

ADELIC DIVISORS ON ARITHMETIC VARIETIES

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ABSTRACT. In this article, we generalize several fundamental results for arithmetic divisors, such as the continuity of the volume function, the generalized Hodge index theorem, Fujita's approximation theorem for arithmetic divisors, Zariski decompositions for arithmetic divisors on arithmetic surfaces and a special case of Dirichlet's unit theorem on arithmetic varieties, to the case of the adelic arithmetic divisors.

CONTENTS

Introduction	1
1. Preliminaries	8
2. Adelic $\mathbb{R}$ -Cartier divisors over a discrete valuation field	20
3. Local and global density theorems	33
4. Adelic arithmetic $\mathbb{R}$ -Cartier divisors	39
5. Continuity of the volume function	53
6. Zariski decompositions of adelic arithmetic divisors on arithmetic surfaces	65
7. Characterization of nef adelic arithmetic divisors on arithmetic surfaces	76
8. Dirichlet's unit theorem for adelic arithmetic divisors	87
Appendix A. Characterization of relatively nef Cartier divisors	93
References	101
Index	103

INTRODUCTION

The theory of birational Arakelov geometry has advanced tremendously over the last decade, such as the continuity of the volume function, the generalized Hodge index theorem, Fujita's approximation theorem for arithmetic divisors, Zariski decompositions for arithmetic divisors on arithmetic surfaces and so on. Besides them, non-Archimedean Arakelov geometry is also well-developed using Berkovich analytic spaces. In this article, we would like to generalize the fundamental results for arithmetic divisors to the case of adelic arithmetic divisors.

0.1. Birational Arakelov geometry. Let $\mathcal{X}$ be a $(d+1)$ -dimensional, generically smooth, normal and projective arithmetic variety, that is, $\mathcal{X}$ is a projective and flat scheme over $\mathbb{Z}$ such that $\mathcal{X}$ is integral and normal, $\mathcal{X}$ is smooth over $\mathbb{Q}$ and the Krull dimension of $\mathcal{X}$ is $d + 1$. A pair $\overline{\mathcal{D}} = (\mathcal{D}, g_\infty)$ is called an *arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{X}$* if the following conditions are satisfied:

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- (i) The first $\mathcal{D}$ is an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$, that is, $\mathcal{D} = a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$ for some Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on $\mathcal{X}$ and $a_1, \dots, a_r \in \mathbb{R}$.
- (ii) The second g_∞ is a real valued continuous function on $(\mathcal{X} \setminus \text{Supp}(\mathcal{D}))(\mathbb{C})$ such that, for each $x \in \mathcal{X}(\mathbb{C})$, $g_\infty + \sum_{i=1}^r a_i \log |f_i|^2$ extends to a continuous function around x , where $f_1, \dots, f_r$ are local equations of $\mathcal{D}_1, \dots, \mathcal{D}_r$ at x , respectively. In addition, g_∞ is invariant under the complex conjugation map.

Let $\text{Rat}(\mathcal{X})$ be the rational function field of $\mathcal{X}$. We define $H^0(\mathcal{X}, \mathcal{D})$ to be

$$H^0(\mathcal{X}, \mathcal{D}) := \{\phi \in \text{Rat}(\mathcal{X})^\times \mid \mathcal{D} + (\phi) \geq 0\} \cup \{0\}.$$

Note that $H^0(\mathcal{X}, \mathcal{D})$ is a finitely generated $\mathbb{Z}$ -module. For $\phi \in H^0(\mathcal{X}, \mathcal{D})$, we can see that $|\phi| \exp(-g_\infty/2)$ extends to a continuous function ϑ on $\mathcal{X}(\mathbb{C})$, so that

$$\sup \{\vartheta(x) \mid x \in \mathcal{X}(\mathbb{C})\}$$

is denoted by $\|\phi\|_{g_\infty}$. The volume $\widehat{\text{vol}}(\mathcal{D})$ of $\mathcal{D}$, by definition, is given by

$$\widehat{\text{vol}}(\mathcal{D}) := \limsup_{n \rightarrow \infty} \frac{\log \# \{\phi \in H^0(\mathcal{X}, n\mathcal{D}) \mid \|\phi\|_{n g_\infty} \leq 1\}}{n^{d+1}/(d+1)!}.$$

It is known that the volume function $\widehat{\text{vol}}$ has the following fundamental properties (for details, see [30]):

- (1) (Finiteness) $\widehat{\text{vol}}(\mathcal{D}) < \infty$ ([26], [27]).
- (2) (Limit theorem) $\widehat{\text{vol}}(\mathcal{D}) = \lim_{n \rightarrow \infty} \frac{\log \# \{\phi \in H^0(\mathcal{X}, n\mathcal{D}) \mid \|\phi\|_{n g_\infty} \leq 1\}}{n^{d+1}/(d+1)!}$ ([7], [27]).
- (3) (Positive homogeneity) $\widehat{\text{vol}}(a\mathcal{D}) = a^{d+1} \widehat{\text{vol}}(\mathcal{D})$ for $a \in \mathbb{R}_{\geq 0}$ ([26], [27]).
- (4) (Continuity) The volume function $\widehat{\text{vol}}$ is continuous in the following sense: Let $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r, \overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_s$ be arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on $\mathcal{X}$. For a compact subset B in $\mathbb{R}^r$ and a positive number ϵ , there are positive numbers δ and δ' such that

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^s \delta_j \overline{\mathcal{A}}_j + (0, \phi) \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \leq \epsilon$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_s \in \mathbb{R}$ and $\phi \in C^0(X)$ with $(a_1, \dots, a_r) \in B$, $|\delta_1| + \cdots + |\delta_s| \leq \delta$ and $\|\phi\|_{\text{sup}} \leq \delta'$ ([26], [27]).

Here we would like to introduce several kinds of the positivity of an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on $\mathcal{X}$:

- Big: $\widehat{\text{vol}}(\overline{\mathcal{D}}) > 0$.
- Relatively nef: the first Chern current $c_1(\overline{\mathcal{D}})$ is positive and $\overline{\mathcal{D}}$ is relatively nef with respect to $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$, that is, $\deg(\overline{\mathcal{D}}|_C) \geq 0$ for all vertical 1-dimensional closed integral subschemes C of $\mathcal{X}$.
- Nef: $\overline{\mathcal{D}}$ is relatively nef and $\widehat{\deg}(\overline{\mathcal{D}}|_C) \geq 0$ for all horizontal 1-dimensional closed integral subschemes C of $\mathcal{X}$.
- Pseudo-effective: $\overline{\mathcal{D}} + \overline{\mathcal{A}}$ is big for any big arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{A}}$ of C^0 -type on $\mathcal{X}$.

In addition, $\overline{\mathcal{D}}$ is said to be *integrable* if $\overline{\mathcal{D}} = \overline{\mathcal{D}}' - \overline{\mathcal{D}}''$ for some relatively nef arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}'$ and $\overline{\mathcal{D}}''$ of C^0 -type. For integrable arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}$ of C^0 -type, the arithmetic intersection number $\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1})$ is

well-defined (cf. [30, Subsection 6.4], [31, Subsection 2.1]). The following fundamental results were obtained by several authors such as Faltings, Gillet-Soulé, S. Zhang, Moriwaki, H. Chen, X. Yuan and so on:

- (5) (Generalized Hodge index theorem) If $\overline{\mathcal{D}}$ is relatively nef, then

$$\widehat{\deg}(\overline{\mathcal{D}}^{d+1}) \leq \widehat{\text{vol}}(\overline{\mathcal{D}}).$$

Moreover, if $\overline{\mathcal{D}}$ is nef, then $\widehat{\deg}(\overline{\mathcal{D}}^{d+1}) = \widehat{\text{vol}}(\overline{\mathcal{D}})$ ([10], [11], [37], [26], [30]).

- (6) (Fujita's approximation theorem for arithmetic divisors) If $\overline{\mathcal{D}}$ is big, then, for any positive number ϵ , there are a birational morphism $\mu: \mathcal{Y} \rightarrow \mathcal{X}$ of generically smooth, normal and projective arithmetic varieties and a nef arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{P}}$ of C^0 -type on $\mathcal{Y}$ such that

$$\overline{\mathcal{P}} \leq \mu^*(\overline{\mathcal{D}}) \quad \text{and} \quad \widehat{\text{vol}}(\overline{\mathcal{D}}) - \epsilon \leq \widehat{\text{vol}}(\overline{\mathcal{P}}) \leq \widehat{\text{vol}}(\overline{\mathcal{D}})$$

([7], [35], [28]).

- (7) (Zariski decompositions for arithmetic divisors on arithmetic surfaces) We assume that $d = 1$ and $\mathcal{X}$ is regular. Let $\Upsilon(\overline{\mathcal{D}})$ be the set of all nef arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{L}}$ of C^0 -type on $\mathcal{X}$ with $\overline{\mathcal{L}} \leq \overline{\mathcal{D}}$. If $\Upsilon(\overline{\mathcal{D}}) \neq \emptyset$, then there is the greatest element $\overline{\mathcal{P}}$ of $\Upsilon(\overline{\mathcal{D}})$, that is, $\overline{\mathcal{P}} \in \Upsilon(\overline{\mathcal{D}})$ and $\overline{\mathcal{L}} \leq \overline{\mathcal{P}}$ for all $\overline{\mathcal{L}} \in \Upsilon(\overline{\mathcal{D}})$ ([30], [32]).
- (8) (Special case of Dirichlet's unit theorem on arithmetic varieties) If $\mathcal{D}$ is numerically trivial on the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$ and $\overline{\mathcal{D}}$ is pseudo-effective, then there is $\varphi \in \text{Rat}(\mathcal{X})^\times \otimes \mathbb{R}$ such that $\overline{\mathcal{D}} + \widehat{(\varphi)} \geq 0$ ([31]).

The purpose of this article is to generalize the above results to adelic arithmetic divisors.

0.2. Green functions on analytic spaces over a complete discrete valuation field. Let k be a field and v a non-trivial complete discrete absolute value of k . Let X be a projective and geometrically integral variety over k . Let X^{an} be the analytification of X in the sense of Berkovich [2]. Note that X^{an} is a connected compact Hausdorff space. Let $\text{Rat}(X)$ be the rational function field of X . Let D be an $\mathbb{R}$ -Cartier divisor on X , that is, $D = a_1 D_1 + \cdots + a_r D_r$ for some Cartier divisors $D_1, \dots, D_r$ on X and $a_1, \dots, a_r \in \mathbb{R}$. Let $X = \bigcup_{i=1}^N U_i$ be an affine open covering of X such that each D_j is given by $f_{ji} \in \text{Rat}(X)^\times$ on U_i for $j = 1, \dots, r$. We say that a continuous function

$$g : X^{\text{an}} \setminus \bigcup_{j=1}^r \text{Supp}(D_j)^{\text{an}} \rightarrow \mathbb{R}$$

is a D -Green function of C^0 -type on X^{an} if $g + \sum_{j=1}^r a_j \log |f_{ji}|^2$ extends to a continuous function on U_i^{an} for each $i = 1, \dots, N$.

Let $\mathcal{X}$ be a model of X over $\text{Spec}(k^\circ)$, that is, $\mathcal{X}$ is a projective and flat integral scheme over $\text{Spec}(k^\circ)$ such that the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is X , where

$$k^\circ := \{f \in k \mid v(f) \leq 1\}.$$

We assume that there are Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on $\mathcal{X}$ such that $\mathcal{D}_j \cap X = D_j$ for $j = 1, \dots, r$. We set $\mathcal{D} := a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$. The pair $(\mathcal{X}, \mathcal{D})$ is called a *model of (X, D)* . For $x \in X^{\text{an}} \setminus \bigcup_{j=1}^r \text{Supp}(D_j)^{\text{an}}$, let $f_1, \dots, f_r$ be local equations of $\mathcal{D}_1, \dots, \mathcal{D}_r$ at $\xi = r_{\mathcal{X}}(x)$,

respectively, where $r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_o$ is the reduction map and $\mathcal{X}_o$ is the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. We define $g_{(\mathcal{X}, \mathcal{D})}(x)$ to be

$$g_{(\mathcal{X}, \mathcal{D})}(x) := - \sum_{j=1}^r a_j \log |f_j(x)|^2.$$

It is easy to see that $g_{(\mathcal{X}, \mathcal{D})}$ is a D -Green function of C^0 -type on X^{an} . We call it the *D -Green function induced by the model $(\mathcal{X}, \mathcal{D})$* .

We say that a D -Green function g is of $(C^0 \cap \text{PSH})$ -type if D is nef and there is a sequence $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ of models of (X, D) with the following properties:

- (i) For each $n \geq 1$, $\mathcal{D}_n$ is relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$.
- (ii) If we set $\phi_n = g_{(\mathcal{X}_n, \mathcal{D}_n)} - g$, then $\lim_{n \rightarrow \infty} \|\phi_n\|_{\text{sup}} = 0$.

0.3. Adelic arithmetic divisors. Let K be a number field and O_K the ring of integers in K . We denote the set of all maximal ideals of O_K by M_K . For $\mathfrak{p} \in M_K$, the absolute value $v_{\mathfrak{p}}$ of K at $\mathfrak{p}$ is given by

$$v_{\mathfrak{p}}(f) = \#(O_K/\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(f)}.$$

Let $K_{\mathfrak{p}}$ be the completion of K with respect to $v_{\mathfrak{p}}$. Let X be a d -dimensional, projective, normal and geometrically integral variety over K and let $X_{\mathfrak{p}} := X \times_{\text{Spec}(K)} \text{Spec}(K_{\mathfrak{p}})$, which is also a projective, normal and geometrically integral variety over $K_{\mathfrak{p}}$. Let $X(\mathbb{C})$ be the set of all $\mathbb{C}$ -valued points of X , that is,

$$X(\mathbb{C}) := \{x : \text{Spec}(\mathbb{C}) \rightarrow X \mid x \text{ is a morphism of schemes}\}.$$

Let $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ be the complex conjugation map, that is, for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition of morphisms $\text{Spec}(\mathbb{C}) \xrightarrow{c} \text{Spec}(\mathbb{C})$ and $\text{Spec}(\mathbb{C}) \xrightarrow{x} X$, where $\text{Spec}(\mathbb{C}) \xrightarrow{c} \text{Spec}(\mathbb{C})$ is the morphism induced by the complex conjugation. The space of F_∞ -invariant real valued continuous functions on $X(\mathbb{C})$ with respect to the analytic topology is denoted by $C_{F_\infty}^0(X(\mathbb{C}))$, that is,

$$C_{F_\infty}^0(X(\mathbb{C})) := \{\varphi \in C^0(X(\mathbb{C})) \mid \varphi \circ F_\infty = \varphi\}.$$

A pair $\overline{D} = (D, g)$ of an $\mathbb{R}$ -Cartier divisor D on X and a collection of Green functions

$$g = \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K} \cup \{g_\infty\}$$

is called an *adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X* if the following conditions are satisfied:

- (1) For each $\mathfrak{p} \in M_K$, $g_{\mathfrak{p}}$ is a D -Green function of C^0 -type on $X_{\mathfrak{p}}^{\text{an}}$. In addition, there are a non-empty open set U of $\text{Spec}(O_K)$, a normal model $\mathcal{X}_U$ of X over U and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}_U$ on $\mathcal{X}_U$ such that $\mathcal{D}_U \cap X = D$ and $g_{\mathfrak{p}}$ is a D -Green function induced by the model $(\mathcal{X}_U, \mathcal{D}_U)$ for all $\mathfrak{p} \in U \cap M_K$.
- (2) The function g_∞ is an F_∞ -invariant D -Green function of C^0 -type on $X(\mathbb{C})$.

For simplicity, a collection of Green functions $g = \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K} \cup \{g_\infty\}$ is often expressed by the following symbol:

$$g = \sum_{\mathfrak{p} \in M_K} g_{\mathfrak{p}}[\mathfrak{p}] + g_\infty[\infty].$$

Let $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$ denote the space of all adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X .

Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . We define $H^0(X, D)$ to be

$$H^0(X, D) := \{\phi \in \text{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}.$$

For $\phi \in H^0(X, D)$ and $\wp \in M_K \cup \{\infty\}$, $|\phi| \exp(-g_\wp/2)$ extends to a continuous function, so that its supremum is denoted by $\|\phi\|_{g_\wp}$. The set $\hat{H}^0(X, \overline{D})$ of small sections of $\overline{D}$ and the volume $\widehat{\text{vol}}(\overline{D})$ of $\overline{D}$ are defined by

$$\begin{cases} \hat{H}^0(X, \overline{D}) := \{\phi \in H(X, D) \mid \|\phi\|_{g_\wp} \leq 1 \text{ for all } \wp \in M_K \cup \{\infty\}\}, \\ \widehat{\text{vol}}(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^{d+1}/(d+1)!}, \end{cases}$$

respectively. Similarly as given for arithmetic divisors, we can also introduce several kinds of the positivity of $\overline{D}$ as follows:

- Big: $\widehat{\text{vol}}(\overline{D}) > 0$.
- Relatively nef: $g_\wp$ is of $(C^0 \cap \text{PSH})$ -type for all $\wp \in M_K \cup \{\infty\}$.
- Nef: $\overline{D}$ is relatively nef and $\widehat{\text{deg}}(\overline{D}|_x) \geq 0$ for all closed points x of X (for the definition of $\widehat{\text{deg}}(\overline{D}|_x)$, see Subsection 4.2).
- Pseudo-effective: $\overline{D} + \overline{A}$ is big for any big adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{A}$ of C^0 -type on X .

Further, $\overline{D}$ is said to be *integrable* if $\overline{D} = \overline{D}' - \overline{D}''$ for some relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{D}'$ and $\overline{D}''$ of C^0 -type on X . For integrable adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{D}_1, \dots, \overline{D}_{d+1}$ of C^0 -type on X , the arithmetic intersection number

$$\widehat{\text{deg}}(\overline{D}_1 \cdots \overline{D}_{d+1})$$

can be defined (cf. Subsection 4.5).

0.4. Main results. Let X be a d -dimensional, projective, normal and geometrically integral variety over a number field K . The following theorems are the main results of this article. Theorem 0.4.1, Theorem 0.4.2, Theorem 0.4.3, Theorem 0.4.4 and Theorem 0.4.6 are generalizations of (4), (5), (6), (7) and (8), respectively. Properties (1), (2) and (3) also hold for adelic arithmetic divisors (cf. Theorem 5.1.1). Several similar results on arithmetic toric varieties are known. For details, see [3] and [4]. The adelic version of Fujita's approximation theorem has been already established by Boucksom and Chen [5]. In this article, we give another proof of it and generalize it to $\mathbb{R}$ -divisors. Further, Theorem 0.4.5 is a generalization of the result proved in [32]. Note that the numbering of the main results in the introduction differs from one of the corresponding results in the subsequent chapters, which can be found at the end of each theorem.

Theorem 0.4.1 (Continuity of the volume function for adelic arithmetic divisors). *The volume function $\widehat{\text{vol}} : \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \dots, \overline{D}_r, \overline{A}_1, \dots, \overline{A}_{r'}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be a finite subset of M_K . For a compact subset B in $\mathbb{R}^r$ and a positive number ϵ , there are positive numbers δ and δ' such that*

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{\mathfrak{p}_l}[\mathfrak{p}_l] + \varphi_\infty[\infty] \right) \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon$$

holds for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{p_1} \in C^0(X_{p_1}^{\text{an}}), \dots, \varphi_{p_s} \in C^0(X_{p_s}^{\text{an}})$ and $\varphi_\infty \in C_{F_\infty}^0(X(\mathbb{C}))$ with $(a_1, \dots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{p_l}\|_{\text{sup}} + \|\varphi_\infty\|_{\text{sup}} \leq \delta'$ (cf. Theorem 5.2.1).

Theorem 0.4.2 (Generalized Hodge index theorem for adelic arithmetic divisors). *Let $\overline{D}$ be a relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then*

$$\widehat{\deg}(\overline{D}^{d+1}) \leq \widehat{\text{vol}}(\overline{D}).$$

Moreover, if $\overline{D}$ is nef, then $\widehat{\deg}(\overline{D}^{d+1}) = \widehat{\text{vol}}(\overline{D})$ (cf. Theorem 5.3.2).

Theorem 0.4.3 (Fujita's approximation theorem for adelic arithmetic divisors). *Let $\overline{D}$ be a big adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then, for any positive number ϵ , there are a birational morphism $\mu : Y \rightarrow X$ of normal, projective and geometrically integral varieties over K and a nef adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{P}$ of C^0 -type on Y such that $\overline{P} \leq \mu^*(\overline{D})$ and $\widehat{\text{vol}}(\overline{D}) - \epsilon \leq \widehat{\text{vol}}(\overline{P}) \leq \widehat{\text{vol}}(\overline{D})$ (cf. Theorem 5.1.6).*

Theorem 0.4.4 (Zariski decompositions for adelic arithmetic divisors on curves). *We assume $d = 1$. Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let $\Upsilon(\overline{D})$ be the set of all nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{L}$ of C^0 -type on X with $\overline{L} \leq \overline{D}$. If $\Upsilon(\overline{D}) \neq \emptyset$, then there is the greatest element $\overline{P}$ of $\Upsilon(\overline{D})$, that is, $\overline{P} \in \Upsilon(\overline{D})$ and $\overline{L} \leq \overline{P}$ for all $\overline{L} \in \Upsilon(\overline{D})$. Moreover, the natural map $\hat{H}^0(X, a\overline{P}) \rightarrow \hat{H}^0(X, a\overline{D})$ is bijective for $a \in \mathbb{R}_{>0}$. In particular, $\widehat{\text{vol}}(\overline{P}) = \widehat{\text{vol}}(\overline{D})$ (cf. Corollary 6.2.7).*

Theorem 0.4.5 (Numerical characterization of nef adelic arithmetic divisors on curves). *We assume $d = 1$. Let $\overline{D}$ be an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor on X . Then $\overline{D}$ is nef if and only if $\overline{D}$ is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D})$ (cf. Theorem 7.4.1).*

Theorem 0.4.6 (Special case of Dirichlet's unit theorem for adelic arithmetic divisors). *Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . If D is numerically trivial and $\overline{D}$ is pseudo-effective, then there is $\varphi \in \text{Rat}(X)^\times \otimes \mathbb{R}$ such that $\overline{D} + (\varphi) \geq 0$ (cf. Theorem 8.1.2).*

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0.5. Conventions and terminology.

