such that $e_p(x) = \exp(-2\pi\sqrt{-1}x)$ for $x \in \mathbb{Z}[p^{-1}]$ is denoted e_p . Note that e_p is of order 0. We put $e(x) = e(x_\infty) \prod_{p < \infty} e_p(x_p)$ for an adele $x = (x_v)_v \in \mathbb{A}$.

Let $\underline{\chi} = \otimes_v \underline{\chi}_v$ be the Hecke character of $\mathbb{A}^\times/\mathbb{Q}^\times$ determined by χ . Then $\underline{\chi}_v$ is the character corresponding to $\mathbb{Q}_v(\sqrt{-D})/\mathbb{Q}$ and given by

$$\underline{\chi}_v(t) = \left(\frac{-D, t}{\mathbb{Q}_v} \right).$$

Let Q_D be the set of all primes which divides D . For each prime $q \in Q_D$, we put $D_q = q^{\text{ord}_q D}$. We define a primitive Dirichlet character χ_q by

$$\chi_q(n) = \begin{cases} \chi(n') & \text{if } (n, q) = 1 \\ 0 & \text{if } q|n, \end{cases}$$

where n' is an integer such that

$$n' \equiv \begin{cases} n & \text{mod } D_q, \\ 1 & \text{mod } D_q^{-1}D \end{cases}$$

Then we have $\chi = \prod_{q|D} \chi_q$. Note that

$$\chi_q(n) = \left(\frac{\chi_q(-1)D_q, n}{\mathbb{Q}_q} \right) = \prod_{p|n} \left(\frac{\chi_q(-1)D_q, n}{\mathbb{Q}_p} \right)$$

for $q \nmid n$, $n > 0$. One should not confuse χ_q with $\underline{\chi}_q$.

1. Fourier coefficients of Eisenstein series on $\mathcal{H}_m$

In this section, we consider Siegel series associated to non-degenerate hermitian matrices. Fix a prime p . Put $\xi_p = \chi(p)$, i.e.,

$$\xi_p = \begin{cases} 1 & \text{if } -D \in (\mathbb{Q}_p^\times)^2 \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{-D})/\mathbb{Q}_p \text{ is unramified quadratic extension} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{-D})/\mathbb{Q}_p \text{ is ramified quadratic extension.} \end{cases}$$

For $H \in \Lambda_m(\mathcal{O})$, $\det H \neq 0$, we put

$$\begin{aligned} \gamma(H) &= (-D)^{[m/2]} \det(H) \\ \zeta_p(H) &= \underline{\chi}_p(\gamma(H))^{m-1}. \end{aligned}$$

The Siegel series for H is defined by

$$b_p(H, s) = \sum_{R \in \text{Her}_m(K_p)/\text{Her}_m(\mathcal{O}_p)} \mathbf{e}_p(\text{tr}(BR)) p^{-\text{ord}_p(\nu(R))s}, \quad \text{Re}(s) \gg 0.$$

Here, $\text{Her}_m(K_p)$ (resp. $\text{Her}_m(\mathcal{O}_p)$) is the additive group of all hermitian matrices with entries in K_p (resp. $\mathcal{O}_p$). The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined

as follows: Choose a coprime pair $\{C, D\}$, $C, D \in M_{2n}(\mathcal{O}_p)$ such that $C^t \bar{D} = D^t \bar{C}$, and $D^{-1}C = R$. Then $\nu(R) = \det(D)\mathcal{O}_p \cap \mathbb{Z}_p$.

We define a polynomial $t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X]$ by

$$t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i}X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1}\xi_p X).$$

There exists a polynomial $F_p(H; X) \in \mathbb{Z}[X]$ such that

$$F_p(H; p^{-s}) = b_p(H, s)t_p(K/\mathbb{Q}; p^{-s})^{-1}.$$

This is proved in [9].

Moreover, $F_p(H; X)$ satisfies the following functional equation:

$$F_p(H; p^{-2m}X^{-1}) = \zeta_p(H)(p^m X)^{-\text{ord}_p \gamma(H)} F_p(H; X).$$

This functional equation is a consequence of [7], Proposition 3.1. We will discuss it in the next section.