0.5.1. For a topological space M , let $C^0(M)$ denote the set of all real valued continuous functions on M . Note that $C^0(M)$ forms an $\mathbb{R}$ -algebra.

0.5.2. Let k be a field and v a non-Archimedean absolute value of k . We define k° and $k^{\circ\circ}$ to be

$$k^\circ := \{x \in k \mid v(x) \leq 1\} \quad \text{and} \quad k^{\circ\circ} := \{x \in k \mid v(x) < 1\}.$$

Note that k° is a valuation ring and $k^{\circ\circ}$ is its maximal ideal. If v is either “complete” or “discrete and $\text{char}(k) = 0$ ”, then k° is excellent.

0.5.3. Let M be a finitely generated $\mathbb{Z}$ -module and let $\|\cdot\|$ be a norm of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We define $\hat{h}^0(M, \|\cdot\|)$ and $\hat{\chi}(M, \|\cdot\|)$ to be

$$\begin{cases} \hat{h}^0(M, \|\cdot\|) := \log \#\{x \in M \mid \|x\| \leq 1\}, \\ \hat{\chi}(M, \|\cdot\|) := \log \left(\frac{\text{vol}(B(M, \|\cdot\|))}{\text{vol}(M_{\mathbb{R}}/(M/M_{\text{tor}}))} \right) + \log \#(M_{\text{tor}}), \end{cases}$$

where $B(M, \|\cdot\|)$ is the unit ball with respect to $\|\cdot\|$ (i.e. $B(M, \|\cdot\|) := \{x \in M_{\mathbb{R}} \mid \|x\| \leq 1\}$), M_{tor} is the torsion subgroup of M and $\text{vol}(M_{\mathbb{R}}/(M/M_{\text{tor}}))$ is the volume of the fundamental domain of $M_{\mathbb{R}}/(M/M_{\text{tor}})$.

0.5.4. Let S be a noetherian integral scheme. An integral scheme X over S is called a *variety over S* if X is flat, separated and of finite type over S . If S is given by $\text{Spec}(O_K)$ (i.e. K is a number field and O_K is the ring of integers in K), then a variety over S is often called an *arithmetic variety*.

0.5.5. Let S be a noetherian integral scheme and k the rational function field of S . Let X be a projective variety over k . A projective variety $\mathcal{X}$ over S is called a *model of X over S* if the generic fiber of $\mathcal{X} \rightarrow S$ is X . Moreover, if $\mathcal{X}$ is normal (resp. regular), then $\mathcal{X}$ is called a *normal model of X over S* (resp. *regular model of X over S*). Note that if $\mathcal{X}$ is normal (resp. regular), then X is also normal (resp. regular). We assume that S is an excellent Dedekind scheme, $\dim X = 1$ and X is smooth over k . By [21], for any model $\mathcal{X}$ of X over S , there is a regular model $\mathcal{X}'$ of X over S together with a birational morphism $\mathcal{X}' \rightarrow \mathcal{X}$.

0.5.6. Let $f : \mathcal{X} \rightarrow S$ be a proper morphism of noetherian schemes. Let C be a curve on $\mathcal{X}$, that is, a 1-dimensional reduced and irreducible closed subscheme on $\mathcal{X}$. The curve C is said to be *vertical* with respect to $f : \mathcal{X} \rightarrow S$ if $f(C)$ is a closed point of S . For $\mathcal{L} \in \text{Pic}(\mathcal{X}) \otimes \mathbb{R}$, we say that $\mathcal{L}$ is *relatively nef* with respect to $f : \mathcal{X} \rightarrow S$ if $\deg(\mathcal{L}|_C) \geq 0$ for all vertical curves C on $\mathcal{X}$. Let $\mathcal{D}$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$, that is, $\mathcal{D} = a_1 \mathcal{D}_1 + \dots + a_r \mathcal{D}_r$ for some Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on $\mathcal{X}$ and $a_1, \dots, a_r \in \mathbb{R}$. The $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ is said to be *relatively nef* with respect to $f : \mathcal{X} \rightarrow S$ if $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1)^{\otimes a_1} \otimes \dots \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}_r)^{\otimes a_r} \in \text{Pic}(\mathcal{X}) \otimes \mathbb{R}$ is relatively nef with respect to $f : \mathcal{X} \rightarrow S$.

0.5.7. Let (A, m) be a 1-dimensional noetherian local domain. For $x \in A \setminus \{0\}$, we define $\text{ord}_A(x)$ to be $\text{ord}_A(x) := \text{length}_A(A/xA)$. It is easy to see that

$$\text{ord}_A(xy) = \text{ord}_A(x) + \text{ord}_A(y)$$

for $x, y \in A \setminus \{0\}$, so that it extends to $F^\times$ as a homomorphism, where F is the quotient field of A . Further, if we set $F_{\mathbb{R}}^\times := F^\times \otimes_{\mathbb{Z}} \mathbb{R}$, then ord_A also extends to $F_{\mathbb{R}}^\times$ as an $\mathbb{R}$ -linear form. Let X be a noetherian integral scheme and γ a point of X such that $\dim \mathcal{O}_{X, \gamma} = 1$. Then $\text{ord}_{\mathcal{O}_{X, \gamma}}$ is often denoted by ord_γ or ord_Γ , where Γ is the closure of $\{\gamma\}$.

0.5.8. Let X be a regular scheme and let $\text{Div}(X)$ be the group of Cartier divisors on X . We set $\text{Div}(X)_{\mathbb{R}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$ -Cartier divisor. As X is regular, an $\mathbb{R}$ -Cartier divisor D has a unique expression

$$D = \sum_{\Gamma} a_{\Gamma} \Gamma,$$

where $a_\Gamma \in \mathbb{R}$ and Γ runs over all prime divisors on X . For $D_1, \dots, D_r \in \text{Div}(X)_\mathbb{R}$, we set

$$D_1 = \sum_{\Gamma} a_{1,\Gamma} \Gamma, \dots, D_r = \sum_{\Gamma} a_{r,\Gamma} \Gamma.$$

We define $\max\{D_1, \dots, D_r\}$ and $\min\{D_1, \dots, D_r\}$ to be

$$\begin{cases} \max\{D_1, \dots, D_r\} := \sum_{\Gamma} \max\{a_{1,\Gamma}, \dots, a_{r,\Gamma}\} \Gamma, \\ \min\{D_1, \dots, D_r\} := \sum_{\Gamma} \min\{a_{1,\Gamma}, \dots, a_{r,\Gamma}\} \Gamma. \end{cases}$$

0.5.9. Let M be a finitely generated free $\mathbb{Z}$ -module and let V be a subset of M . Let $\langle V \rangle_{\mathbb{Z}}$ be the $\mathbb{Z}$ -submodule generated by V . The *convex lattice hull* of V is defined to be

$$\{x \in \langle V \rangle_{\mathbb{Z}} \mid \exists m \in \mathbb{Z}_{>0} \exists x_1, \dots, x_m \in V \text{ } mx = x_1 + \dots + x_m\}.$$

1. PRELIMINARIES

The goal of this section is to prepare several kinds of materials for the later sections. Subsection 1.1 is devoted to the proofs of several lemmas. In Subsection 1.2, we consider the support of an $\mathbb{R}$ -Cartier divisor. In Subsection 1.3, we quickly review the analytification of an algebraic scheme in the sense of Berkovich [2]. Finally, in Subsection 1.4, we discuss general facts of Green functions on complex varieties.

1.1. Lemmas. In this subsection, we prove nine lemmas, which are non-trivial and indispensable for other subsections.

Lemma 1.1.1. *Let V be a vector space over $\mathbb{Q}$ and W a subspace of V over $\mathbb{Q}$. Let $x_1, \dots, x_r \in V$ and $a_1, \dots, a_r \in \mathbb{R}$ such that $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. If $a_1 x_1 + \dots + a_r x_r \in W \otimes_{\mathbb{Q}} \mathbb{R}$, then $x_1, \dots, x_r \in W$.*

Proof. We set $H := \{\phi \in \text{Hom}_{\mathbb{Q}}(V, \mathbb{Q}) \mid \phi|_W = 0\} = \text{Hom}_{\mathbb{Q}}(V/W, \mathbb{Q})$. For $\phi \in H$, the natural extension to $\text{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R})$ is denoted by $\phi_{\mathbb{R}}$. As $a_1 x_1 + \dots + a_r x_r \in W \otimes_{\mathbb{Q}} \mathbb{R}$, we have

$$0 = \phi_{\mathbb{R}}(a_1 x_1 + \dots + a_r x_r) = a_1 \phi(x_1) + \dots + a_r \phi(x_r)$$

for all $\phi \in H$. Thus $\phi(x_1) = \dots = \phi(x_r) = 0$ for all $\phi \in H$ because $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$ and $\phi(x_1), \dots, \phi(x_r) \in \mathbb{Q}$. Therefore, $x_1, \dots, x_r \in W$. $\square$

Lemma 1.1.2. *Let V be a vector space over $\mathbb{R}$ and P a convex cone in V . We assume that $P \cap W$ is open in W for any finite dimensional subspace W of V . Let $x_1, \dots, x_r \in V$, $a_1, \dots, a_r \in \mathbb{R}_{>0}$ and $x := a_1 x_1 + \dots + a_r x_r$. If $x \in P$, then there are $x'_1, \dots, x'_r \in (\mathbb{Q}_{>0} x_1 + \dots + \mathbb{Q}_{>0} x_r) \cap P$ and $a'_1, \dots, a'_r \in \mathbb{R}_{>0}$ such that $x = a'_1 x'_1 + \dots + a'_r x'_r$.*

Proof. Let us begin with the following claim:

Claim 1.1.2.1. *Let $x_1, \dots, x_r \in V$, $a_1, \dots, a_r \in \mathbb{R}_{>0}$ and x be the same as above. We assume that $x \in P$. Fix an integer s with $1 \leq s \leq r$. Then there are $c_1, \dots, c_{s-1}, c_{s+1}, \dots, c_r \in \mathbb{Q}_{>0}$ such that $a_i - c_i a_s > 0$ for all $i \neq s$ and*

$$x_s + \sum_{i \neq s} c_i x_i \in P.$$

Proof. As P is cone, we have $x_s + \sum_{i \neq s} (a_i/a_s)x_i \in P$, so that, by using the openness of P , we can find $c_1, \dots, c_{s-1}, c_{s+1}, \dots, c_r \in \mathbb{Q}_{>0}$ such that

$$x_s + \sum_{i \neq s} c_i x_i \in P \quad \text{and} \quad c_i < a_i/s_s \ (\forall i \neq s),$$

as required. $\square$

We set $I = \{i \mid x_i \notin P\}$. We prove the lemma by induction on $\#(I)$. In the case where $\#(I) = 0$, the assertion is obvious, so that we may assume that $I \neq \emptyset$. We choose $s \in I$. Renumbering $x_1, \dots, x_r$, we may assume that

$$\{i \mid x_i \in P\} = \{0 \leq i < s\} \quad \text{and} \quad I = \{s \leq i \leq r\}.$$

By the previous claim, there are $c_1, \dots, c_{s-1}, c_{s+1}, \dots, c_r \in \mathbb{Q}_{>0}$ such that $a_i - c_i a_s > 0$ for all $i \neq s$ and

$$x_s + \sum_{i \neq s} c_i x_i \in P.$$

Note that

$$x = \sum_{i < s} (a_i - c_i a_s) x_i + a_s \left(x_s + \sum_{i \neq s} c_i x_i \right) + \sum_{j > s} (a_j - c_j a_s) x_j.$$

Therefore, by hypothesis of induction, there are

$$x'_1, \dots, x'_r \in \mathbb{Q}_{>0} x_1 + \dots + \mathbb{Q}_{>0} x_{s-1} + \mathbb{Q}_{>0} \left(x_s + \sum_{i \neq s} c_i x_i \right) + \mathbb{Q}_{>0} x_{s+1} + \dots + \mathbb{Q}_{>0} x_r$$

and $a'_1, \dots, a'_r \in \mathbb{R}_{>0}$ such that $x = a'_1 x'_1 + \dots + a'_r x'_r$ and $x'_i \in P$ for all $i = 1, \dots, r$, and hence the lemma follows because

$$\begin{aligned} \mathbb{Q}_{>0} x_1 + \dots + \mathbb{Q}_{>0} x_{s-1} + \mathbb{Q}_{>0} \left(x_s + \sum_{i \neq s} c_i x_i \right) + \mathbb{Q}_{>0} x_{s+1} + \dots + \mathbb{Q}_{>0} x_r \\ \subseteq \mathbb{Q}_{>0} x_1 + \dots + \mathbb{Q}_{>0} x_r. \end{aligned}$$

$\square$

Lemma 1.1.3. *Let M be a finitely generated $\mathbb{Z}$ -module and let $\|\cdot\|$ and $\|\cdot\|'$ be norms of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let M' be a submodule of M such that M/M' is a torsion group, so that $M_{\mathbb{R}} = M'_{\mathbb{R}} := M' \otimes_{\mathbb{Z}} \mathbb{R}$. If $\|\cdot\| \leq \|\cdot\|'$, then*

$$(1.1.3.1) \quad \hat{\chi}(M', \|\cdot\|') \leq \hat{\chi}(M, \|\cdot\|).$$

Moreover, for $\lambda \in \mathbb{R}_{\geq 0}$, the following formulae hold:

$$(1.1.3.2) \quad \hat{h}^0(M, \exp(-\lambda)\|\cdot\|) \leq \hat{h}^0(M, \|\cdot\|) + \lambda \operatorname{rk} M + \log(3) \operatorname{rk} M,$$

$$(1.1.3.3) \quad \hat{\chi}(M, \exp(-\lambda)\|\cdot\|) = \hat{\chi}(M, \|\cdot\|) + \lambda \operatorname{rk} M,$$

$$(1.1.3.4) \quad \hat{h}^0(M, \|\cdot\|) \leq \hat{h}^0(M', \|\cdot\|) + \log \#(M/M') + \log(6) \operatorname{rk} M,$$

$$(1.1.3.5) \quad \hat{\chi}(M, \|\cdot\|) = \hat{\chi}(M', \|\cdot\|) + \log \#(M/M').$$

(For the definition of $\hat{h}^0$ and $\hat{\chi}$, see Conventions and terminology 0.5.3.)

Proof. (1.1.3.1) is obvious. (1.1.3.2) follows from [28, Lemma 1.2.2]. (1.1.3.3) is also obvious because $B(M, \exp(-\lambda)\|\cdot\|) = \exp(\lambda)B(M, \|\cdot\|)$.

Let us consider (1.1.3.4). Let $\pi : M \rightarrow M/M'$ be the canonical homomorphism. Let us choose $x_1, \dots, x_N \in M$ with the following properties:

- (i) $\|x_i\| \leq 1$ for all $i = 1, \dots, N$.
- (ii) $\pi(x_i) \neq \pi(x_j)$ for $i \neq j$.
- (iii) For any $x \in M$ with $\|x\| \leq 1$, there is x_i with $\pi(x) = \pi(x_i)$.

Then we can see that

$$\{x \in M \mid \|x\| \leq 1\} \subseteq \{x_i + x' \mid x' \in M' \text{ and } \|x'\| \leq 2\},$$

and hence,

$$\hat{h}^0(M, \|\cdot\|) \leq \hat{h}^0(M', (1/2)\|\cdot\|) + \log \#(M/M').$$

Therefore, (1.1.3.4) follows from (1.1.3.2).

For (1.1.3.5), let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'/M'_{\text{tor}} & \longrightarrow & M/M_{\text{tor}} & \longrightarrow & (M/M_{\text{tor}})/(M'/M'_{\text{tor}}) & \longrightarrow & 0, \end{array}$$

which shows that we may assume that M is torsion free. Let $\omega_1, \dots, \omega_r$ be a free basis of M such that $a_1\omega_1, \dots, a_r\omega_r$ form a free basis of M' for some $a_1, \dots, a_r \in \mathbb{Z}_{>0}$. Then

$$\text{vol}(M_{\mathbb{R}}/M') = a_1 \cdots a_r \text{vol}(M_{\mathbb{R}}/M).$$

Thus (1.1.3.5) follows because $\#(M/M') = a_1 \cdots a_r$. □

Lemma 1.1.4. *Let A and M be $\mathbb{Z}$ -modules and let $f : M^n \rightarrow A$ be a multi-linear map, that is,*

$$f(x_1, \dots, x_i - x'_i, \dots, x_n) = f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x'_i, \dots, x_n)$$

for all $i = 1, \dots, n$ and $x_1, \dots, x_i, x'_i, \dots, x_n \in M$. Then, for $x_1, \dots, x_n, x'_1, \dots, x'_n \in M$,

$$f(x'_1, \dots, x'_n) = f(x_1, \dots, x_n) + \sum_{i=1}^n f(x'_1, \dots, x'_{i-1}, \delta_i, x_{i+1}, \dots, x_n),$$

where $\delta_i = x'_i - x_i$.

Proof. We prove it by induction on n . In the case where $n = 1$, the above means that $f(x'_1) = f(x_1) + f(x'_1 - x_1)$, which is obvious. In general, using the induction hypothesis, we have

$$f(x'_1, \dots, x'_n) = f(x'_1, \dots, x'_{n-1}, x_n) + f(x'_1, \dots, x'_{n-1}, \delta_n)$$

and

$$\begin{aligned} f(x'_1, \dots, x'_{n-1}, x_n) &= f(x_1, \dots, x_{n-1}, x_n) \\ &\quad + \sum_{i=1}^{n-1} f(x'_1, \dots, x'_{i-1}, \delta_i, x_{i+1}, \dots, x_{n-1}, x_n). \end{aligned}$$

Thus we have the assertion. □

Lemma 1.1.5. *Let V be a vector space over $\mathbb{R}$ and W a subspace of V over $\mathbb{R}$. Let*

$$\langle \rangle : W \times V^{l-1} \rightarrow \mathbb{R}$$

be a multi-linear map over $\mathbb{R}$. We assume that $\langle \rangle$ is symmetric, that is,

$$\langle x_1, x_2, \dots, x_l \rangle = \langle x_2, x_1, \dots, x_l \rangle$$

for all $x_1, x_2 \in W$ and $x_3, \dots, x_l \in V$ and

$$\langle x_1, \dots, x_i, \dots, x_j, \dots, x_l \rangle = \langle x_1, \dots, x_j, \dots, x_i, \dots, x_l \rangle$$

for all $x_1 \in W$, $x_2, \dots, x_l \in V$ and $2 \leq i < j \leq l$. For simplicity, $\langle x_1, \dots, x_l \rangle$ is denoted by $\langle x_1 \cdots x_l \rangle$ or $\langle \prod_i x_i \rangle$. Then

$$\begin{aligned} & \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} (a_i + b_i) \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &= \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} b_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle + \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} a_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} (x_j + b_j) \right\rangle \end{aligned}$$

holds for $a_1, \dots, a_l, b_1, \dots, b_l \in W$ and $x_1, \dots, x_l \in V$.

Proof. If we set

$$\begin{aligned} A := \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} & \left\langle \prod_{i \in I} (a_i + b_i) \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ & - \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} b_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle, \end{aligned}$$

then

$$\begin{aligned} A &= \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \sum_{L \subseteq I} \left\langle \prod_{i \in L} a_i \cdot \prod_{i' \in I \setminus L} b_{i'} \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &\quad - \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} b_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &= \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \sum_{\emptyset \neq L \subseteq I} \left\langle \prod_{i \in L} a_i \cdot \prod_{i' \in I \setminus L} b_{i'} \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &= \sum_{\emptyset \neq L \subseteq \{1, \dots, l\}} \sum_{M \sqcup M' = \{1, \dots, l\} \setminus L} \left\langle \prod_{i \in L} a_i \cdot \prod_{m \in M} b_m \cdot \prod_{m' \in M'} x_{m'} \right\rangle \\ &= \sum_{\emptyset \neq L \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in L} a_i \cdot \prod_{j \in \{1, \dots, l\} \setminus L} (x_j + b_j) \right\rangle, \end{aligned}$$

as desired. $\square$

Lemma 1.1.6. *Let S be a connected Dedekind scheme and k the rational function field of S . Let X be a projective variety over k . Then we have the following:*

- (1) *There exists a model of X over S (cf. Conventions and terminology 0.5.5).*
- (2) *Let J be an invertible fractional ideal sheaf on X . Then there are a model $\mathcal{X}$ of X over S and an invertible fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ such that $\mathcal{J} \cap X = J$.*
- (3) *Let D be an $\mathbb{R}$ -Cartier divisor on X . Then there are a model $\mathcal{X}$ of X over S and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{D} \cap X = D$.*

Proof. (1) As X is projective over k , there is a closed embedding $\iota : X \hookrightarrow \mathbb{P}_k^N$. Let $\mathcal{X}$ be the closure of $\iota(X)$ in $\mathbb{P}_S^N$. Then $\mathcal{X}$ is integral, projective and flat over S because S is a connected Dedekind scheme.

(2) Let $\mathcal{X}'$ be a model of X over S . Then we can find a non-empty open set U of S and an invertible fractional ideal sheaf $\mathcal{J}'_U$ on $\mathcal{X}'_U$ such that $\mathcal{J}'_U \cap X = J$. Therefore, as $\mathcal{X}'$ is noetherian, by using the extension theorem of coherent sheaves (cf. [13, Théorème 6.9.7]), we have a fractional ideal sheaf $\mathcal{J}'$ on $\mathcal{X}'$ such that $\mathcal{J}' \cap \mathcal{X}'_U = \mathcal{J}'_U$. Let $\pi : \mathcal{X} = \text{Proj}(\bigoplus_{m=0}^{\infty} \mathcal{J}'^m) \rightarrow \mathcal{X}'$ be the blowing-up by the fractional ideal sheaf $\mathcal{J}'$. Then, as $\mathcal{J} := \mathcal{J}' \mathcal{O}_{\mathcal{X}}$ is invertible, the assertion of (2) follows.

(3) is a consequence of (2). $\square$

Lemma 1.1.7. *Let k be a field and v an absolute value of k . Let k_v be the completion of k with respect to v . By abuse of notation, the unique extension of v to k_v is also denoted by v . Then we have the following:*

- (1) *We assume that v is discrete. Let X be a projective and geometrically integral variety over k and let $\mathcal{X}$ be a model of X over $\text{Spec}(k^\circ)$. We set*

$$X_v := X \times_{\text{Spec}(k)} \text{Spec}(k_v) \quad \text{and} \quad \mathcal{X}_v := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k_v^\circ).$$

Let $\pi : \mathcal{X}_v \rightarrow \mathcal{X}$ be the projection, and let $(\mathcal{X}_v)_\circ$ and $\mathcal{X}_\circ$ be the central fibers of

$$\mathcal{X}_v \rightarrow \text{Spec}(k_v^\circ) \quad \text{and} \quad \mathcal{X} \rightarrow \text{Spec}(k^\circ),$$

respectively (cf. Conventions and terminology 0.5.2). If we choose $\xi_v \in (\mathcal{X}_v)_\circ$ and $\xi \in \mathcal{X}_\circ$ with $\pi(\xi_v) = \xi$, then we have the following:

- (1.1) *$\mathcal{X}_v$ is a model of X_v over $\text{Spec}(k_v^\circ)$.*
- (1.2) *$\mathcal{O}_{\mathcal{X}, \xi}$ is regular if and only if $\mathcal{O}_{\mathcal{X}_v, \xi_v}$ is regular.*
- (1.3) *We assume that k° is excellent. Then $\mathcal{O}_{\mathcal{X}, \xi}$ is normal if and only if $\mathcal{O}_{\mathcal{X}_v, \xi_v}$ is normal.*
- (2) *Let A be a k -algebra and $A_v := A \otimes_k k_v$. Let $x = |\cdot|_x$ and $x' = |\cdot|_{x'}$ be seminorms of A_v . If $|a \otimes 1|_x = |a \otimes 1|_{x'}$ for all $a \in A$, then $x = x'$.*

Proof. (1) We need to see that $\mathcal{X}_v$ is integral. As $\mathcal{X}_v \rightarrow \text{Spec}(k_v^\circ)$ is flat and the generic fiber of $\mathcal{X}_v \rightarrow \text{Spec}(k_v^\circ)$ is integral, (1.1) follows from [24, Lemma 4.2].

Before starting the proofs of (1.2) and (1.3), let us see $m_{\xi_v} = m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v}$. Let $\kappa(\xi)$ be the residue field at ξ . Here we consider the exact sequence

$$\kappa(\xi) \otimes_{k^\circ} k_v^{\circ\circ} \xrightarrow{\alpha} \kappa(\xi) \otimes_{k^\circ} k_v^\circ \longrightarrow \kappa(\xi) \otimes_{k^\circ} (k_v^\circ / k_v^{\circ\circ}) \longrightarrow 0$$

induced by $0 \rightarrow k_v^{\circ\circ} \rightarrow k_v^\circ \rightarrow k_v^\circ / k_v^{\circ\circ} \rightarrow 0$. Note that $\alpha = 0$ because $\alpha(a \otimes \varpi b) = a \otimes \varpi b = \varpi a \otimes b = 0$ for $a \in \kappa(\xi)$ and $b \in k_v^\circ$. Therefore,

$$\kappa(\xi) \otimes_{k^\circ} k_v^\circ \simeq \kappa(\xi) \otimes_{k^\circ} (k_v^\circ / k_v^{\circ\circ}) \simeq \kappa(\xi) \otimes_{k^\circ} (k^\circ / k^{\circ\circ}) \simeq \kappa(\xi).$$

On the other hand, applying $\otimes_{\mathcal{O}_{\mathcal{X}, \xi}} \mathcal{O}_{\mathcal{X}_v, \xi_v}$ to the exact sequence $0 \rightarrow m_\xi \rightarrow \mathcal{O}_{\mathcal{X}, \xi} \rightarrow \kappa(\xi) \rightarrow 0$, we obtain

$$0 \rightarrow m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v} \rightarrow \mathcal{O}_{\mathcal{X}_v, \xi_v} \rightarrow \kappa(\xi) \otimes_{\mathcal{O}_{\mathcal{X}, \xi}} \mathcal{O}_{\mathcal{X}_v, \xi_v} \rightarrow 0.$$

As π induces the isomorphism $(\mathcal{X}_v)_\circ \xrightarrow{\sim} \mathcal{X}_\circ$, $\pi^{-1}(\xi) = \{\xi_v\}$, so that $\mathcal{O}_{\mathcal{X}_v, \xi_v} = \mathcal{O}_{\mathcal{X}, \xi} \otimes_{k^\circ} k_v^\circ$. Therefore,

$$\kappa(\xi) \otimes_{\mathcal{O}_{\mathcal{X}, \xi}} \mathcal{O}_{\mathcal{X}_v, \xi_v} = \kappa(\xi) \otimes_{\mathcal{O}_{\mathcal{X}, \xi}} (\mathcal{O}_{\mathcal{X}, \xi} \otimes_{k^\circ} k_v^\circ) \simeq \kappa(\xi) \otimes_{k^\circ} k_v^\circ \simeq \kappa(\xi),$$

which shows that $m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v}$ is the maximal ideal of $\mathcal{O}_{\mathcal{X}_v, \xi_v}$, and hence $m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v} = m_{\xi_v}$.

Since $\mathcal{O}_{\mathcal{X}, \xi} \subseteq \mathcal{O}_{\mathcal{X}_v, \xi_v} \subseteq \widehat{\mathcal{O}_{\mathcal{X}, \xi}}$ and $m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v} = m_{\xi_v}$, by [22, Chapter 1, Theorem 3.16], we have $\widehat{\mathcal{O}_{\mathcal{X}_v, \xi_v}} \simeq \widehat{\mathcal{O}_{\mathcal{X}, \xi}}$. Thus (1.2) follows from [1, Proposition 11.24]. Further, (1.3) follows from [12, VI, 7.8.3, (v)].

(2) For $\alpha \in A_v$, we set $\alpha = a_1 \otimes \lambda_1 + \cdots + a_r \otimes \lambda_r$, where $a_1, \dots, a_r \in A$ and $\lambda_1, \dots, \lambda_r \in k_v$. Then we can find sequences $\{\lambda_{1,n}\}_{n=1}^\infty, \dots, \{\lambda_{r,n}\}_{n=1}^\infty$ in k such that $\lambda_i = \lim_{n \rightarrow \infty} \lambda_{i,n}$ for $i = 1, \dots, r$. Here we set

$$\alpha_n = a_1 \otimes \lambda_{1,n} + \cdots + a_r \otimes \lambda_{r,n} = (\lambda_{1,n} a_1 + \cdots + \lambda_{r,n} a_r) \otimes 1.$$

Then,

$$\begin{aligned} \|\alpha_n|_x - |\alpha|_x\| &\leq \|\alpha_n - \alpha\|_x = \|a_1 \otimes (\lambda_{1,n} - \lambda_1) + \cdots + a_r \otimes (\lambda_{r,n} - \lambda_r)\|_x \\ &\leq \|a_1 \otimes (\lambda_{1,n} - \lambda_1)\|_x + \cdots + \|a_r \otimes (\lambda_{r,n} - \lambda_r)\|_x \\ &= \|(a_1 \otimes 1) \cdot (1 \otimes (\lambda_{1,n} - \lambda_1))\|_x + \cdots + \|(a_r \otimes 1) \cdot (1 \otimes (\lambda_{r,n} - \lambda_r))\|_x \\ &= \|a_1 \otimes 1\|_x v(\lambda_{1,n} - \lambda_1) + \cdots + \|a_r \otimes 1\|_x v(\lambda_{r,n} - \lambda_r), \end{aligned}$$

and hence $\lim_{n \rightarrow \infty} \|\alpha_n\|_x = \|\alpha\|_x$. In the same way, $\lim_{n \rightarrow \infty} \|\alpha_n\|_{x'} = \|\alpha\|_{x'}$. On the other hand, by our assumption,

$$\|\alpha_n\|_x = \|(\lambda_{1,n} a_1 + \cdots + \lambda_{r,n} a_r) \otimes 1\|_x = \|(\lambda_{1,n} a_1 + \cdots + \lambda_{r,n} a_r) \otimes 1\|_{x'} = \|\alpha_n\|_{x'}$$

for all $n \geq 1$. Therefore, $\|\alpha\|_x = \|\alpha\|_{x'}$, and hence $x = x'$. $\square$

Lemma 1.1.8. *Let M be a d -dimensional compact Kähler manifold. Let $D_1, \dots, D_l$ be $\mathbb{R}$ -Cartier divisors on M . For each $i = 1, \dots, l$, let g_i be a D_i -Green function of C^∞ -type. Let U be an open set of M such that U is not empty on each connected component of M . Then there are constants $C_1, \dots, C_l \geq 1$ such that C_i depends only on g_i and U , and that*

$$\sup_{x \in M} \{|s|_{t+m_1 g_1 + \cdots + m_l g_l}(x)\} \leq C_1^{m_1} \cdots C_l^{m_l} \sup_{x \in U} \{|s|_{t+m_1 g_1 + \cdots + m_l g_l}(x)\}.$$

for all $m_1, \dots, m_l \in \mathbb{R}_{\geq 0}$, $s \in H^0(M, m_1 D_1 + \cdots + m_l D_l)$ and any locally constant function t on M . Here we use the same notation as in [31, Lemma 1.3.1].

Proof. This is a slight generalization of [31, Lemma 1.3.1]. Let $M = M_1 \cup \cdots \cup M_r$ be the decomposition into connected components and let $U_j := U \cap M_j$ ($j = 1, \dots, r$). By [31, Lemma 1.3.1], there are constants $C_{j1}, \dots, C_{jl} \geq 1$ such that C_{ji} depends only on g_i and U_j , and that

$$\sup_{x \in M_j} \{|s|_{m_1 g_1 + \cdots + m_l g_l}(x)\} \leq C_{j1}^{m_1} \cdots C_{jl}^{m_l} \sup_{x \in U_j} \{|s|_{m_1 g_1 + \cdots + m_l g_l}(x)\}.$$

for all $m_1, \dots, m_l \in \mathbb{R}_{\geq 0}$ and $s \in H^0(M, m_1 D_1 + \cdots + m_l D_l)$, and hence

$$\sup_{x \in M_j} \{|s|_{t+m_1 g_1 + \cdots + m_l g_l}(x)\} \leq C_{j1}^{m_1} \cdots C_{jl}^{m_l} \sup_{x \in U_j} \{|s|_{t+m_1 g_1 + \cdots + m_l g_l}(x)\}.$$

for any locally constant function t on M . We set $C_i = \max\{C_{1i}, \dots, C_{ri}\}$. We choose $x_0 \in M$ such that $\sup_{x \in M} \{|s|_{t+m_1g_1+\dots+m_lg_l}(x)\} = |s|_{t+m_1g_1+\dots+m_lg_l}(x_0)$. If $x_0 \in M_j$, then

$$\begin{aligned} \sup_{x \in M} \{|s|_{t+m_1g_1+\dots+m_lg_l}(x)\} &= \sup_{x \in M_j} \{|s|_{t+m_1g_1+\dots+m_lg_l}(x)\} \\ &\leq C_{j1}^{m_1} \cdots C_{jl}^{m_l} \sup_{x \in U_j} \{|s|_{t+m_1g_1+\dots+m_lg_l}(x)\} \\ &\leq C_1^{m_1} \cdots C_l^{m_l} \sup_{x \in U} \{|s|_{t+m_1g_1+\dots+m_lg_l}(x)\}, \end{aligned}$$

as required. $\square$

Lemma 1.1.9. *Let X be a normal scheme of finite type over a perfect field k . If k' is an extension field of k , then $X' := X \times_{\text{Spec}(k)} \text{Spec}(k')$ is also a normal scheme of finite type over k'*

Proof. Let A be a finitely generated k -algebra and $A' := A \otimes_k k'$. Then we have the following:

- (1) $\dim A = \dim A'$.
- (2) If $f_1, \dots, f_r$ are regular sequence of A , then $f_1 \otimes 1, \dots, f_r \otimes 1$ are also regular sequence of A' .
- (3) If A is regular, then A' is regular.

Indeed, (1) is obvious. (2) is also obvious because k' is flat over k . For (3), as k is perfect, A is smooth over k , so that A' is also smooth over k' , and hence A' is regular.

The normality of X' follows from (1), (2) and (3) together with Serre's criterion of normality. $\square$

1.2. $\mathbb{R}$ -Cartier divisors on a noetherian integral scheme. Let A be a noetherian integral domain and F the quotient field of A . Let $\mathbb{K}$ be either $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$. We set

$$F_{\mathbb{K}}^{\times} := (F^{\times}, \times) \otimes_{\mathbb{Z}} \mathbb{K} \quad \text{and} \quad (A_p^{\times})_{\mathbb{K}} := (A_p^{\times}, \times) \otimes_{\mathbb{Z}} \mathbb{K}$$

for $p \in \text{Spec}(A)$. As $\mathbb{K}$ is flat over $\mathbb{Z}$, we have $(A_p^{\times})_{\mathbb{K}} \subseteq F_{\mathbb{K}}^{\times}$. For $f \in F_{\mathbb{K}}^{\times}$, we define $V_{\mathbb{K}}(f)$ to be

$$V_{\mathbb{K}}(f) := \{p \in \text{Spec}(A) \mid f \notin (A_p^{\times})_{\mathbb{K}}\}.$$

Let us begin with the following proposition:

- Proposition 1.2.1.** (1) $V_{\mathbb{R}}(f) = V_{\mathbb{Q}}(f)$ for $f \in F_{\mathbb{Q}}^{\times}$.
 (2) If $f \in F^{\times}$, then $V_{\mathbb{Q}}(f) = \bigcap_{n=1}^{\infty} V_{\mathbb{Z}}(f^n)$. Moreover, if A is normal, then $V_{\mathbb{Q}}(f) = V_{\mathbb{Z}}(f)$.
 (3) For $f \in F_{\mathbb{K}}^{\times}$, the set $V_{\mathbb{K}}(f)$ is closed in $\text{Spec}(A)$.

Proof. (1) By Lemma 1.1.1 in Subsection 1.1, $(A_p^{\times})_{\mathbb{Q}} = F_{\mathbb{Q}}^{\times} \cap (A_p^{\times})_{\mathbb{R}}$, and hence (1) follows.

(2) As $F_{\mathbb{Q}}^{\times} / (A_p^{\times})_{\mathbb{Q}} = (F^{\times} / A_p^{\times}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $f = 1$ in $F_{\mathbb{Q}}^{\times} / (A_p^{\times})_{\mathbb{Q}}$ if and only if $f^n = 1$ in $F^{\times} / A_p^{\times}$ for some $n \in \mathbb{Z}_{>0}$. Thus the first assertion follows. The second assertion is obvious because $V_{\mathbb{Z}}(f) = V_{\mathbb{Z}}(f^n)$ if A is normal.

(3) First we prove that $V_{\mathbb{Z}}(f)$ is closed for $f \in F^{\times}$. We set

$$I = \{a \in A \mid af \in A\} \quad \text{and} \quad J = If.$$

Clearly I and J are ideals of A . Note that $I_p = \{a \in A_p \mid af \in A_p\}$ by [1, Corollary 3.15]). Thus,

$$f \in A_p^{\times} \iff I_p = A_p \text{ and } J_p = A_p,$$

so that $V_{\mathbb{Z}}(f) = \text{Supp}(\text{Spec}(A/I)) \cup \text{Supp}(\text{Spec}(A/J))$, which is closed.

Next let us see that $V(f)_{\mathbb{Q}}$ is closed for $f \in F_{\mathbb{Q}}^{\times}$. Clearly we may assume that $f \in F^{\times}$ because, for $n \in \mathbb{Z}_{>0}$, $f \in (A_p^{\times})_{\mathbb{Q}}$ if and only if $f^n \in (A_p^{\times})_{\mathbb{Q}}$. Thus, by (2), $V_{\mathbb{Q}}(f)$ is closed.

Finally we consider the case $\mathbb{K} = \mathbb{R}$. We can find $f_1, \dots, f_r \in F^{\times}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that $f = f_1^{a_1} \cdots f_r^{a_r}$ and $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. Then, by Lemma 1.1.1,

$$f \in (A_p^{\times})_{\mathbb{R}} \iff f_1, \dots, f_r \in (A_p^{\times})_{\mathbb{Q}},$$

and hence $V_{\mathbb{R}}(f) = \bigcup_{i=1}^r V_{\mathbb{Q}}(f_i)$, which is closed by the previous observation. $\square$

Definition 1.2.2. Let X be a noetherian integral scheme and let $\text{Rat}(X)$ be the rational function field of X . Let $\text{Div}(X)$ be the group of Cartier divisors on X , that is,

$$\text{Div}(X) := H^0(X, \text{Rat}(X)^{\times} / \mathcal{O}_X^{\times}).$$

We set

$$\text{Div}(X)_{\mathbb{K}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K} \quad \text{and} \quad \text{Rat}(X)_{\mathbb{K}}^{\times} := \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K}.$$

An element of $\text{Div}(X)_{\mathbb{K}}$ (resp. $\text{Rat}(X)_{\mathbb{K}}^{\times}$) is called a $\mathbb{K}$ -Cartier divisor on X (resp. $\mathbb{K}$ -rational function on X). A $\mathbb{K}$ -rational function $f \in \text{Rat}(X)_{\mathbb{K}}^{\times}$ naturally gives rise to a $\mathbb{K}$ -Cartier divisor, which is called the $\mathbb{K}$ -principal divisor of f and is denoted by $(f)_{\mathbb{K}}$. Occasionally, $(f)_{\mathbb{K}}$ is denoted by (f) for simplicity. For $D \in \text{Div}(X)_{\mathbb{K}}$ (i.e., $D = a_1 D_1 + \cdots + a_r D_r$ for some $D_1, \dots, D_r \in \text{Div}(X)$ and $a_1, \dots, a_r \in \mathbb{K}$), there is an affine open covering $X = \bigcup_{i=1}^N \text{Spec}(A_i)$ of X such that D is given by some $f_i \in \text{Rat}(X)_{\mathbb{K}}^{\times}$ on U_i and $f_i/f_j \in (\mathcal{O}_{X,p}^{\times})_{\mathbb{K}} := \mathcal{O}_{X,p}^{\times} \otimes_{\mathbb{Z}} \mathbb{K}$ for all $p \in U_i \cap U_j$, so that $V_{\mathbb{K}}(f_i) = V_{\mathbb{K}}(f_j)$ on $U_i \cap U_j$, where $U_i = \text{Spec}(A_i)$ for $i = 1, \dots, N$. Therefore, we have a unique closed set Z on X such that $Z|_{U_i} = V_{\mathbb{K}}(f_i)$ for all $i = 1, \dots, N$. It is called the $\mathbb{K}$ -support of D and is denoted by $\text{Supp}_{\mathbb{K}}(D)$. By Proposition 1.2.1, $\text{Supp}_{\mathbb{R}}(D) = \text{Supp}_{\mathbb{Q}}(D)$ for $D \in \text{Div}(X)_{\mathbb{Q}}$ and $\text{Supp}_{\mathbb{Q}}(D) = \bigcap_{n=1}^{\infty} \text{Supp}_{\mathbb{Z}}(nD)$ for $D \in \text{Div}(X)$.

From now on, we assume that X is normal. Then $\text{Supp}_{\mathbb{Q}}(D) = \text{Supp}_{\mathbb{Z}}(D)$ for $D \in \text{Div}(X)$. For a $\mathbb{K}$ -Cartier divisor D on X , the associated $\mathbb{K}$ -Weil divisor D_W of D is defined by

$$D_W := \sum_{\Gamma : \text{prime divisor}} \text{ord}_{\Gamma}(f_{\Gamma}) \Gamma,$$

where f_{Γ} is a local equation of D at Γ . The support of D as a Weil-divisor is denoted by $\text{Supp}_W(D)$, that is,

$$\text{Supp}_W(D) := \bigcup_{\text{ord}_{\Gamma}(f_{\Gamma}) \neq 0} \Gamma.$$

Proposition 1.2.3. We assume that X is normal. Let D be a $\mathbb{K}$ -Cartier divisor on X . Then $\text{Supp}_W(D) \subseteq \text{Supp}_{\mathbb{K}}(D)$. Further, if X is regular, then $\text{Supp}_W(D) = \text{Supp}_{\mathbb{K}}(D)$.

Proof. We use the same notation as in Definition 1.2.2. Let $p \in U_i \setminus \text{Supp}_{\mathbb{K}}(D)$. Then $f_i \in (\mathcal{O}_{X,p}^{\times})_{\mathbb{K}}$. In particular, $\text{ord}_{\Gamma}(f_i) = 0$ for all prime divisors Γ with $p \in \Gamma$, and hence $p \notin \text{Supp}_W(D)$, as desired.