The functional equation implies that $\deg F_p(H; X) = \text{ord}_p \gamma(H)$. In particular, if $p \nmid \gamma(H)$, then $F_p(H; X) = 1$. Put

$$\tilde{F}_p(H; X) = X^{-\text{ord}_p \gamma(H)} F_p(H; p^{-m}X^2).$$

Then following lemma is a immediate consequence of the functional equation of $F(H; X)$.

Lemma 1. *We have*

$$\tilde{F}_p(H; X^{-1}) = \tilde{F}_p(H; X), \quad \text{if } m \text{ is odd.}$$

$$\tilde{F}_p(H; \xi_p X^{-1}) = \tilde{F}_p(H; X), \quad \text{if } m \text{ is even and } \xi_p \neq 0.$$

Let k be a sufficiently large integer. Put $n = [m/2]$. The Eisenstein series $E_{2k+2n}^{(m)}(Z)$ of weight $2k + 2n$ on $\mathcal{H}_m$ is defined by

$$E_{2k+2n}^{(m)}(Z) = \sum_{\{C, D\}/\sim} \det(CZ + D)^{-2k-2n},$$

where $\{C, D\}/\sim$ extends over coprime pairs $\{C, D\}$, $C, D \in M_{2n}(\mathcal{O})$ such that $C^t \bar{D} = D^t \bar{C}$ modulo the action of $GL_m(\mathcal{O})$. We put

$$\mathcal{E}_{2k+2n}^{(m)}(Z) = A_m^{-1} \prod_{i=1}^m L(1+i-2k-2n, \chi^{i-1}) E_{2k+2n}^{(m)}(Z).$$

Here

$$A_m = \begin{cases} 2^{-4n^2-4n} \mathbf{D}^{2n^2+n} & \text{if } m = 2n+1, \\ (-1)^n 2^{-4n^2+4n} \mathbf{D}^{2n^2-n} & \text{if } m = 2n. \end{cases}$$

Then the H -th Fourier coefficient of $\mathcal{E}_{2k+2n}^{(2n+1)}(Z)$ is equal to

$$\begin{aligned} |\gamma(H)|^{2k-1} \prod_{p|\gamma(H)} F_p(H; p^{-2k-2n}) &= |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k+(1/2)}) \\ &= |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{k-(1/2)}) \end{aligned}$$

for any $H \in \Lambda_{2n+1}(\mathcal{O})^+$ and any sufficiently large integer k .

The H -th Fourier coefficient of $\mathcal{E}_{2k+2n}^{(2n)}(Z)$ is equal to

$$|\gamma(H)|^{2k} \prod_{p|\gamma(H)} F_p(H; p^{-2k-2n}) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k})$$

for any $H \in \Lambda_{2n}(\mathcal{O})^+$ and any sufficiently large integer k .

2. Main theorems

We first consider the case when $m = 2n$ is even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k+1}(\Gamma_0(\mathbf{D}), \chi)$ be a primitive form, whose L -function is given by

$$\begin{aligned} L(f, s) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_{p \mid \mathbf{D}} (1 - a(p)p^{-s} + \chi(p)p^{2k-2s})^{-1} \prod_{q \nmid \mathbf{D}} (1 - a(q)q^{-s})^{-1}. \end{aligned}$$

For each prime $p \nmid \mathbf{D}$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \chi(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X).$$

For $q \mid \mathbf{D}$, we put $\alpha_q = q^{-k}a(q)$.

Put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n}(\mathcal{O})^+$$

$$F(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})^+} A(H) \mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n}.$$

Then our first main theorem is as follows:

Theorem 1. Assume that $m = 2n$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n)})$. Moreover, F is a Hecke eigenform. $F = 0$ if and only if $f(\tau)$ comes from a Hecke character of K and n is odd.

Now we consider the case when $m = 2n + 1$ is odd.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, whose L -function is given by

$$\begin{aligned} L(f, s) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1} \end{aligned}$$

For each prime p , we define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n+1}(\mathcal{O})^+$$

$$F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n+1}.$$

Theorem 2. Assume that $m = 2n + 1$ is odd. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n+1)})$. Moreover, F is a non-zero Hecke eigenform.

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