We assume that X is regular. Let $p \in U_i \cap \text{Supp}_{\mathbb{K}}(D)$. As $\mathcal{O}_{X,p}$ is a UFD, there are distinct prime elements $h_1, \dots, h_r \in \mathcal{O}_{X,p}$ modulo $\mathcal{O}_{X,p}^{\times}$, $u \in (\mathcal{O}_{X,p}^{\times})_{\mathbb{K}}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that $f_i = u h_1^{a_1} \cdots h_r^{a_r}$. If $a_1 = \cdots = a_r = 0$, then $f_i \in (\mathcal{O}_{X,p}^{\times})_{\mathbb{K}}$, which contracts to $p \in U_i \cap \text{Supp}_{\mathbb{K}}(D)$, so that we may assume that $a_1, \dots, a_r \in \mathbb{R}_{\neq 0}$. Since $h_1, \dots, h_r$ are distinct modulo $\mathcal{O}_{X,p}^{\times}$, $\Gamma_1 = \text{Spec}(\mathcal{O}_{X,p}/h_1), \dots, \Gamma_r = \text{Spec}(\mathcal{O}_{X,p}/h_r)$ give rise to distinct

prime divisors. In addition, $\text{ord}_\Gamma(f_i) = a_j$ for $j = 1, \dots, r$. Therefore, $p \in \text{Supp}_W(D)$. $\square$

Finally let us consider Hartogs' lemma for $\mathbb{R}$ -rational functions.

Lemma 1.2.4 (Hartogs' lemma for $\mathbb{R}$ -rational functions). *Let A be a normal and noetherian domain and F the quotient field of A . For $x \in F^\times$, if $\text{ord}_\Gamma(x) \geq 0$ for all prime divisors Γ of A , then there are $x_1, \dots, x_r \in A \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $x = x_1^{a_1} \cdots x_r^{a_r}$.*

Proof. Let Σ be the set of all prime divisors of A . Clearly we can find $y_1, \dots, y_r \in F^\times$ and $c_1, \dots, c_r \in \mathbb{R}_{>0}$ such that $x = y_1^{c_1} \cdots y_r^{c_r}$ and $c_1, \dots, c_r$ are linearly independent over $\mathbb{Q}$. We set $\Sigma' = \{\Gamma \in \Sigma \mid \text{ord}_\Gamma(x) > 0\}$. Let us see the following:

Claim 1.2.4.1. Σ' is a finite set and $\text{ord}_\Gamma(y_i) = 0$ for all $\Gamma \in \Sigma \setminus \Sigma'$ and $i = 1, \dots, r$.

Proof. If $\Gamma \notin \text{Supp}_\mathbb{Z}((x)_\mathbb{R})$, then $\text{ord}_\Gamma(x) = 0$, so that Σ' is a finite set. Moreover, if $\Gamma \in \Sigma \setminus \Sigma'$, then

$$0 = \text{ord}_\Gamma(x) = c_1 \text{ord}_\Gamma(y_1) + \cdots + c_r \text{ord}_\Gamma(y_r),$$

and hence $\text{ord}_\Gamma(y_1) = \cdots = \text{ord}_\Gamma(y_r) = 0$ by the linear independency of $c_1, \dots, c_r$ over $\mathbb{Q}$. $\square$

If we set $P = \{z \in F^\times \mid \text{ord}_\Gamma(z) > 0 \text{ for all } \Gamma \in \Sigma'\}$, then P is a convex cone and $P \cap W$ is open for any finite dimensional subspace W of $F^\times$, so that, by virtue of Lemma 1.1.2 in Subsection 1.1, there are $e_{ij} \in \mathbb{Q}_{>0}$ ($i, j = 1, \dots, r$) and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ such that, if we set $x_i = y_1^{e_{i1}} \cdots y_r^{e_{ir}}$ for $i = 1, \dots, r$, then $x = x_1^{a_1} \cdots x_r^{a_r}$ and $x_i \in P$ for $i = 1, \dots, r$. Replacing e_{ij} by $e e_{ij}$ and a_i by a_i/e for some $e \in \mathbb{Z}_{>0}$, we may assume that $e_{ij} \in \mathbb{Z}$ for all i, j . In particular, $x_i \in F^\times$ for $i = 1, \dots, r$. Note that, for $\Gamma \in \Sigma \setminus \Sigma'$,

$$\text{ord}_\Gamma(x_i) = e_{i1} \text{ord}_\Gamma(y_1) + \cdots + e_{ir} \text{ord}_\Gamma(y_r) = 0,$$

and hence $\text{ord}_\Gamma(x_i) \geq 0$ for all $\Gamma \in \Sigma$ and $i = 1, \dots, r$. Therefore, by algebraic Hartogs' lemma (cf. [23, Theorem 11.5]), $x_i \in A \setminus \{0\}$ for $i = 1, \dots, r$, as required. $\square$

1.3. Analytification of algebraic schemes over a complete valuation field. Throughout this subsection, k is a field and v is a complete absolute value of k . Here we quickly review the analytification of algebraic schemes over k in the sense of Berkovich [2].

Let A be a k -algebra. We say that a map $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ is a *multiplicative semi-norm over k* if the following conditions are satisfied:

- (1) $|a + b| \leq |a| + |b|$ for all $a, b \in A$.
- (2) $|ab| = |a||b|$ for all $a, b \in A$.
- (3) $|a| = v(a)$ for all $a \in k$.

Let $x = |\cdot|_x$ be a multiplicative semi-norm over k . We set $p_x := \{a \in A \mid |a|_x = 0\}$, which is a prime ideal of A . We call p_x the *associated prime of x* . The residue field at p_x is denoted by $k(x)$. Clearly x descends to an absolute value v_x of $k(x)$ such that $v_x(a) = v(a)$ for all $a \in k$. The field $k(x)$ and the absolute value v_x are called the *residue field of x* and the *associated absolute value of x* , respectively. Conversely, let v' be an absolute value of the residue field $k(p)$ at $p \in \text{Spec}(A)$ such that $v'(a) = v(a)$ for all $a \in k$. If we set $|a| := v'(a \bmod p)$ for $a \in A$, then $|\cdot|$ yields a multiplicative semi-norm over k whose residue field and associated absolute value are $k(p)$ and v' , respectively. In particular, this observation shows that if v is non-Archimedean, then $|\cdot|$ is also non-Archimedean, that is, $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in A$.

Let $\text{Spec}_k^{\text{an}}(A)$ denote the set of all multiplicative semi-norms over k . For $x = |\cdot|_x \in \text{Spec}_k^{\text{an}}(A)$, $|a|_x$ is often denoted by $|a(x)|$. We equip the weakest topology to $\text{Spec}_k^{\text{an}}(A)$ such that the map $\text{Spec}_k^{\text{an}}(A) \rightarrow \mathbb{R}_{\geq 0}$ given by $x \mapsto |a(x)|$ is continuous for every $a \in A$, that is, the collection

$$\left\{ \{x \in \text{Spec}_k^{\text{an}}(A) \mid |a(x)| \in U\} \right\}_{a, U} \quad (\text{where } a \in A \text{ and } U \text{ is an open set in } \mathbb{R}_{\geq 0})$$

forms a subbasis of the topology. The map $\text{Spec}_k^{\text{an}}(A) \rightarrow \text{Spec}(A)$ given by $x \mapsto p_x$ is denoted by p . It is easy to see that $p : \text{Spec}_k^{\text{an}}(A) \rightarrow \text{Spec}(A)$ is continuous.

Let $f : A \rightarrow B$ be a homomorphism of k -algebras. We define a map

$$f^{\text{an}} : \text{Spec}_k^{\text{an}}(B) \rightarrow \text{Spec}_k^{\text{an}}(A)$$

to be $|a|_{f^{\text{an}}(y)} = |f(a)|_y$ for $y = |\cdot|_y \in \text{Spec}_k^{\text{an}}(B)$ and $a \in A$. We can easily check that $f^{\text{an}} : \text{Spec}_k^{\text{an}}(B) \rightarrow \text{Spec}_k^{\text{an}}(A)$ is continuous. Let s be a non-nilpotent element of A . Let $\iota : A \rightarrow A_s$ be the canonical homomorphism. Then we can see that ι^{an} yields a homeomorphism

$$(1.3.1) \quad \text{Spec}_k^{\text{an}}(A_s) \xrightarrow[\text{homeo}]{\approx} \{x \in \text{Spec}_k^{\text{an}}(A) \mid |s(x)| \neq 0\}.$$

Let X be an algebraic scheme over k , that is, a scheme separated and of finite type over k . If $X = \text{Spec}(A)$ is an affine scheme over k , then $X^{\text{an}} := \text{Spec}_k^{\text{an}}(A)$. In general, if $X = \bigcup_{i=1}^N U_i$ is an affine open covering of X , then X^{an} is defined by gluing together U_i^{an} as a topological space (cf. (1.3.1)). For each i , we can define $p : U_i^{\text{an}} \rightarrow U_i$, which can be extended to a continuous map $p : X^{\text{an}} \rightarrow X$. Let $f : X \rightarrow Y$ be a morphism of algebraic schemes over k . We can see that f induces a natural continuous map $f^{\text{an}} : X^{\text{an}} \rightarrow Y^{\text{an}}$.

From now on, we assume that v is non-Archimedean and X is proper over k . Let $\mathcal{X}$ be a proper and flat scheme over $\text{Spec}(k^\circ)$ such that the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is X . Let $\mathcal{X}_\circ$ be the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$, that is, $\mathcal{X}_\circ = \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k^\circ/k^\circ)$ (for the definitions of k° and k° , see Conventions and terminology 0.5.2). Let

$$r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_\circ$$

be the *reduction map* induced by $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$, which can be defined in the following way: For $x \in X^{\text{an}}$, let $k(x)$ be the residue field of x . Then, by using the valuation criterion of properness, there is a morphism $t : \text{Spec}(k(x)^\circ) \rightarrow \mathcal{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{Spec}(k(x)) & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow t & \downarrow \\ \text{Spec}(k(x)^\circ) & \longrightarrow & \text{Spec}(k^\circ) \end{array}$$

Then $r_{\mathcal{X}}(x)$ is given by $t(\text{Spec}(k(x)^\circ))$. The morphism $t : \text{Spec}(k(x)^\circ) \rightarrow \mathcal{X}$ yields a homomorphism $\mathcal{O}_{\mathcal{X}, r_{\mathcal{X}}(x)} \rightarrow k(x)^\circ$. In particular,

$$(1.3.2) \quad |f|_x \leq 1 \text{ for all } f \in \mathcal{O}_{\mathcal{X}, r_{\mathcal{X}}(x)}.$$

It is well-known that $r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_\circ$ is anti-continuous, that is, for any open set U of $\mathcal{X}_\circ$, $r_{\mathcal{X}}^{-1}(U)$ is closed (cf. [2, Section 2.4]). Let Y be another proper algebraic scheme over k and $\mu : Y \rightarrow X$ a morphism over k . Let $\mathcal{Y} \rightarrow \text{Spec}(k^\circ)$ be a proper and flat scheme over $\text{Spec}(k^\circ)$ such that the generic fiber of $\mathcal{Y} \rightarrow \text{Spec}(k^\circ)$ is Y and there is

a morphism $\tilde{\mu} : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec}(k^\circ)$ as an extension of μ . It is easy to see that the following diagram is commutative:

$$(1.3.3) \quad \begin{array}{ccc} Y^{\text{an}} & \xrightarrow{r_{\mathcal{Y}}} & \mathcal{Y}_\circ \\ \mu^{\text{an}} \downarrow & \square & \downarrow \tilde{\mu} \\ X^{\text{an}} & \xrightarrow{r_{\mathcal{X}}} & \mathcal{X}_\circ \end{array}$$

1.4. Green functions on complex varieties. In this subsection, we consider general facts of Green functions on complex varieties. Let X be a complex variety (i.e. a variety over $\mathbb{C}$). A continuous function φ on X is called a *continuous plurisubharmonic function* if there is a desingularization $\mu : Y \rightarrow X$ such that $\varphi \circ \mu$ is a continuous plurisubharmonic function on Y .

Proposition 1.4.1. *For a continuous function φ on X , the following are equivalent:*

- (1) *φ is a continuous plurisubharmonic function on X .*
- (2) *For any desingularization $\nu : Z \rightarrow X$, $\varphi \circ \nu$ is a continuous plurisubharmonic function on Z .*

Proof. It is sufficient to show that (1) $\implies$ (2). By our assumption, there is a desingularization $\mu : Y \rightarrow X$ such that $\varphi \circ \mu$ is a continuous plurisubharmonic function on Y . We can find a non-singular complex variety W together with proper birational morphisms $\mu' : W \rightarrow Z$ and $\nu' : W \rightarrow Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} W & \xrightarrow{\nu'} & Y \\ \mu' \downarrow & & \downarrow \mu \\ Z & \xrightarrow{\nu} & X \end{array}$$

Then, as $\varphi \circ \nu \circ \mu' = \varphi \circ \mu \circ \nu'$ and $\varphi \circ \mu$ is plurisubharmonic, by [19, Corollary 2.9.5], we can see that $\varphi \circ \nu \circ \mu'$ is plurisubharmonic on W . Let Z_0 be a non-empty Zariski open set of Z such that μ' is an isomorphism over Z_0 . Then $\varphi \circ \nu$ is plurisubharmonic on Z_0 , so that, by [19, Theorem 2.9.22], $\varphi \circ \nu$ is plurisubharmonic on Z because $\varphi \circ \nu$ is continuous on Z . $\square$

Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let D be a $\mathbb{K}$ -Cartier divisor on X , that is,

$$D = a_1 D_1 + \cdots + a_r D_r$$

for some Cartier divisors $D_1, \dots, D_r$ on X and $a_1, \dots, a_r \in \mathbb{K}$. For details of $\mathbb{K}$ -Cartier divisors, see Subsection 1.2. Let $X = \bigcup_{i=1}^N \text{Spec}(A_i)$ be an affine open covering of X and let $f_i \in \text{Rat}(X)_{\mathbb{K}}^\times$ be a local equation of D on $\text{Spec}(A_i)$ for each $i = 1, \dots, N$. Let g be a continuous function on $X \setminus \text{Supp}_{\mathbb{K}}(D)$. We say that g is a *D -Green function of C^0 -type* (resp. *D -Green function of $(C^0 \cap \text{PSH})$ -type*) if $g + \log |f_i|^2$ extends to a continuous function (resp. a continuous plurisubharmonic function) on $\text{Spec}(A_i)$ for each $i = 1, \dots, N$. Note that if g and g' are D -Green function of C^0 -type on X , then $g - g'$ extends to a continuous function on X .

Proposition 1.4.2. *We assume that $\mathbb{K} = \mathbb{R}$ and X is normal. Let g be a D -Green function of C^0 -type on X and $\phi \in \text{Rat}(X)_{\mathbb{R}}^\times$. If $D + (\phi) \geq 0$, then $|\phi| \exp(-g/2)$ extends to a continuous function on X .*

Proof. This is a local question, so that we may assume that $X = \text{Spec}(A)$ and $D = (f)$ for some $f \in A_{\mathbb{R}}^{\times}$. Then $\text{ord}_{\Gamma}(f\phi) \geq 0$ for all prime divisors Γ on X , and hence, by Lemma 1.2.4, there are $x_1, \dots, x_r \in A \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $f\phi = x_1^{a_1} \cdots x_r^{a_r}$. Thus $|f\phi|$ is a continuous function on X . On the other hand,

$$|\phi| \exp(-g/2) = |f\phi| \exp(-(g + \log|f|^2)/2).$$

Therefore, the assertion follows. $\square$

Let $c : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{C})$ be the morphism given by the complex conjugation map $z \mapsto \bar{z}$. Let X^c denote the fiber product $X \times_{\text{Spec}(\mathbb{C})}^c \text{Spec}(\mathbb{C})$ in terms of c and let $F : X^c \rightarrow X$ be the projection morphism. For $x \in X^c(\mathbb{C})$, the composition

$$\text{Spec}(\mathbb{C}) \xrightarrow{c} \text{Spec}(\mathbb{C}) \xrightarrow{x} X^c \xrightarrow{F} X$$

yields a $\mathbb{C}$ -valued point of X , so that we define $F_{\infty} : X^c(\mathbb{C}) \rightarrow X(\mathbb{C})$ to be

$$F_{\infty}(x) = F \circ x \circ c.$$

Proposition 1.4.3. *Let g be a D -Green function of C^0 -type on X . If we set $D^c = F^*(D)$ and $g^c = F_{\infty}^*(g)$, then g^c is a D^c -Green function of C^0 -type on X^c . Moreover, if g is of $(C^0 \cap \text{PSH})$ -type, then g^c is also of $(C^0 \cap \text{PSH})$ -type.*

Proof. First of all, note that

$$F^*(\psi)(x) = x^*(F^*(\psi)) = c^*(c^*(x^*(F^*(\psi)))) = \overline{(F_{\infty}(x))^*(\psi)} = \overline{\psi(F_{\infty}(x))}$$

for $x \in X^c(\mathbb{C})$ and $\psi \in \text{Rat}(X)^{\times}$. Let f be a local equation of D . Then $g + \log|f|^2$ is a continuous function, so that $F_{\infty}^*(g + \log|f|^2) = g^c + F_{\infty}^*(\log|f|^2)$ is also a continuous function. Moreover, by the previous observation, for $x \in X^c(\mathbb{C})$,

$$F_{\infty}^*(\log|f|^2)(x) = \log|f(F_{\infty}(x))|^2 = \log|F^*(f)(x)|^2.$$

Therefore, g^c is an D^c -Green function of C^0 -type because $F^*(f)$ is a local equation of D^c .

For the last assertion, it is sufficient to show that if u is a continuous plurisubharmonic function on X , then $F_{\infty}^*(u)$ is also a continuous plurisubharmonic function on X^c . First of all, we may assume that X is non-singular by Proposition 1.4.1. As the assertion is a local question, we may further assume that

$$X = \text{Spec}(\mathbb{C}[x_1, \dots, x_N]/(f_1, \dots, f_l))$$

for some $f_1, \dots, f_l \in \mathbb{C}[x_1, \dots, x_N]$. For $f = \sum a_{e_1, \dots, e_N} x_1^{e_1} \cdots x_N^{e_N} \in \mathbb{C}[x_1, \dots, x_N]$, we define $\bar{f}$ to be

$$\bar{f} := \sum \overline{a_{e_1, \dots, e_N}} x_1^{e_1} \cdots x_N^{e_N}.$$

Then

$$X^c = \text{Spec}(\mathbb{C}[x_1, \dots, x_N]/(\bar{f}_1, \dots, \bar{f}_l)).$$

Moreover, $F : X^c \rightarrow X$ is the morphism arising from the homomorphism $f \mapsto \bar{f}$ and $F_{\infty} : X^c(\mathbb{C}) \rightarrow X(\mathbb{C})$ is given by $(a_1, \dots, a_N) \mapsto (\overline{a_1}, \dots, \overline{a_N})$. We need to show that $F_{\infty}^*(u)$ is plurisubharmonic around any point x of $X^c(\mathbb{C})$. Clearly we may assume that $x = 0$. Changing the order of $x_1, \dots, x_N$, we may further assume that $x_1, \dots, x_n$ ($n \leq N$) form a local parameter of X at 0, and hence $x_1, \dots, x_n$ also form a local parameter of X^c at 0. Here let us consider the projections $\pi : X(\mathbb{C}) \rightarrow \mathbb{C}^n$ and $\pi^c : X^c(\mathbb{C}) \rightarrow \mathbb{C}^n$ given by

$(a_1, \dots, a_N) \mapsto (a_1, \dots, a_n)$. Then π and π^c is étale at 0. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} X^c(\mathbb{C}) & \xrightarrow{F_\infty} & X(\mathbb{C}) \\ \pi^c \downarrow & & \downarrow \pi \\ \mathbb{C}^n & \xrightarrow{F_\infty} & \mathbb{C}^n, \end{array}$$

where $F_\infty : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is given by $(a_1, \dots, a_n) \mapsto (\overline{a_1}, \dots, \overline{a_n})$. We choose a neighborhood U of 0 such that π is an isomorphism on U . Then π^c is also an isomorphism on $F_\infty^{-1}(U)$ at 0. Therefore, we can reduced the problem to a neighborhood of $\mathbb{C}^n$ at the origin and hence the assertion follows from the same argument as in [30, Lemma 5.1.1]. $\square$

Definition 1.4.4. Let $\mathcal{X}$ be a normal and projective arithmetic variety. Note that $\mathcal{X}$ is not necessarily generically smooth. A pair $\overline{\mathcal{D}} = (\mathcal{D}, g)$ of an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ and an F_∞ -invariant $\mathcal{D}$ -Green function g of C^0 -type on $\mathcal{X}(\mathbb{C})$ is called an *arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{X}$* . Moreover, if $\mathcal{D}$ is relatively nef and g is of $(C^0 \cap \text{PSH})$ -type, then $\overline{\mathcal{D}}$ is said to be *relatively nef*.

2. ADELIC $\mathbb{R}$ -CARTIER DIVISORS OVER A DISCRETE VALUATION FIELD

In this section, we introduce an adelic $\mathbb{R}$ -Cartier divisor on a projective variety over a discrete valuation field and study their basic properties. Roughly speaking, an adelic $\mathbb{R}$ -Cartier divisor is a pair of an $\mathbb{R}$ -Cartier divisor and a Green function on the analytification of the given variety, which is an analogue of Arakelov divisors (i.e. arithmetic divisors) on an arithmetic variety.

Throughout this section, let k be a field and v a discrete absolute value. We set

$$k^\circ := \{a \in k \mid v(a) \leq 1\} \quad \text{and} \quad k^\circ\circ := \{a \in k \mid v(a) < 1\}.$$

Let ϖ be a uniformizing parameter of v , that is, $k^\circ\circ = \varpi k^\circ$. Note that v might be trivial, so that we do not exclude the case where $\varpi = 0$. Let us begin with Green functions on analytic spaces over a complete discrete valuation field.

2.1. Green functions on analytic spaces over a discrete valuation field. We assume that v is complete. Let X be a projective and geometrically integral variety over k . Let $\text{Rat}(X)$ be the rational function field of X . Let $U = \text{Spec}(A)$ be an affine open set of X . Let $p \in U$ and $x = |\cdot|_x \in U^{\text{an}}$ such that $p_x \subseteq p$, where p_x is the associated prime of x (cf. Subsection 1.3). Then we have the natural extension $|\cdot|_x : A_p \rightarrow \mathbb{R}_{\geq 0}$ of $|\cdot|_x$ on A given by $|a/s|_x = |a|_x/|s|_x$ for $a \in A$ and $s \in A \setminus p$, which yields the group homomorphism $|\cdot|_x : A_p^\times \rightarrow \mathbb{R}_{>0}$. Thus we obtain a canonical extension $(A_p^\times)_{\mathbb{R}} \rightarrow \mathbb{R}_{>0}$, which is also denoted by $|\cdot|_x$ by abuse of notation. Let $f \in \text{Rat}(X)_{\mathbb{R}}^\times$ and $x \in U^{\text{an}} \setminus V_{\mathbb{R}}(f)^{\text{an}}$ (see Subsection 1.2 for the definitions of $\text{Rat}(X)_{\mathbb{R}}^\times$ and $V_{\mathbb{R}}(f)$). As $p_x \notin V_{\mathbb{R}}(f)$, we get $f \in (A_{p_x}^\times)_{\mathbb{R}}$, and hence $|f(x)| \in \mathbb{R}_{>0}$. Therefore, we have the map $U^{\text{an}} \setminus V_{\mathbb{R}}(f)^{\text{an}} \rightarrow \mathbb{R}$ given by $x \mapsto \log|f(x)|^2$. We denote it by $\log|f|^2$. Clearly $\log|f|^2$ is continuous on $U^{\text{an}} \setminus V_{\mathbb{R}}(f)^{\text{an}}$.

Definition 2.1.1. Let D be an $\mathbb{R}$ -Cartier divisor on X , that is, $D \in \text{Div}(X)_{\mathbb{R}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $X = \bigcup_{i=1}^N U_i$ be an affine open covering of X such that D is given by $f_i \in \text{Rat}(X)_{\mathbb{R}}^\times$ on U_i . A continuous function $g : X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}} \rightarrow \mathbb{R}$ is called a *D -Green function of C^0 -type on X^{an}* if $g + \log|f_i|^2$ extends to a continuous function on U_i^{an} for each $i = 1, \dots, N$.

For example, for $f \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, $-\log|f|^2$ is an $(f)_{\mathbb{R}}$ -Green function of C^0 -type on X^{an} , where $(f)_{\mathbb{R}}$ is the $\mathbb{R}$ -principal divisor of f (cf. Subsection 1.2). We set

$$C_{\eta}^0(X^{\text{an}}) := \varinjlim_{U: \text{Zariski open set of } X} C^0(U^{\text{an}}).$$

The space of all Green functions forms a subspace of $C_{\eta}^0(X^{\text{an}})$ over $\mathbb{R}$. More precisely, we have the following proposition:

Proposition 2.1.2. *Let D and D' be $\mathbb{R}$ -Cartier divisors on X . Let g be a D -Green function of C^0 -type on X^{an} and g' a D' -Green function of C^0 -type on X^{an} . Then we have the following:*

- (1) *For $a, b \in \mathbb{R}$, $ag + bg'$ is an $(aD + bD')$ -Green function of C^0 -type.*
- (2) *If $D = D'$, then $\max\{g, g'\}$ and $\min\{g, g'\}$ are D -Green functions of C^0 -type.*

Proof. (1) Let $X = \bigcup_{i=1}^N U_i$ be an affine open covering of X such that D and D' are given by f_i and f'_i on U_i , respectively. By our assumption, there are continuous functions φ and φ' on U_i^{an} such that $g = -\log|f_i|^2 + \varphi$ and $g' = -\log|f'_i|^2 + \varphi'$. Thus

$$ag + bg' = -\log|f_i^a f_i'^b|^2 + a\varphi + b\varphi'.$$

Note that $f_i^a f_i'^b$ is a local equation of $aD + bD'$ on U_i . Thus (1) follows.

(2) Note that

$$\max\{g, g'\} = -\log|f_i|^2 + \max\{\varphi, \varphi'\} \quad \text{and} \quad \min\{g, g'\} = -\log|f_i|^2 + \min\{\varphi, \varphi'\}$$

on $U_i^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Moreover, $\max\{\varphi, \varphi'\}$ and $\min\{\varphi, \varphi'\}$ are continuous on U_i^{an} , as required. $\square$

Next let us consider a norm arising from a Green function.

Proposition 2.1.3. *We assume that X is normal. We set*

$$H^0(X, D) := \{\phi \in \text{Rat}(X)^{\times} \mid (\phi) + D \geq 0\} \cup \{0\}.$$

Let g be a D -Green function of C^0 -type on X^{an} . Then we have the following:

- (1) *For $\phi \in H^0(X, D)$, $|\phi| \exp(-g/2)$ extends to a continuous function ϑ on X^{an} . We denote $\|\vartheta\|_{\text{sup}}$ by $\|\phi\|_g$.*
- (2) *The following formulae hold:*
 - (2.1) *$\|a\phi\|_g = \nu(a)\|\phi\|_g$ for all $a \in k$ and $\phi \in H^0(X, D)$.*
 - (2.2) *$\|\phi_1 + \phi_2\|_g \leq \max\{\|\phi_1\|_g, \|\phi_2\|_g\}$ for all $\phi_1, \phi_2 \in H^0(X, D)$.*

Proof. (1) Clearly we may assume that $\phi \neq 0$. Let $X = \bigcup_{i=1}^N \text{Spec}(A_i)$ be an affine open covering of X such that D is given by $h_i \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ on $\text{Spec}(A_i)$. Since $D + (\phi)$ is effective as a Weil divisor, $\text{ord}_{\Gamma}(\phi h_i) \geq 0$ for any prime divisor Γ on $\text{Spec}(A_i)$. Thus, by virtue of Hartogs' lemma for $\mathbb{R}$ -rational functions (cf. Lemma 1.2.4), there are $u_1, \dots, u_r \in A_i \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $\phi h_i = u_1^{a_1} \cdots u_r^{a_r}$. In particular, $|\phi h_i| = |u_1|^{a_1} \cdots |u_r|^{a_r}$ is continuous on $\text{Spec}_k^{\text{an}}(A_i)$. On the other hand, there is a continuous function φ_i on $\text{Spec}_k^{\text{an}}(A_i)$ such that $g = -\log|h_i|^2 + \varphi_i$ on $\text{Spec}_k^{\text{an}}(A_i)$. Therefore,

$$|\phi| \exp(-g/2) = |\phi h_i| \exp(-\varphi_i/2)$$

is continuous on $\text{Spec}_k^{\text{an}}(A_i)$.

(2.1) is obvious. (2.2) is also obvious because $|\phi_1 + \phi_2| \leq \max\{|\phi_1|, |\phi_2|\}$ on some dense open set. $\square$

A pair $(\mathcal{X}, \mathcal{D})$ is called a *model of (X, D)* if $\mathcal{X}$ is a model of X over $\text{Spec}(k^\circ)$ (cf. Conventions and terminology 0.5.5) and $\mathcal{D}$ is an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$ with $\mathcal{D} \cap X = D$. The $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ is often called a *model of D on $\mathcal{X}$* . For $x \in X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, let f be a local equation of $\mathcal{D}$ at $\xi = r_{\mathcal{X}}(x)$, where $r_{\mathcal{X}}$ is the reduction map $X^{\text{an}} \rightarrow \mathcal{X}_\circ$ (cf. Subsection 1.3). As $p_x \in \text{Spec}(\mathcal{O}_{\mathcal{X}, \xi})$ and $f \in (\mathcal{O}_{\mathcal{X}, p_x}^\times)_{\mathbb{R}}$, we have $|f(x)| \neq 0$, so that we can define $g_{(\mathcal{X}, \mathcal{D})}(x)$ to be

$$g_{(\mathcal{X}, \mathcal{D})}(x) := -\log |f(x)|^2.$$

Let f' be another local equation of $\mathcal{D}$ at ξ . Then there is $u \in (\mathcal{O}_{\mathcal{X}, \xi}^\times)_{\mathbb{R}}$ such that $f' = f u$, and hence $|f(x)| = |f'(x)|$ because $|u(x)| = 1$ (cf. (1.3.2)). Therefore, $g_{(\mathcal{X}, \mathcal{D})}(x)$ does not depend on the choice of f . Let us see the following proposition:

- Proposition 2.1.4.** (1) *The function $g_{(\mathcal{X}, \mathcal{D})}$ is a D -Green function of C^0 -type on X^{an} .*
 (2) *Let Y be another projective and geometrically integral variety over k and let $\nu : Y \rightarrow X$ be a morphism over k . Let $\mathcal{Y}$ be a model of Y such that there is a morphism $\tilde{\nu} : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec}(k^\circ)$ as an extension of ν . Then $g_{(\mathcal{Y}, \tilde{\nu}^*(\mathcal{D}))} = g_{(\mathcal{X}, \mathcal{D})} \circ \nu^{\text{an}} \circ (\nu^{\text{an}})^{-1}$ on $(\nu^{\text{an}})^{-1}(X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}})$.*
 (3) *Let $\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}$ be the normalization of $\mathcal{X}$. Let X' the generic fiber of $\mathcal{X}' \rightarrow \text{Spec}(k^\circ)$ and $\mu : X' \rightarrow X$ the induced morphism (note that X' is normal). We assume that the associated $\mathbb{R}$ -Weil divisor of $\mu^*(D)$ is effective. Then the associated $\mathbb{R}$ -Weil divisor of $\tilde{\mu}^*(\mathcal{D})$ is effective if and only if $g_{(\mathcal{X}, \mathcal{D})} \geq 0$ on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$.*

Proof. (1) Let $\mathcal{X} = \bigcup_{i=1}^N \text{Spec}(\mathcal{A}_i)$ be an affine open covering of $\mathcal{X}$ such that we have a local equation $f_i \in \text{Rat}(X)_{\mathbb{R}}^\times$ of $\mathcal{D}$ on $\mathcal{U}_i := \text{Spec}(\mathcal{A}_i)$, where $\mathcal{A}_i$ is a k° -algebra for each $i = 1, \dots, N$. We set $C_i = r_{\mathcal{X}}^{-1}(\mathcal{U}_i \cap \mathcal{X}_\circ)$. Then C_i is closed (cf. [2, Section 2.4]) and $\bigcup_{i=1}^N C_i = X^{\text{an}}$.

First let us see that $g_{(\mathcal{X}, \mathcal{D})} : X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}} \rightarrow \mathbb{R}$ is continuous. By our construction, $g_{(\mathcal{X}, \mathcal{D})}(x) = -\log |f_i(x)|^2$ for $x \in C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Thus $g_{(\mathcal{X}, \mathcal{D})}$ is continuous on $C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Let Z be a closed subset in $\mathbb{R}_{\geq 0}$. As

$$\left(g_{(\mathcal{X}, \mathcal{D})} \Big|_{C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}} \right)^{-1}(Z) = g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}),$$

$g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}})$ is closed in $C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, and hence

$$g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}})$$

is closed in $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Note that

$$\begin{aligned} & \bigcup_{i=1}^N \left(g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}) \right) \\ &= g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap \bigcup_{i=1}^N (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}) = g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z). \end{aligned}$$

Thus $g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z)$ is closed in $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, so that $g_{(\mathcal{X}, \mathcal{D})}$ is continuous on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$.

Since f_i is a local equation of D on $U_i = \mathcal{U}_i \cap X$, in order to see that $g_{(\mathcal{X}, \mathcal{D})}$ is a D -Green function of C^0 -type, it is sufficient to see that $\psi = g_{(\mathcal{X}, \mathcal{D})} + \log |f_i|^2$ extends to a continuous function on U_i^{an} , which is obvious because $\psi = 0$ on U_i^{an} .

(2) First note that $\text{Supp}_{\mathbb{R}}(\nu^*(D)) \subseteq \nu^{-1}(\text{Supp}_{\mathbb{R}}(D))$, so that

$$(\nu^{\text{an}})^{-1}(X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}) \subseteq Y^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(\nu^*(D))^{\text{an}}.$$

We set $C'_i = r_{\mathcal{Y}}^{-1}(\tilde{\nu}^{-1}(\mathcal{U}_i) \cap \mathcal{Y}_o)$. Let $y \in C'_i \setminus (\nu^{\text{an}})^{-1}(\text{Supp}_{\mathbb{R}}(D)^{\text{an}})$, $\xi' = r_{\mathcal{Y}}(y)$ and $\xi = r_{\mathcal{X}}(\nu^{\text{an}}(y))$. Note that $\xi = \tilde{\nu}(\xi') \in \mathcal{U}_i \cap \mathcal{X}_o$ by (1.3.2). Then, as $\tilde{\nu}^*(f_i)$ is a local equation of $\tilde{\nu}^*(\mathcal{D})$ at ξ' ,

$$g_{(\mathcal{Y}, \tilde{\nu}^*(\mathcal{D}))}(y) = -\log |\tilde{\nu}^*(f_i)|_y^2 = -\log |f_i|_{\nu^{\text{an}}(y)}^2 = g_{(\mathcal{X}, \mathcal{D})}(\nu^{\text{an}}(y)),$$

as required.

(3) By virtue of (2), $g_{(\mathcal{X}', \tilde{\mu}^*(\mathcal{D}))} = g_{(\mathcal{X}, \mathcal{D})} \circ \mu^{\text{an}}$ on $(X')^{\text{an}} \setminus (\mu^{\text{an}})^{-1}(\text{Supp}_{\mathbb{R}}(D)^{\text{an}})$. Moreover, $\mu^{\text{an}} : (X')^{\text{an}} \rightarrow X^{\text{an}}$ is surjective by [2, Proposition 3.4.6], and hence $g_{(\mathcal{X}', \tilde{\mu}^*(\mathcal{D}))} \geq 0$ on $(X')^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(\mu^*(D))^{\text{an}}$. Thus we may assume that $\mathcal{X}$ is normal.

First we assume that $\mathcal{D}$ is effective as a Weil divisor. Then, $\text{ord}_{\Gamma}(f_i) \geq 0$ for any prime divisor Γ on $\mathcal{U}_i$. Thus, by Hartogs' lemma for $\mathbb{R}$ -rational functions (cf. Lemma 1.2.4), there are $h_1, \dots, h_r \in \mathcal{A}_i \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $f_i = h_1^{a_1} \cdots h_r^{a_r}$. Note that $|h_j|_x \leq 1$ for $j = 1, \dots, r$ and $x \in C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, and hence $|f_i|_x \leq 1$, as required.

Next we assume that $g_{(\mathcal{X}, \mathcal{D})} \geq 0$ on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Let us see that $\mathcal{D}$ is effective as a Weil divisor. If ν is trivial, then $\mathcal{D} = D$ is effective, so that we may assume that ν is non-trivial. It is sufficient to show that the coefficient of $\mathcal{D}_W$ with respect to a vertical prime divisor Γ is non-negative. Let us consider the multiplicative seminorm $x_{\Gamma} = |\cdot|_{x_{\Gamma}}$ given by

$$|f|_{x_{\Gamma}} := \nu(\varpi)^{\text{ord}_{\Gamma}(f)/\text{ord}_{\Gamma}(\varpi)}$$

for $f \in \text{Rat}(X)$, where ϖ is a uniformizing parameter of k° . As $x_{\Gamma} \in X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, if $x_{\Gamma} \in C_i$, then

$$0 \leq g_{(\mathcal{X}, \mathcal{D})}(x_{\Gamma}) = -\log |f_i|_{x_{\Gamma}}^2 = -2 \log \nu(\varpi) \frac{\text{ord}_{\Gamma}(f_i)}{\text{ord}_{\Gamma}(\varpi)},$$

and hence $\text{ord}_{\Gamma}(f_i) \geq 0$, as desired. $\square$

Here we discuss a more sophisticated maximum problem of Green functions on a smooth projective curve than (2) in Proposition 2.1.2.

Proposition 2.1.5. *We assume that X is a smooth projective curve over k . Let $D_1, \dots, D_r$ be $\mathbb{R}$ -Cartier divisors on X and let $D := \max\{D_1, \dots, D_r\}$ (Conventions and terminology 0.5.8). For each $i = 1, \dots, r$, let g_i be a D_i -Green function of C^0 -type on X^{an} . We set*

$$g := \max\{g_1, \dots, g_r\}$$

on $X^{\text{an}} \setminus (\text{Supp}_{\mathbb{R}}(D_1)^{\text{an}} \cup \dots \cup \text{Supp}_{\mathbb{R}}(D_r)^{\text{an}})$. Then g extends to a continuous function on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, and g yields a D -Green function of C^0 -type on X^{an} , which is also denoted by

$$\max\{g_1, \dots, g_r\}$$

by abuse of notation.

Proof. Let $x_1, \dots, x_N$ be closed points of X such that, for each $i = 1, \dots, r$,

$$D_i = a_{i1}x_1 + \dots + a_{iN}x_N$$

for some $a_{ij} \in \mathbb{R}$. If we set $a_j = \max\{a_{1j}, \dots, a_{rj}\}$ for $j = 1, \dots, N$, then

$$D = a_1x_1 + \dots + a_Nx_N.$$

Let $\tilde{x}_j$ be a unique point of X^{an} such that $\{\tilde{x}_j\} = \{x_j\}^{\text{an}}$.

Claim 2.1.5.1. *If $D = 0$, then g is continuous on X^{an} .*

Proof. Clearly g is continuous on $X^{\text{an}} \setminus \{\tilde{x}_1, \dots, \tilde{x}_N\}$, so that we need to show that g is continuous at each $\tilde{x}_j$. Let us choose an affine open set U of X such that

$$\{x_1, \dots, x_N\} \cap U = \{x_j\}.$$

As $a_{ij} \leq 0$, we can see that g_i is upper-semicontinuous on U^{an} . Moreover, g_i is continuous on U^{an} if $a_{ij} = 0$. We set

$$I := \{i = 1, \dots, r \mid a_{ij} = 0\} \quad \text{and} \quad I' := \{i = 1, \dots, r \mid a_{ij} < 0\}.$$

Note that $I \neq \emptyset$ because $\max\{a_{1j}, \dots, a_{rj}\} = a_j = 0$, so that we choose $i_0 \in I$. Here we set

$$V := \{x \in U^{\text{an}} \mid g_{i_0}(x) > g_{i_0}(\tilde{x}_j) - 1 \text{ and } g_{i'}(x) < g_{i_0}(\tilde{x}_j) - 1 \text{ for all } i' \in I'\}.$$

Then V is an open set and $\tilde{x}_j \in V$. Further

$$\max\{g_i(x) \mid i \in I\} \geq g_{i_0}(x) > g_{i_0}(\tilde{x}_j) - 1 > \max\{g_{i'}(x) \mid i' \in I'\}$$

on $V \setminus \{\tilde{x}_j\}$, and hence $g(x) = \max\{g_i(x) \mid i \in I\}$ on $V \setminus \{\tilde{x}_j\}$. Therefore, the claim follows because g_i is continuous on U^{an} for $i \in I$. $\square$

Let g' be a D -Green function of C^0 -type on X^{an} . It is sufficient to see that $g - g'$ extends to a continuous function on X^{an} . Clearly $g - g'$ is continuous on $X^{\text{an}} \setminus \{\tilde{x}_1, \dots, \tilde{x}_N\}$ and

$$g - g' = \max\{g_1 - g', \dots, g_r - g'\}$$

on $X^{\text{an}} \setminus \{\tilde{x}_1, \dots, \tilde{x}_N\}$. Note that $g_i - g'$ is a $(D_i - D)$ -Green function of C^0 -type on X^{an} and $\max\{D_1 - D, \dots, D_r - D\} = 0$, so that the assertion follows from the above claim. $\square$

Finally let us consider a Green function of $(C^0 \cap \text{PSH})$ -type, which is a counterpart of a semipositive metric.

Definition 2.1.6. Let g be a D -Green function of C^0 -type on X^{an} . We say that g is of $(C^0 \cap \text{PSH})$ -type if D is nef and there is a sequence $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^{\infty}$ of models of (X, D) with the following properties:

- (1) For each $n \geq 1$, $\mathcal{D}_n$ is relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$ (cf Conventions and terminology 0.5.6).
- (2) If we set $\phi_n = g_{(\mathcal{X}_n, \mathcal{D}_n)} - g$, then $\lim_{n \rightarrow \infty} \|\phi_n\|_{\text{sup}} = 0$.

As an application of results in Appendix (cf. Corollary A.3.2), we have the following characterization of relatively nef divisors.

Proposition 2.1.7. *Let $\mathcal{X}$ be a normal model of X and let $\mathcal{D}$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$. Then $g_{(\mathcal{X}, \mathcal{D})}$ is of $(C^0 \cap \text{PSH})$ -type if and only if $\mathcal{D}$ is relatively nef.*

In addition, we have the following propositions.

Proposition 2.1.8. *We assume that v is non-trivial. Let D be a nef $\mathbb{R}$ -Cartier divisor on X . Let g be a D -Green function of $(C^0 \cap \text{PSH})$ -type. Then there are sequences $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^{\infty}$ and $\{(\mathcal{X}_n, \mathcal{D}'_n)\}_{n=1}^{\infty}$ of models of (X, D) with the following properties:*

- (1) *For all $n \geq 1$, $\mathcal{D}_n$ and $\mathcal{D}'_n$ are relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$.*
- (2) *$g_{(\mathcal{X}_n, \mathcal{D}_n)} \leq g \leq g_{(\mathcal{X}_n, \mathcal{D}'_n)}$ for all $n \geq 1$.*

(3) If we set $\phi_n = g_{(\mathcal{X}_n, \mathcal{D}_n)} - g$ and $\phi'_n = g_{(\mathcal{X}_n, \mathcal{D}'_n)} - g$, then

$$\lim_{n \rightarrow \infty} \|\phi_n\|_{\sup} = \lim_{n \rightarrow \infty} \|\phi'_n\|_{\sup} = 0.$$

Proof. By its definition, there is a sequence $\{(\mathcal{X}_n, \mathcal{D}''_n)\}_{n=1}^{\infty}$ of models of (X, D) with the following properties:

- (i) For all $n \geq 1$, $\mathcal{D}''_n$ is relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$.
- (ii) If we set $\phi''_n = g_{(\mathcal{X}_n, \mathcal{D}''_n)} - g$, then $\lim_{n \rightarrow \infty} \|\phi''_n\|_{\sup} = 0$.

Here we set

$$\mathcal{D}_n := \mathcal{D}''_n - \frac{\|\phi''_n\|_{\sup}}{-2 \log \nu(\varpi)} (\mathcal{X}_n)_\circ \quad \text{and} \quad \mathcal{D}'_n := \mathcal{D}''_n + \frac{\|\phi''_n\|_{\sup}}{-2 \log \nu(\varpi)} (\mathcal{X}_n)_\circ,$$

where $(\mathcal{X}_n)_\circ$ is the central fiber of $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$. Then

$$g_{(\mathcal{X}_n, \mathcal{D}_n)} = g_{(\mathcal{X}_n, \mathcal{D}''_n)} - \|\phi''_n\|_{\sup} \quad \text{and} \quad g_{(\mathcal{X}_n, \mathcal{D}'_n)} = g_{(\mathcal{X}_n, \mathcal{D}''_n)} + \|\phi''_n\|_{\sup},$$

and hence

$$g_{(\mathcal{X}_n, \mathcal{D}_n)} - g = \phi''_n - \|\phi''_n\|_{\sup} \leq 0 \quad \text{and} \quad g_{(\mathcal{X}_n, \mathcal{D}'_n)} - g = \phi''_n + \|\phi''_n\|_{\sup} \geq 0,$$

as required. $\square$

2.2. Definition of adelic $\mathbb{R}$ -Cartier divisors. We assume that k° is excellent. Let X be a d -dimensional, normal, projective and geometrically integral variety over k . Let k_ν be the completion of k with respect to ν . By abuse of notation, the unique extension of ν to k_ν is also denoted by ν . We set

$$X_\nu := X \times_{\text{Spec}(k)} \text{Spec}(k_\nu),$$

which is also a projective and geometrically integral variety over k_ν . Note that X_ν is normal. Indeed, let $\mathcal{X}$ be a normal model of X over $\text{Spec}(k^\circ)$. If we set

$$\mathcal{X}_\nu := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k_\nu^\circ),$$

then $\mathcal{X}_\nu$ is normal at every point of the central fiber $(\mathcal{X}_\nu)_\circ$ by Lemma 1.1.7. The set of normal points of $\mathcal{X}_\nu$ is open because k_ν° is excellent. Thus $\mathcal{X}_\nu$ is normal, and hence X_ν is normal.

A pair $\overline{D} = (D, g)$ is called an *adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X* if D is an $\mathbb{R}$ -Cartier divisor on X and g is a D -Green function of C^0 -type on X_ν^{an} . If D is nef and g is of $(\text{PSH} \cap C^0)$ -type, then $\overline{D}$ is said to be *relatively nef*. Moreover, we say that $\overline{D}$ is *integrable* if there are relatively nef adelic $\mathbb{R}$ -Cartier divisors $\overline{D}'$ and $\overline{D}''$ of C^0 -type on X such that $\overline{D} = \overline{D}' - \overline{D}''$. We say that a continuous function ϕ on X_ν^{an} is *integrable* if $(0, \phi)$ is integrable as an adelic $\mathbb{R}$ -Cartier divisor. Let $\overline{D}' = (D', g')$ be another adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . For $a, a' \in \mathbb{R}$, we define $a\overline{D} + a'\overline{D}'$ to be

$$a\overline{D} + a'\overline{D}' := (aD + a'D', ag + a'g').$$

The space of all adelic $\mathbb{R}$ -Cartier divisors of C^0 -type is denoted by $\text{Div}_{C^0}^a(X)_{\mathbb{R}}$, which forms a vector space over $\mathbb{R}$ by the above formula. For $\overline{D}_1 = (D_1, g_1), \overline{D}_2 = (D_2, g_2) \in \text{Div}_{C^0}^a(X)_{\mathbb{R}}$, we define $\overline{D}_1 \leq \overline{D}_2$ to be

$$\overline{D}_1 \leq \overline{D}_2 \stackrel{\text{def}}{\iff} D_1 \leq D_2 \text{ and } g_1 \leq g_2.$$

Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(k^\circ)$ and let $\mathcal{D}$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$. Let $\mathcal{X}_\nu := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k_\nu^\circ)$ and $\pi : \mathcal{X}_\nu \rightarrow \mathcal{X}$ the projection. As we checked

before, $\mathcal{X}_v$ is normal. We denote $\pi^*(\mathcal{D})$ by $\mathcal{D}_v$. The pair $(\mathcal{X}, \mathcal{D})$ gives rise to an adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X , that is, the pair $(\mathcal{D} \cap X, g_{(\mathcal{X}_v, \mathcal{D}_v)})$ of $\mathcal{D} \cap X$ and $g_{(\mathcal{X}_v, \mathcal{D}_v)}$. We denote it by $\mathcal{D}^a$ and it is called the *associated adelic $\mathbb{R}$ -Cartier divisor with $\mathcal{D}$* . By abuse of notation, we often use the notations $\mathcal{D} \leq \overline{D}_2$ and $\overline{D}_1 \leq \mathcal{D}$ instead of $\mathcal{D}^a \leq \overline{D}_2$ and $\overline{D}_1 \leq \mathcal{D}^a$, respectively

Proposition 2.2.1. *Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(k^\circ)$ and let $\text{Div}(\mathcal{X})_{\mathbb{R}}$ be the group of $\mathbb{R}$ -Cartier divisors on $\mathcal{X}$. Let $\iota : \text{Div}(\mathcal{X})_{\mathbb{R}} \rightarrow \text{Div}_{C^0}^a(X)_{\mathbb{R}}$ be the map given by $\mathcal{D} \mapsto \mathcal{D}^a$. Then we have the following:*

- (1) *The map $\iota : \text{Div}(\mathcal{X})_{\mathbb{R}} \rightarrow \text{Div}_{C^0}^a(X)_{\mathbb{R}}$ is an injective homomorphism of $\mathbb{R}$ -vector spaces.*
- (2) $\mathcal{D}_1 \leq \mathcal{D}_2 \iff \mathcal{D}_1^a \leq \mathcal{D}_2^a$.

Proof. Clearly ι is a homomorphism of $\mathbb{R}$ -vector spaces. The injectivity of ι follows from (2). Let us consider (2). It is sufficient to see that

$$\mathcal{D} \geq 0 \iff \mathcal{D}^a \geq (0, 0)$$

for $\mathcal{D} \in \text{Div}(\mathcal{X})_{\mathbb{R}}$. The implication $\implies$ is obvious. Conversely we assume that $\mathcal{D}^a \geq (0, 0)$, that is, $\mathcal{D} \cap X \geq 0$ and $g_{(\mathcal{X}_v, \mathcal{D}_v)} \geq 0$. By Proposition 2.1.4, $\mathcal{D}_v = \pi^*(\mathcal{D}) \geq 0$, so that we can easily see that $\mathcal{D} \geq 0$ by using the valuations along the central fiber $\mathcal{X}_\circ = (\mathcal{X}_v)_\circ$. $\square$

2.3. Local degree. We assume that k° is excellent and k is perfect. We use the same notation as in Subsection 2.2. Let x be a closed point of X with $x \notin \text{Supp}_{\mathbb{R}}(D)$. Let $k(x)$ be the residue field at x . As $k(x)$ is separable over k , we have $k(x) \otimes_k k_v = k_1 \oplus \cdots \oplus k_l$ for some finite separable extensions $k_1, \dots, k_l$ over k_v . Note that each k_i has the unique extension v_i of v . The *local degree of $\overline{D}$ along x over v* is defined by

$$\widehat{\deg}_v(\overline{D}|_x) := \sum_{i=1}^l \frac{[k_i : k_v]}{2} g(v_i).$$

Here we assume that k is a number field and $v(f) = \#(O_k/\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(f)}$, where O_k is the ring of integers in k and $\mathfrak{p}$ is a maximal ideal of O_k . Let $\mathcal{X}$ be a normal model of X over $\text{Spec}((O_k)_{\mathfrak{p}})$. Let $\mathcal{D} = a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$ such that $\mathcal{D} \cap X = D$, $a_1, \dots, a_r \in \mathbb{R}$ and $\mathcal{D}_1, \dots, \mathcal{D}_r$ are effective Cartier divisors on $\mathcal{X}$. We assume that $g = g_{(\mathcal{X}, \mathcal{D})}$ and $x \notin \text{Supp}_{\mathbb{Z}}(\mathcal{D}_1) \cup \cdots \cup \text{Supp}_{\mathbb{Z}}(\mathcal{D}_r)$. Let $O_{k(x)}$ be the ring of integers in $k(x)$. Then it is easy to see that

$$(2.3.1) \quad \widehat{\deg}_v(\overline{D}|_x) = \sum_{j=1}^r a_j \log \# \left((O_{k(x)}(\mathcal{D}_j) / O_{k(x)})_{\mathfrak{p}} \right).$$

2.4. Local intersection number. We assume that k° is excellent and v is non-trivial. We use the same notation as in Subsection 2.2. Let ϕ be a continuous function on X_v^{an} . Let $\mathcal{X}$ be a normal model of X and let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be $\mathbb{R}$ -Cartier divisors on $\mathcal{X}$. Note that $d = \dim X$. Let $\Gamma_1, \dots, \Gamma_r$ be the irreducible components of the central fiber $\mathcal{X}_\circ$ of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. Let v_j be the discrete absolute value arising from Γ_j such that $v_j|_k = v$. By Lemma 1.1.7, there is a unique $\tilde{v}_j \in X_v^{\text{an}}$ such that the restriction of $\tilde{v}_j$ to $\text{Rat}(X)$ is v_j . For each i and j , we can choose a unique real number λ_{ij} such that

$\Gamma_j \not\subseteq \text{Supp}_W(\mathcal{L}_i + \lambda_{ij}\mathcal{X}_\circ)$ (for Supp_W , see Definition 1.2.2). Then the number given by

$$\sum_{j=1}^r \frac{\phi(\tilde{v}_j) \text{ord}_{\Gamma_j}(\varpi)}{-2 \log v(\varpi)} \deg_{k^\circ/k^\infty} \left((\mathcal{L}_1 + \lambda_{1j}\mathcal{X}_\circ)|_{\Gamma_j} \cdots (\mathcal{L}_d + \lambda_{dj}\mathcal{X}_\circ)|_{\Gamma_j} \right) =$$

$$\deg_{k^\circ/k^\infty} \left(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot \sum_{j=1}^r \frac{\phi(\tilde{v}_j) \text{ord}_{\Gamma_j}(\varpi)}{-2 \log v(\varpi)} \Gamma_j \right)$$

is denoted by $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$, where $\deg_{k^\circ/k^\infty}$ is the degree over k°/k^∞ . Obviously, $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$ is multi-linear with respect to $\mathcal{L}_1, \dots, \mathcal{L}_d$. In addition,

$$\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; a\phi + a'\phi') = a\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi) + a'\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi')$$

for $a, a' \in \mathbb{R}$ and $\phi, \phi' \in C^0(X_v^{\text{an}})$. Let $\mathcal{E}$ be a vertical $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$ and let $\mathcal{E} = a_1\Gamma_1 + \cdots + a_r\Gamma_r$ be the irreducible decomposition of $\mathcal{E}$ as a Weil divisor. Let $\phi_\mathcal{E}$ be the continuous function arising from $\mathcal{E}$, that is, $\phi_\mathcal{E} = g_{(\mathcal{X}, \mathcal{E})}$. Then

$$\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi_\mathcal{E}) = \sum_{j=1}^r a_j \deg_{k^\circ/k^\infty} \left((\mathcal{L}_1 + \lambda_{1j}\mathcal{X}_\circ)|_{\Gamma_j} \cdots (\mathcal{L}_d + \lambda_{dj}\mathcal{X}_\circ)|_{\Gamma_j} \right)$$

$$= \deg_{k^\circ/k^\infty}(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot \mathcal{E}).$$

If $\deg_{k^\circ/k^\infty}(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot \Gamma_j) \geq 0$ for all j (for example, $\mathcal{L}_1, \dots, \mathcal{L}_d$ are relatively nef) and $\phi \leq \phi'$, then

$$(2.4.1) \quad \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi) \leq \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi').$$

If $\mathcal{X}'$ is another normal model of X and $\mu : \mathcal{X}' \rightarrow \mathcal{X}$ is a birational morphism, then, by the projection formula,

$$\widehat{\deg}_v(\mu^*(\mathcal{L}_1) \cdots \mu^*(\mathcal{L}_d); \phi) = \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi).$$

Let us begin with the following lemma:

Lemma 2.4.2. *Let $\mathcal{L}_1, \dots, \mathcal{L}_{d+2}, \mathcal{L}'_1, \dots, \mathcal{L}'_{d+2}$ be relatively nef $\mathbb{R}$ -Cartier divisors on $\mathcal{X}$. We assume that there are $a_1, \dots, a_d, a_{d+1} \in \mathbb{R}_{\geq 0}$ with the following properties:*

- (1) *For all $i = 1, \dots, d$, $\mathcal{L}_i \cap X = \mathcal{L}'_i \cap X$ and $-a_i\mathcal{X}_\circ \leq \mathcal{L}'_i - \mathcal{L}_i \leq a_i\mathcal{X}_\circ$.*
- (2) *$\mathcal{L}_{d+1} \cap X = \mathcal{L}_{d+2} \cap X$ and $\mathcal{L}'_{d+1} \cap X = \mathcal{L}'_{d+2} \cap X$. Moreover,*

$$-2a_{d+1}g_{(\mathcal{X}, \mathcal{X}_\circ)} \leq \psi' - \psi \leq 2a_{d+1}g_{(\mathcal{X}, \mathcal{X}_\circ)},$$

$$\text{where } \psi := g_{(\mathcal{X}, \mathcal{L}_{d+1} - \mathcal{L}_{d+2})} \text{ and } \psi' := g_{(\mathcal{X}, \mathcal{L}'_{d+1} - \mathcal{L}'_{d+2})}.$$

Then we have the following inequality:

$$\left| \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi') \right|$$

$$\leq \sum_{i=1}^{d+1} 2a_i \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_{d+1}),$$

where $L_i := \mathcal{L}_i \cap X$ for $i = 1, \dots, d+1$.

Proof. As $\mathcal{L}'_1, \dots, \mathcal{L}'_d$ are relatively nef and $-2a_{d+1}g_{(\mathcal{X}, \mathcal{X}_\circ)} \leq \psi' - \psi \leq 2a_{d+1}g_{(\mathcal{X}, \mathcal{X}_\circ)}$, by using (2.4.1), we have

$$\left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi') - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi) \right| \leq 2a_{d+1} \deg(L_1 \cdots L_d),$$

and hence,

$$\begin{aligned} & \left| \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi') \right| \\ & \leq \left| \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi) \right| + 2a_{d+1} \deg(L_1 \cdots L_d). \end{aligned}$$

On the other hand, by Lemma 1.1.4,

$$\begin{aligned} \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi) &= \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) \\ &+ \sum_{i=1}^d \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{E}_i \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \psi), \end{aligned}$$

where $\mathcal{E}_i = \mathcal{L}'_i - \mathcal{L}_i$. Let ϕ_i be the continuous function arising from $\mathcal{E}_i$. Then, as

$$\begin{aligned} \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{E}_i \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \psi) \\ = \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot (\mathcal{L}_{d+1} - \mathcal{L}_{d+2}) \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \phi_i) \end{aligned}$$

and $-a_i g(\mathcal{X}, \mathcal{X}_\circ) \leq \phi_i \leq a_i g(\mathcal{X}, \mathcal{X}_\circ)$, by using (2.4.1), we can see that

$$\begin{aligned} & \left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{E}_i \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \psi) \right| \\ & \leq \left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{L}_{d+1} \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \phi_i) \right| \\ & \quad + \left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{L}_{d+2} \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \phi_i) \right| \\ & \leq 2a_i \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_d \cdot L_{d+1}), \end{aligned}$$

as desired. $\square$

The next proposition guarantees the intersection pairing of integrable adelic $\mathbb{R}$ -Cartier divisors along an integrable continuous function.

Proposition-Definition 2.4.3. *Let $\bar{L}_1 = (L_1, g_1), \dots, \bar{L}_d = (L_d, g_d)$ be relatively nef adelic $\mathbb{R}$ -Cartier divisors on X , and let ϕ be an integrable continuous function on X_v^{an} . Then there are sequences*

$$\begin{aligned} & \{(\mathcal{X}_{1,n}, \mathcal{L}_{1,n})\}_{n=1}^\infty, \dots, \{(\mathcal{X}_{d,n}, \mathcal{L}_{d,n})\}_{n=1}^\infty, \\ & \{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})\}_{n=1}^\infty, \{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})\}_{n=1}^\infty \end{aligned}$$

with the following properties:

- (1) $\mathcal{X}_{i,n}$ is a normal model of X over $\text{Spec}(k^\circ)$ and $\mathcal{L}_{i,n}$ is a relatively nef $\mathbb{R}$ -Cartier divisor on $\mathcal{X}_{i,n}$ for $i = 1, \dots, d+2$ and $n \geq 1$.
- (2) $\mathcal{L}_{i,n} \cap X = L_i$ for $i = 1, \dots, d$ and $n \geq 1$.
- (3) There is an $\mathbb{R}$ -Cartier divisor L_{d+1} on X such that $L_{d+1} = \mathcal{L}_{d+1,n} \cap X = \mathcal{L}_{d+2,n} \cap X$ for all $n \geq 1$.
- (4) If we set $\phi_{i,n} := g_i - g(\mathcal{X}_{i,n}, \mathcal{L}_{i,n})$, then $\lim_{n \rightarrow \infty} \|\phi_{i,n}\|_{\text{sup}} = 0$ for $i = 1, \dots, d$.
- (5) If we set $\psi_n := g(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n}) - g(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})$, then

$$\lim_{n \rightarrow \infty} \|\psi_n - \phi\|_{\text{sup}} = 0.$$

For sequences

$$\{(\mathcal{X}_{1,n}, \mathcal{L}_{1,n})\}_{n=1}^{\infty}, \dots, \{(\mathcal{X}_{d,n}, \mathcal{L}_{d,n})\}_{n=1}^{\infty}, \\ \{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})\}_{n=1}^{\infty}$$

satisfying the above properties, let $\mathcal{Y}_n$ be a normal model of X over $\text{Spec}(k^\circ)$ together with birational morphisms

$$\mu_{i,n} : \mathcal{Y}_n \rightarrow \mathcal{X}_{i,n}$$

for $i = 1, \dots, d$. Then the following limits

$$\begin{cases} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \phi), \\ \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \psi_n) \end{cases}$$

exist and

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \phi) \\ = \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \psi_n). \end{aligned}$$

Moreover, the above limits do not depend on the choice of the sequences

$$\{(\mathcal{X}_{1,n}, \mathcal{L}_{1,n})\}_{n=1}^{\infty}, \dots, \{(\mathcal{X}_{d,n}, \mathcal{L}_{d,n})\}_{n=1}^{\infty}, \\ \{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})\}_{n=1}^{\infty},$$

so that this limit is denoted by $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$.

Proof. The existence of sequences are obvious by the relative nefness of $\bar{L}_1, \dots, \bar{L}_d$ and the integrability of ϕ .

We set

$$\begin{cases} A_n = \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \phi), \\ B_n = \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \psi_n). \end{cases}$$

For a positive number ϵ , there is N such that

$$\|\psi_n - \phi\|_{\sup} \leq 2\epsilon(-\log v(\varpi)) \quad \text{and} \quad \|\phi_{i,n}\|_{\sup} \leq \epsilon(-\log v(\varpi))$$

for all $n \geq N$ and $i = 1, \dots, d$. Then, for $n, m \geq N$,

$$\begin{aligned} \left| \mathcal{G}(\mathcal{X}_{i,n}, \mathcal{L}_{i,n}) - \mathcal{G}(\mathcal{X}_{i,m}, \mathcal{L}_{i,m}) \right| &\leq \left| g_i - \mathcal{G}(\mathcal{X}_{i,n}, \mathcal{L}_{i,n}) \right| \\ &\quad + \left| g_i - \mathcal{G}(\mathcal{X}_{i,m}, \mathcal{L}_{i,m}) \right| \leq 2\epsilon(-\log v(\varpi)) \end{aligned}$$

for $i = 1, \dots, d$. Let us choose a normal model $\mathcal{X}_{n,m}$ of X together with birational morphisms $\tau_n : \mathcal{X}_{n,m} \rightarrow \mathcal{Y}_n$ and $\tau_m : \mathcal{X}_{n,m} \rightarrow \mathcal{Y}_m$. Then the above inequality implies

$$-\epsilon \mathcal{G}(\mathcal{X}_{n,m}, (\mathcal{X}_{n,m})_\circ) \leq \mathcal{G}(\mathcal{X}_{n,m}, \tau_n^*(\mu_{i,n}^*(\mathcal{L}_{i,n})) - \tau_m^*(\mu_{i,m}^*(\mathcal{L}_{i,m}))) \leq \epsilon \mathcal{G}(\mathcal{X}_{n,m}, (\mathcal{X}_{n,m})_\circ),$$

so that, by Proposition 2.2.1,

$$-\epsilon(\mathcal{X}_{n,m})_\circ \leq \tau_n^*(\mu_{i,n}^*(\mathcal{L}_{i,n})) - \tau_m^*(\mu_{i,m}^*(\mathcal{L}_{i,m})) \leq \epsilon(\mathcal{X}_{n,m})_\circ$$

for $i = 1, \dots, d$. On the other hand,

$$\|\psi_n - \psi_m\|_{\sup} \leq \|\psi_n - \phi\|_{\sup} + \|\psi_m - \phi\|_{\sup} \leq 4\epsilon(-\log v(\varpi)).$$

Therefore, by using Lemma 2.4.2, we have

$$|B_n - B_m| \leq 2\epsilon \sum_{i=1}^{d+1} \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_{d+1})$$

for $n, m \geq N$, which shows that the sequence $\{B_n\}_{n=1}^\infty$ is a Cauchy sequence, so that its limit exists. Further, as $0 \leq |\phi - \psi_n| \leq 2\epsilon(-\log v(\varpi))$, by (2.4.1), we have

$$|A_n - B_n| \leq \epsilon \deg(L_1 \cdots L_d),$$

so that $\lim_{n \rightarrow \infty} A_n$ exists and $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} B_n$.

Let $\{(\mathcal{X}'_{1,n}, \mathcal{L}'_{1,n})\}_{n=1}^\infty, \dots, \{(\mathcal{X}'_{d+2,n}, \mathcal{L}'_{d+2,n})\}_{n=1}^\infty$ be other sequences satisfying the above properties (1), (2), (3), (4) and (5). For the above sequences, L_{d+1} in property (3), $\phi_{i,n}$ in property (4) and ψ_n in property (5) are denoted by L'_{d+1} , $\phi'_{i,n}$ and ψ'_n , respectively. Replacing $\mathcal{Y}_n$ by a suitable model of X , we may assume that there are birational morphisms $\mu'_{i,n} : \mathcal{Y}_n \rightarrow \mathcal{X}'_{i,n}$ ($i = 1, \dots, d$). For a positive number ϵ , there is N such that

$$\begin{cases} \|\psi_n - \phi\|_{\sup} \leq 2\epsilon(-\log v(\varpi)), & \|\phi_{i,n}\|_{\sup} \leq \epsilon(-\log v(\varpi)), \\ \|\psi'_n - \phi\|_{\sup} \leq 2\epsilon(-\log v(\varpi)), & \|\phi'_{i,n}\|_{\sup} \leq \epsilon(-\log v(\varpi)) \end{cases}$$

for all $n \geq N$ and $i = 1, \dots, d$. Then,

$$\begin{aligned} & \left| \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}_{i,n}))} - \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}'_{i,n}))} \right| \\ & \leq \left| \mathcal{G}_i - \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}_{i,n}))} \right| + \left| \mathcal{G}_i - \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}'_{i,n}))} \right| \leq 2\epsilon(-\log v(\varpi)) \end{aligned}$$

for $n \geq N$ and $i = 1, \dots, d$. Therefore, in the similar way as before,

$$-\epsilon(\mathcal{Y}_n)_\circ \leq \mu_{i,n}^*(\mathcal{L}_{i,n}) - \mu_{i,n}^*(\mathcal{L}'_{i,n}) \leq \epsilon(\mathcal{Y}_n)_\circ.$$

Moreover,

$$\|\psi_n - \psi'_n\|_{\sup} \leq \|\psi_n - \phi\|_{\sup} + \|\psi'_n - \phi\|_{\sup} \leq 4\epsilon(-\log v(\varpi))$$

for $n \geq N$, and hence the uniqueness of the limit follows from Lemma 2.4.2. $\square$

Let ϕ and ϕ' be integrable continuous functions on X_v^{an} . Let $\bar{L}_1, \dots, \bar{L}_i, \bar{L}'_i, \dots, \bar{L}_d$ be relatively nef adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $\phi \geq 0$, then, by Proposition-Definition 2.4.3,

$$(2.4.4) \quad \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) \geq 0.$$

Moreover, it is easy to see that

$$(2.4.5) \quad \begin{cases} \widehat{\deg}_v(\bar{L}_1 \cdots (a\bar{L}_i + a'\bar{L}'_i) \cdots \bar{L}_d; \phi) \\ \quad = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}'_i \cdots \bar{L}_d; \phi) \quad (a, a' \in \mathbb{R}_{\geq 0}), \\ \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; a\phi + a'\phi') \\ \quad = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi') \quad (a, a' \in \mathbb{R}) \end{cases}$$

and that

$$(2.4.6) \quad \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_j \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_j \cdots \bar{L}_i \cdots \bar{L}_d; \phi).$$

Let $\bar{L}_1, \dots, \bar{L}_d$ be integrable adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . For each i , we choose relatively nef adelic $\mathbb{R}$ -Cartier divisors $\bar{L}_{i,+1}$ and $\bar{L}_{i,-1}$ of C^0 -type such that $\bar{L}_i = \bar{L}_{i,+1} - \bar{L}_{i,-1}$. By using (2.4.5), it is not difficult to see that the quantity

$$\sum_{\epsilon_1, \dots, \epsilon_{d+1} \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_{d+1} \widehat{\deg}_v(\bar{L}_{1, \epsilon_1} \cdots \bar{L}_{d, \epsilon_d}; \phi)$$

does not depend on the choice of $\bar{L}_{1,+1}, \bar{L}_{1,-1}, \dots, \bar{L}_{d,+1}, \bar{L}_{d,-1}$, so that it is denoted by $\widehat{\deg}(\bar{L}_1 \cdots \bar{L}_d; \phi)$. Further, (2.4.5) and (2.4.6) extend to the following formula:

$$(2.4.7) \quad \begin{cases} \widehat{\deg}_v(\bar{L}_1 \cdots (a\bar{L}_i + a'\bar{L}'_i) \cdots \bar{L}_d; \phi) \\ \quad = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}'_i \cdots \bar{L}_d; \phi), \\ \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; a\phi + a'\phi') \\ \quad = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi') \end{cases}$$

and

$$(2.4.8) \quad \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_j \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_j \cdots \bar{L}_i \cdots \bar{L}_d; \phi)$$

for $a, a' \in \mathbb{R}$ and integrable adelic $\mathbb{R}$ -Cartier divisors $\bar{L}_1, \dots, \bar{L}_i, \bar{L}'_i, \dots, \bar{L}_d$ of C^0 -type on X . Here let us consider a consequence of Proposition-Definition 2.4.3.

Proposition 2.4.9. *Let $\bar{L}_1 = (L_1, g_1), \dots, \bar{L}_d = (L_d, g_d)$ be integrable adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X , and let ϕ be an integrable continuous function on X_v^{an} . If $L_i = 0$, then*

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots (0, \phi) \cdots \bar{L}_d; g_i).$$

Proof. For $j = 1, \dots, d$, let

$$\bar{L}_{j,+1} = (L_{j,+1}, g_{j,+1}) \quad \text{and} \quad \bar{L}_{j,-1} = (L_{j,-1}, g_{j,-1})$$

be relatively nef adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X such that $\bar{L}_j = \bar{L}_{j,+1} - \bar{L}_{j,-1}$. Moreover, we choose relative nef adelic $\mathbb{R}$ -Cartier divisors

$$\bar{L}_{d+1,+1} = (L_{d+1,+1}, g_{d+1,+1}) \quad \text{and} \quad \bar{L}_{d+1,-1} = (L_{d+1,-1}, g_{d+1,-1})$$

of C^0 -type on X such that $\bar{L}_{d+1,+1} - \bar{L}_{d+1,-1} = (0, \phi)$. Then there are sequences

$$\{(\mathcal{X}_{1,+1,n}, \mathcal{L}_{1,+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{1,-1,n}, \mathcal{L}_{1,-1,n})\}_{n=1}^{\infty}, \dots, \\ \{(\mathcal{X}_{d+1,+1,n}, \mathcal{L}_{d+1,+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{d+1,-1,n}, \mathcal{L}_{d+1,-1,n})\}_{n=1}^{\infty}$$

satisfying the following conditions:

- (a) $\mathcal{X}_{j,\epsilon,n}$ is a normal model of X over $\text{Spec}(k^\circ)$ for $j = 1, \dots, d+1, \epsilon = \pm 1$ and $n \geq 1$.
- (b) $\mathcal{L}_{j,\epsilon,n}$ is a nef $\mathbb{R}$ -Cartier divisor on $\mathcal{X}_{j,\epsilon,n}$ such that $\mathcal{L}_{j,\epsilon,n} \cap X = L_{j,\epsilon}$ for $j = 1, \dots, d+1, \epsilon = \pm 1$ and $n \geq 1$.
- (c) If we set $\phi_{j,\epsilon,n} = g_{j,\epsilon} - g(\mathcal{X}_{j,\epsilon,n}, \mathcal{L}_{j,\epsilon,n})$, then $\lim_{n \rightarrow \infty} \|\phi_{j,\epsilon,n}\|_{\text{sup}} = 0$ for $j = 1, \dots, d+1$ and $\epsilon = \pm 1$.

Here we set

$$\begin{cases} \psi_n = g(\mathcal{X}_{d+1,+1,n}, \mathcal{L}_{d+1,+1,n}) - g(\mathcal{X}_{d+1,-1,n}, \mathcal{L}_{d+1,-1,n}), \\ \theta_n = g(\mathcal{X}_{i,+1,n}, \mathcal{L}_{i,+1,n}) - g(\mathcal{X}_{i,-1,n}, \mathcal{L}_{i,-1,n}). \end{cases}$$

Then, by Proposition-Definition 2.4.3,

$$\lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,\epsilon_1,n} \cdots \mathcal{L}_{i,\epsilon_i,n} \cdots \mathcal{L}_{d,\epsilon_d,n}; \psi_n) = \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{i,\epsilon_i} \cdots \bar{L}_{d,\epsilon_d}; \phi)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,\epsilon_1,n} \cdots \mathcal{L}_{d+1,\epsilon_{d+1},n} \cdots \mathcal{L}_{d,\epsilon_d,n}; \theta_n) \\ = \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{d+1,\epsilon_{d+1}} \cdots \bar{L}_{d,\epsilon_d}; g_i). \end{aligned}$$

Therefore, if we set $\mathcal{L}_{i,n} = \mathcal{L}_{i+1,n} - \mathcal{L}_{i-1,n}$ for $i = 1, \dots, d+1$, then

$$\begin{cases} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{i,n} \cdots \mathcal{L}_{d,n}; \psi_n) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi), \\ \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{d+1,n} \cdots \mathcal{L}_{d,n}; \theta_n) = \widehat{\deg}_v(\bar{L}_1 \cdots (0, \phi) \cdots \bar{L}_d; g_i). \end{cases}$$

On the other hand, note that

$$\widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{i,n} \cdots \mathcal{L}_{d,n}; \psi_n) = \widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{d+1,n} \cdots \mathcal{L}_{d,n}; \theta_n).$$

Thus the assertion of the proposition follows. $\square$

The results of this subsection leads to the following definition:

Definition 2.4.10. Let $\bar{L}_1 = (L_1, g_1), \dots, \bar{L}_{d+1} = (L_{d+1}, g_{d+1})$ be integrable adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . By Proposition 2.4.9 and (2.4.8), if $L_i = 0$ for some i , then

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_{d+1})$$

is well-defined, that is,

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_{d+1}) := \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_{i-1} \cdot \bar{L}_{i+1} \cdots \bar{L}_{d+1}; g_i).$$

Moreover, it is symmetric and multi-linear.

Proposition 2.4.11. Let $\bar{L}_1, \dots, \bar{L}_d$ be integrable adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X , and let ϕ and ϕ' be integrable continuous functions on X_v^{an} . Then we have the following:

- (1) For $f \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, $\widehat{\deg}_v(((f), -\log|f|^2) \cdot \bar{L}_2 \cdots \bar{L}_d; \phi) = 0$.
- (2) If $\bar{L}_1, \dots, \bar{L}_d$ are relatively nef and $\phi' \leq \phi$ on X_v^{an} , then

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi') \leq \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi).$$

- (3) Let

$$\begin{aligned} \bar{L}_{1,1} = (L_{1,1}, g_{1,1}), \bar{L}_{1,-1} = (L_{1,-1}, g_{1,-1}), \dots, \\ \bar{L}_{d,1} = (L_{d,1}, g_{d,1}), \bar{L}_{d,-1} = (L_{d,-1}, g_{d,-1}) \end{aligned}$$

be relatively nef adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X such that $\bar{L}_i = \bar{L}_{i,1} - \bar{L}_{i,-1}$ for $i = 1, \dots, d$. Then

$$\left| \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) \right| \leq \frac{\|\phi\|_{\sup}}{-2 \log v(\varpi)} \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \deg(L_{1,\epsilon_1} \cdots L_{d,\epsilon_d}).$$

- (4) If $\mu: X' \rightarrow X$ is a birational morphism of normal, projective and geometrically integral varieties over k , then

$$\widehat{\deg}_v(\mu^*(\bar{L}_1) \cdots \mu^*(\bar{L}_d); (\mu^{\text{an}})^*(\phi)) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi).$$

Proof. Let $\mathcal{X}$ be a normal model of X over $\mathrm{Spec}(k^\circ)$ and let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_d$ be $\mathbb{R}$ -Cartier divisors on $\mathcal{X}$.

(1) By the definition of $\widehat{\deg}_v$, we have $\widehat{\deg}_v((f)_{\mathcal{X}} \cdot \mathcal{L}_2 \cdots \mathcal{L}_d; \phi) = 0$. Thus (1) follows.

(2) If $\mathcal{L}_1, \dots, \mathcal{L}_d$ are relatively nef, then $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi) \leq \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$ (cf. (2.4.1)), so that we have (2).

(3) First of all, note that

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) = \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_d \widehat{\deg}_v(\bar{L}_{1, \epsilon_1} \cdots \bar{L}_{d, \epsilon_d}; \phi).$$

Therefore, by using (2),

$$\begin{aligned} \left| \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) \right| &\leq \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \left| \widehat{\deg}_v(\bar{L}_{1, \epsilon_1} \cdots \bar{L}_{d, \epsilon_d}; \phi) \right| \\ &\leq \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \widehat{\deg}_v(\bar{L}_{1, \epsilon_1} \cdots \bar{L}_{d, \epsilon_d}; \|\phi\|_{\sup}) \\ &= \frac{\|\phi\|_{\sup}}{-2 \log v(\varpi)} \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \deg(L_{1, \epsilon_1} \cdots L_{d, \epsilon_d}), \end{aligned}$$

as required.

(4) Let $\mathcal{X}$ and $\mathcal{X}'$ be normal models of X and X' , respectively. We assume there is a birational morphism $\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}$ such that $\tilde{f}|_{\mathcal{X}} = f$. Let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be $\mathbb{R}$ -Cartier divisors on $\mathcal{X}$. Then, by using the projection formula, we can see that

$$\widehat{\deg}_v(\mu^*(\mathcal{L}_1) \cdots \mu^*(\mathcal{L}_d); (\mu^{\mathrm{an}})^*(\phi)) = \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi),$$

so that, by our definition of $\widehat{\deg}_v(\mu^*(\bar{L}_1) \cdots \mu^*(\bar{L}_d); (\mu^{\mathrm{an}})^*(\phi))$ and $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$ (cf. Proposition-Definition 2.4.3), the assertion follows. $\square$

3. LOCAL AND GLOBAL DENSITY THEOREMS

The density theorem in terms of global model functions was established by Gubler [14, Theorem 7.12] and X. Yuan [34, Lemma 3.5]. Recently an elementary proof was found by Boucksom, Favre and Jonsson [6], in which they work over a valuation field whose residue field is of characteristic zero. Fortunately their proof is valid even if the characteristic of the residue field is positive. In this section, for reader's convenience, we will give a proof of the density theorem according to the ideas in [6].

Let k be a field and v a non-trivial discrete absolute value of k . Note that v is multiplicative. Let ϖ be a uniformizing parameter of k° , that is, $k^\circ = \varpi k^\circ$ (for the definition of k° and k° , see Conventions and terminology 0.5.2). Let X be a projective and geometrically integral variety over k and let $\mathrm{Rat}(X)$ be the rational function field of X .

3.1. Vertical fractional ideal sheaves and birational system of models. Let $\mathcal{X}$ be a model of X over $\mathrm{Spec}(k^\circ)$ (cf. Conventions and terminology 0.5.5). The central fiber of $\mathcal{X} \rightarrow \mathrm{Spec}(k^\circ)$ is denoted by $\mathcal{X}_\circ$, that is, $\mathcal{X}_\circ := \mathcal{X} \times_{\mathrm{Spec}(k^\circ)} \mathrm{Spec}(k^\circ/k^\circ)$. A non-zero coherent subsheaf $\mathcal{J}$ of $\mathrm{Rat}(X)$ on $\mathcal{X}$ is called a *fractional ideal sheaf* on $\mathcal{X}$. It is said to be *vertical* if there is $m \in \mathbb{Z}_{\geq 0}$ such that $\varpi^m \mathcal{J}$ is an ideal sheaf and $\mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\varpi^m \mathcal{J}) \subseteq \mathcal{X}_\circ$. Let $\mathcal{D}$ be a vertical Cartier divisor on $\mathcal{X}$, that is, $\mathrm{Supp}_{\mathbb{Z}}(\mathcal{D}) \subseteq \mathcal{X}_\circ$ (for the definition of $\mathrm{Supp}_{\mathbb{Z}}$, see Subsection 1.2). Note that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is a vertical fractional sheaf. Indeed, let $\xi \in \mathcal{X}_\circ$ and f a local equation of $\mathcal{D}$ at ξ . Then f is a unit element of $((\mathcal{O}_{\mathcal{X}, \xi})_S)_p$ for all

$p \in \text{Spec}((\mathcal{O}_{\mathcal{X},\xi})_S)$, where S is the multiplicative set given by $\{1, \varpi, \varpi^2, \dots\}$, and hence $f \in (\mathcal{O}_{\mathcal{X},\xi})_S$. Thus we can find $m_\xi \in \mathbb{Z}_{\geq 0}$ such that $\varpi^{m_\xi} f \in \mathcal{O}_{\mathcal{X},\xi}$. This observation shows that $\varpi^m \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \subseteq \mathcal{O}_{\mathcal{X}}$ for some $m \in \mathbb{Z}_{\geq 0}$, as required.

A set Ψ of models of X is called a *birational system of models of X* if the following conditions are satisfied:

- (1) For any $\mathcal{X} \in \Psi$ and any vertical fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$, there is $\mathcal{X}' \in \Psi$ together with a birational morphism $\nu: \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathcal{J} \mathcal{O}_{\mathcal{X}'}$ is invertible.
- (2) For any $\mathcal{X}_1, \mathcal{X}_2 \in \Psi$, there is $\mathcal{X}_3 \in \Psi$ together with birational morphisms $\nu_1: \mathcal{X}_3 \rightarrow \mathcal{X}_1$ and $\nu_2: \mathcal{X}_3 \rightarrow \mathcal{X}_2$.

Remark 3.1.1. The set of all models of X is a birational system of models of X . In addition, fixing a model $\mathcal{Z}$ of X , the set of all models $\mathcal{X}$ over $\mathcal{Z}$ (that is, there is a birational morphism $\mathcal{X} \rightarrow \mathcal{Z}$) forms a birational system of models of X .

3.2. Model functions. We assume that ν is complete. Let $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ be a model of X . Let $\mathcal{J}$ be a vertical fractional ideal sheaf on $\mathcal{X}$. According to [6, Subsection 2.3], we define a function $\log|\mathcal{J}|$ on X^{an} to be

$$\log|\mathcal{J}|(x) := \log \max \{ |h(x)| \mid h \in \mathcal{J}_{r_{\mathcal{X}}(x)} \},$$

where $r_{\mathcal{X}}: X^{\text{an}} \rightarrow \mathcal{X}_\circ$ is the reduction map induced by the model $\mathcal{X}$. For example, $\log|\varpi \mathcal{O}_{\mathcal{X}}| = \log \nu(\varpi)$. Note that $\log|\mathcal{J}| = \log|\mathcal{J} \mathcal{O}_{\mathcal{X}'}|$ for a birational morphism $\nu: \mathcal{X}' \rightarrow \mathcal{X}$ of models of X and a vertical fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$. Moreover, if $\mathcal{J}$ is invertible, then $\log|\mathcal{J}| = g_{(\mathcal{X}, \mathcal{J})}/2$ (cf. Subsection 2.1). Let us fix a birational system Ψ of models of X . A function φ on X^{an} is called a *model function with respect to Ψ* if there are $\mathcal{X} \in \Psi$ and a vertical fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ such that $\varphi = \log|\mathcal{J}|$. The set of all model functions with respect to Ψ is denoted by $\mathbb{M}(X^{\text{an}}; \Psi)$. Then we have the following generalization of [6, Proposition 2.2].

- Proposition 3.2.1.** (1) For $\varphi \in \mathbb{M}(X^{\text{an}}; \Psi)$, there are $\mathcal{X} \in \Psi$ and a vertical Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\varphi = \log|\mathcal{O}_{\mathcal{X}}(\mathcal{D})|$. In particular, $\mathbb{M}(X^{\text{an}}; \Psi)$ forms an abelian group and $\mathbb{M}(X^{\text{an}}; \Psi) \subseteq C^0(X^{\text{an}})$ (cf. Proposition 2.1.4).
 (2) For $\varphi_1, \varphi_2 \in \mathbb{M}(X^{\text{an}}; \Psi)$, $\max\{\varphi_1, \varphi_2\} \in \mathbb{M}(X^{\text{an}}; \Psi)$.
 (3) For $x, y \in X^{\text{an}}$ with $x \neq y$, there is $\varphi \in \mathbb{M}(X^{\text{an}}; \Psi)$ such that $\varphi(x) \neq \varphi(y)$.

Proof. The assertions of the proposition can be proved by the ideas of [6, Proposition 2.2].

(1) We choose $\mathcal{X} \in \Psi$ and a vertical fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ such that $\varphi = \log|\mathcal{J}|$. By our assumption, there is $\mathcal{X}' \in \Psi$ together with a birational morphism $\nu: \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathcal{J} \mathcal{O}_{\mathcal{X}'}$ is invertible, that is, there is a vertical Cartier divisor $\mathcal{D}'$ on $\mathcal{X}'$ such that $\mathcal{J} \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(\mathcal{D}')$. Clearly $\log|\mathcal{J}| = \log|\mathcal{O}_{\mathcal{X}'}(\mathcal{D}')|$, as desired.

(2) By (1) and the property of Ψ , we can find $\mathcal{X} \in \Psi$ and vertical Cartier divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{X}$ such that $\varphi_1 = \log|\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1)|$ and $\varphi_2 = \log|\mathcal{O}_{\mathcal{X}}(\mathcal{D}_2)|$. If we set $\mathcal{J} = \mathcal{O}_{\mathcal{X}}(\mathcal{D}_1) + \mathcal{O}_{\mathcal{X}}(\mathcal{D}_2)$ in $\text{Rat}(X)$, then $\mathcal{J}$ is a vertical fractional ideal sheaf of $\mathcal{X}$ and $\log|\mathcal{J}| = \max\{\varphi_1, \varphi_2\}$.

(3) Fix $\mathcal{X} \in \Psi$. First we assume that $r_{\mathcal{X}}(x) \neq r_{\mathcal{X}}(y)$. Let $\mathfrak{m}$ be the maximal ideal at $r_{\mathcal{X}}(x)$. Then $\log|\mathfrak{m}|(x) < 0$ and $\log|\mathfrak{m}|(y) \geq 0$, as desired. Next we assume that $\xi = r_{\mathcal{X}}(x) = r_{\mathcal{X}}(y)$. Let $\mathcal{U} = \text{Spec}(\mathcal{A})$ be an affine open neighborhood of ξ . We can find $f \in \mathcal{A}$ such that $|f|_x \neq |f|_y$. Since $\mathcal{X}$ is noetherian, there is an ideal sheaf $\mathcal{J}$ on $\mathcal{X}$

such that $\mathcal{I}_\xi = f \mathcal{O}_{\mathcal{X}, \xi}$. For each $m \in \mathbb{Z}_{>0}$, we set $\mathcal{J}_m = \mathcal{I} + \varpi^m \mathcal{O}_{\mathcal{X}}$. Note that $\mathcal{J}_m$ is a vertical ideal sheaf on $\mathcal{X}$. Moreover,

$$\begin{cases} \log |\mathcal{J}_m|(x) = \max\{\log |f|_x, m \log v(\varpi)\}, \\ \log |\mathcal{J}_m|(y) = \max\{\log |f|_y, m \log v(\varpi)\}, \end{cases}$$

where $\log(0) = -\infty$. As $|f|_x \neq |f|_y$ and $v(\varpi) < 1$, if m is sufficiently large, then

$$\log |\mathcal{J}_m|(x) \neq \log |\mathcal{J}_m|(y),$$

as required. $\square$

3.3. Density theorems. Let k_v be the completion of k with respect to v . By abuse of notation, the unique extension of v to k_v is also denoted by v . We set $X_v = X \times_{\text{Spec}(k)} \text{Spec}(k_v)$, which is also a projective and geometrically integral variety over k_v . For a model $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ of X , $\mathcal{X}_v := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k_v^\circ)$ is also a model of X_v by (1.1) in Lemma 1.1.7. The projection $\mathcal{X}_v \rightarrow \mathcal{X}$ is denoted by $\pi_{\mathcal{X}}$. Let Ψ be a system of models of X .

Proposition 3.3.1. $\Psi_v := \{\mathcal{X}_v \mid \mathcal{X} \in \Psi\}$ forms a system of models of X_v .

Proof. The assertion follows from (2) in the following lemma. $\square$

Lemma 3.3.2. (1) Let $\mathcal{J}_v$ be a vertical ideal sheaf on $\mathcal{X}_v$. Then there is $n \in \mathbb{Z}_{\geq 0}$ such that $\varpi^n \mathcal{O}_{\mathcal{X}_v} \subseteq \mathcal{J}_v$.

(2) Let $\mathcal{J}_v$ be a vertical fractional ideal sheaf on $\mathcal{X}_v$. Then there is a vertical fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ such that $\mathcal{J} \mathcal{O}_{\mathcal{X}_v} = \mathcal{J}_v$.

Proof. (1) For $\xi \in (\mathcal{X}_v)_\circ$, $(\mathcal{O}_{\mathcal{X}_v, \xi} / (\mathcal{J}_v)_\xi)_S = 0$ because $\text{Supp}(\mathcal{O}_{\mathcal{X}_v} / \mathcal{J}_v) \subseteq (\mathcal{X}_v)_\circ$, where S is the multiplicative set given by $\{1, \varpi, \varpi^2, \dots\}$. Therefore,

$$\varpi^{m_\xi} (\mathcal{O}_{\mathcal{X}_v, \xi} / (\mathcal{J}_v)_\xi) = 0$$

for some $m_\xi \in \mathbb{Z}_{\geq 0}$, and hence $\varpi^{m_\xi} \mathcal{O}_{\mathcal{X}_v, \xi} \subseteq (\mathcal{J}_v)_\xi$. Thus the assertion follows.

(2) Clearly we may assume that $\mathcal{J}_v$ is an ideal sheaf. Then, by (1), there is $n \in \mathbb{Z}_{\geq 0}$ such that $\varpi^n \mathcal{O}_{\mathcal{X}_v} \subseteq \mathcal{J}_v$. As

$$\mathcal{J}_v / \varpi^n \mathcal{O}_{\mathcal{X}_v} \subseteq \mathcal{O}_{\mathcal{X}_v} / \varpi^n \mathcal{O}_{\mathcal{X}_v} \xleftarrow[\pi_{\mathcal{X}}^*]{\sim} \mathcal{O}_{\mathcal{X}} / \varpi^n \mathcal{O}_{\mathcal{X}},$$

we can find an ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ such that $\varpi^n \mathcal{O}_{\mathcal{X}} \subseteq \mathcal{J}$ and $\mathcal{J}_v / \varpi^n \mathcal{O}_{\mathcal{X}_v} \simeq \mathcal{J} / \varpi^n \mathcal{O}_{\mathcal{X}}$, so that we can easily see that $\mathcal{J} \mathcal{O}_{\mathcal{X}_v} = \mathcal{J}_v$. $\square$

Theorem 3.3.3 (Local density theorem). $\mathbb{M}(X_v^{\text{an}}; \Psi_v) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dense in $C^0(X_v^{\text{an}})$ with respect to the supremum norm $\|\cdot\|_{\text{sup}}$.

Proof. We set $T = X_v^{\text{an}}$ and $\Sigma = \mathbb{M}(X_v^{\text{an}}; \Psi_v) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is well known that T is a compact Hausdorff space (cf. [2, Theorem 3.4.8]). Thus, if we can check conditions (1) – (4) in Lemma 3.3.4, we have the assertion.

(1) follows from (3) in Proposition 3.2.1.

(2) Note that $\log v(\varpi) \in \mathbb{M}(X_v^{\text{an}}; \Psi_v) \subseteq \Sigma$.

(3) is obvious.

(4) Let us check that $\max\{\psi_1, \psi_2\} \in \Sigma$ for $\psi_1, \psi_2 \in \Sigma$. We choose $n \in \mathbb{Z}_{>0}$ such that $n\varphi_1, n\varphi_2 \in \mathbb{M}(X_v^{\text{an}}; \Psi_v)$. Then, by (2) in Proposition 3.2.1,

$$n \max\{\psi_1, \psi_2\} = \max\{n\psi_1, n\psi_2\} \in \Sigma,$$

and hence $\max\{\psi_1, \psi_2\} \in \Sigma$. $\square$

Lemma 3.3.4. *Let T be a compact Hausdorff space. Let Σ be a subset of $C^0(T)$ with the following properties:*

- (1) *For any $x, y \in T$ with $x \neq y$, there is $f \in \Sigma$ such that $f(x) \neq f(y)$.*
- (2) *$\mathbb{R}^\times \cap \Sigma \neq \emptyset$.*
- (3) *Σ forms a $\mathbb{Q}$ -vector space.*
- (4) *For $f, g \in \Sigma$, $\max\{f, g\} \in \Sigma$.*

Then Σ is dense in $C^0(T)$ with respect to the supremum norm $\|\cdot\|_{\sup}$.

Proof. Let $\bar{\Sigma}$ be the closure of Σ with respect to $\|\cdot\|_{\sup}$. Then it is easy to see that $\bar{\Sigma}$ has the following properties:

- (1)' *For any $x, y \in T$ with $x \neq y$, there is $f \in \bar{\Sigma}$ such that $f(x) \neq f(y)$.*
- (2)' *$1 \in \bar{\Sigma}$.*
- (3)' *$\bar{\Sigma}$ forms an $\mathbb{R}$ -vector space.*
- (4)' *For $f, g \in \bar{\Sigma}$, $\max\{f, g\} \in \bar{\Sigma}$.*

Thus, by [17, Theorem 7.29], $\bar{\Sigma}$ is dense in $C^0(T)$. Note that $\bar{\Sigma} = \overline{\bar{\Sigma}}$. Therefore the assertion follows. $\square$

Definition 3.3.5. A function φ on X_v^{an} is called a *global model function with respect to Ψ* if there are $\mathcal{X} \in \Psi$ and a vertical fractional ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ such that $\varphi = \log|\mathcal{J} \mathcal{O}_{\mathcal{X}_v}|$. The set of all global model functions with respect to Ψ is denoted by $\mathbb{M}(X; \Psi)$.

By using (2) in Lemma 3.3.2, we can see

$$\mathbb{M}(X; \Psi) = \mathbb{M}(X_v^{\text{an}}; \Psi_v),$$

Thus the local density theorem (cf. Theorem 3.3.3) implies the following main result of this section.

Theorem 3.3.6 (Global density theorem). *$\mathbb{M}(X; \Psi) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dense in $C^0(X_v^{\text{an}})$ with respect to the supremum norm $\|\cdot\|_{\sup}$.*

The following theorem is an application of the global density theorem.

Theorem 3.3.7 (Approximation theorem of adelic $\mathbb{R}$ -divisors). *We assume that k° is excellent and X is normal. Let $\overline{D} = (D, g)$ be an adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . For any positive number $\epsilon > 0$, there exist a normal model $\mathcal{X}$ of X over $\text{Spec}(k^\circ)$, and $\mathbb{R}$ -Cartier divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{X}$ such that*

$$\overline{D} - (0, \epsilon) \leq \mathcal{D}_1 \leq \overline{D} \leq \mathcal{D}_2 \leq \overline{D} + (0, \epsilon).$$

Proof. Let us begin with the following claim:

Claim 3.3.7.1. *For a positive number ϵ , there exist a normal model $\mathcal{X}$ of X over $\text{Spec}(k^\circ)$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{D} \cap X = D$ and*

$$\|g_{(\mathcal{X}, \mathcal{D})} - g\|_{\sup} \leq \epsilon/2.$$

Proof. First let us choose a model $\mathcal{X}_0$ of X over $\text{Spec}(k^\circ)$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}_0$ on $\mathcal{X}_0$ such that $D = \mathcal{D}_0 \cap X$. Let g_0 be the D -Green function of C^0 -type on X_v^{an} arising from the model $(\mathcal{X}_0, \mathcal{D}_0)$. We set $\phi := g - g_0$. Then ϕ is a continuous function on X_v^{an} . Let Ψ be the set of all models of X over $\mathcal{X}_0$ (cf. Remark 3.1.1). By the global density theorem (cf. Theorem 3.3.6), there is a global model function φ on X_v^{an} with respect

to Ψ such that $\|\phi - \varphi\|_{\sup} \leq \epsilon/2$, that is, there are a model $\mathcal{X}'$ in Ψ together with a birational morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}_0$, a vertical Cartier divisor $\mathcal{E}'$ on $\mathcal{X}'$ and $a \in \mathbb{Q}$ such that $\varphi = a g_{(\mathcal{X}', \mathcal{E}')}.$ Let $\pi : \mathcal{X} \rightarrow \mathcal{X}'$ be the normalization of $\mathcal{X}'$ and $\mathcal{D} := a\pi^*(\mathcal{E}') + \pi^*(\mu^*(\mathcal{D}_0)).$ As k° is excellent, π is a finite morphism, that is, $\mathcal{X} \in \Psi$. In addition, $g_{(\mathcal{X}, \mathcal{D})} = \varphi + g_0.$ Therefore, we have the assertion of the claim. $\square$

We set $g_1 = g_{(\mathcal{X}, \mathcal{D})} - \epsilon/2$ and $g_2 = g_{(\mathcal{X}, \mathcal{D})} + \epsilon/2.$ Then

$$g - \epsilon \leq g_1 \leq g \leq g_2 \leq g + \epsilon$$

on $X_v^{\text{an}}.$ Moreover, note that the global model function arising from the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is a constant function. Thus the assertion follows. $\square$

Theorem 3.3.8. *We assume that v is complete and X is normal. Let $X_{\text{div}}^{\text{an}}$ denote the subset of X^{an} given by*

$$X_{\text{div}}^{\text{an}} := \left\{ x \in X^{\text{an}} \mid \begin{array}{l} x \text{ is a point arising from an irreducible component of} \\ \text{the central fiber of some normal model of } X \text{ over } k^\circ \end{array} \right\}.$$

Then $X_{\text{div}}^{\text{an}}$ is dense in $X^{\text{an}}.$

Proof. Let ϕ be a continuous function $X^{\text{an}}.$ It is sufficient to show that if $\phi|_{X_{\text{div}}^{\text{an}}} = 0,$ then $\phi = 0.$ For any $\epsilon > 0,$ by Theorem 3.3.3, there is $\theta \in \mathbb{M}(X^{\text{an}}; \Psi) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $|\phi - \theta| \leq \epsilon.$ Thus $|\theta(x)| \leq \epsilon$ for all $x \in X_{\text{div}}^{\text{an}},$ which implies that $|\theta| \leq \epsilon$ on $X^{\text{an}}.$ Indeed, let Θ be a vertical $\mathbb{Q}$ -Cartier divisor on some normal model $\mathcal{X}$ of X such that θ is given by Θ (i.e. $\theta = g_{(\mathcal{X}, \Theta)}).$ Let $\Gamma_1, \dots, \Gamma_r$ be irreducible components of the central fiber $\mathcal{X}_0.$ and let $x_1, \dots, x_r$ be the corresponding points of $X_{\text{div}}^{\text{an}}$ to $\Gamma_1, \dots, \Gamma_r,$ respectively. If we set $\Theta = c_1\Gamma_1 + \dots + c_r\Gamma_r,$ then $c_i = \text{ord}_{\Gamma_i}(\varpi)\theta(x_i)/(-2\log v(\varpi)),$ so that

$$-(\epsilon/(-2\log v(\varpi)))\mathcal{X}_0 \leq \Theta \leq (\epsilon/(-2\log v(\varpi)))\mathcal{X}_0,$$

which means that $|\theta| \leq \epsilon$ on $X^{\text{an}}.$ Thus $|\phi| \leq 2\epsilon,$ and hence $\phi = 0.$ $\square$

3.4. Zariski's lemma for integrable functions. In this Subsection, let us consider Zariski's lemma for integrable functions. We assume that X is normal, $\dim X = 1$ and v is complete. Let $C_{\text{int}}^0(X^{\text{an}})$ be the space of integrable continuous functions on X^{an} (cf. Subsection 2.2). Let

$$\langle \cdot, \cdot \rangle : C_{\text{int}}^0(X^{\text{an}}) \times C_{\text{int}}^0(X^{\text{an}}) \rightarrow \mathbb{R}$$

be the map given by $\langle \varphi, \psi \rangle := \widehat{\deg}_v((0, \varphi); \psi) = \widehat{\deg}_v((0, \varphi) \cdot (0, \psi)),$ which is bilinear and symmetric (see Definition 2.4.10).

Lemma 3.4.1 (Zariski's lemma for integrable functions). *The above pairing $\langle \cdot, \cdot \rangle$ is negative semi-definite. Moreover, for $\varphi \in C_{\text{int}}^0(X^{\text{an}}),$ $\langle \varphi, \varphi \rangle = 0$ if and only if φ is a constant function.*

Proof. The proof can be found in [36]. For reader's convenience, we prove it here. Let $\mathbb{M}(X^{\text{an}})$ be the group of model functions on X^{an} (cf. Subsection 3.2). Then, by virtue of Zariski's lemma (cf. [31, Lemma 1.1.4]), for $\theta \in \mathbb{M}(X^{\text{an}}) \otimes_{\mathbb{Z}} \mathbb{R},$ $\langle \theta, \theta \rangle \leq 0.$ Therefore, by the density theorem (cf. Theorem 3.3.3) together with Proposition-Definition 2.4.3, we can see that $\langle \varphi, \varphi \rangle \leq 0$ for all $\varphi \in C_{\text{int}}^0(X^{\text{an}}).$ Thus $\langle \cdot, \cdot \rangle$ is negative semi-definite. Clearly if φ is a constant function, then $\langle \varphi, \varphi \rangle = 0.$ Conversely, we assume $\langle \varphi, \varphi \rangle = 0.$ Then, by [31, Lemma 1.1.3], $\langle \psi, \varphi \rangle = 0$ for all $\psi \in C_{\text{int}}^0(X^{\text{an}}).$

Let $\mathcal{X}$ be a regular model of X over $\text{Spec}(k^\circ)$ and $\mathcal{X}_\circ = a_1 C_1 + \cdots + a_r C_r$ the irreducible decomposition of the central fiber $\mathcal{X}_\circ$ as a cycle. Let x_j be the point of X^{an} corresponding to C_j . For $i = 1, \dots, r$, if $\psi_i = g_{(\mathcal{X}, C_i)}$, then

$$\begin{aligned} \deg\left(C_i \cdot \sum_{j=1}^r \varphi(x_j) a_j C_j\right) &= -2 \log v(\varpi) \sum_{j=1}^r \frac{\varphi(x_j) a_j}{-2 \log v(\varpi)} \deg(C_i \cdot C_j) \\ &= -2 \log v(\varpi) \langle \psi_i, \varphi \rangle = 0. \end{aligned}$$

Therefore, by the equality condition of Zariski's lemma (cf. [31, Lemma 1.1.4]), we have

$$\varphi(x_1) = \cdots = \varphi(x_r).$$

Let $X_{\text{div}}^{\text{an}}$ denote the subset of X^{an} given by

$$X_{\text{div}}^{\text{an}} := \left\{ x \in X^{\text{an}} \mid \begin{array}{l} x \text{ is a point arising from an irreducible component of} \\ \text{the central fiber of some normal model of } X \text{ over } k^\circ \end{array} \right\}.$$

The above observation shows that $\varphi|_{X_{\text{div}}^{\text{an}}}$ is a constant function. On the other hand, by Theorem 3.3.8, $X_{\text{div}}^{\text{an}}$ is dense in X^{an} , so that φ is a constant function on X^{an} . $\square$

3.5. Radon measure arising from local intersection number. We assume that X is normal and d -dimensional. Let $\bar{L}_1, \dots, \bar{L}_d$ be integrable adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Let ϕ be a continuous function on X_v^{an} . In Subsection 2.4, we define the local intersection number $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$ under the assumption that ϕ is integrable. In this subsection, we define $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$ for a general continuous function ϕ by using the density theorem (cf. Theorem 3.3.6). Let us begin with the following proposition:

Proposition-Definition 3.5.1. *Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of model functions on X_v^{an} such that $\lim_{n \rightarrow \infty} \|\varphi_n - \phi\|_{\text{sup}} = 0$. Then $\lim_{n \rightarrow \infty} \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \varphi_n)$ exists and the limit does not depend on the choice of the sequence $\{\varphi_n\}_{n=1}^\infty$. We denote the limit by $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$.*

Proof. By (3) in Proposition 2.4.11, there is a constant C depending only on $\bar{L}_1, \dots, \bar{L}_d$ and v such that

$$\left| \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \varphi) - \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \varphi') \right| \leq C \|\varphi - \varphi'\|_{\text{sup}}$$

for all model functions φ and φ' on X_v^{an} . Thus we can see that $\{\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \varphi_n)\}_{n=1}^\infty$ forms a Cauchy sequence, so that the limit exists. Let $\{\varphi'_n\}_{n=1}^\infty$ be another sequence of model functions with $\lim_{n \rightarrow \infty} \|\varphi'_n - \phi\|_{\text{sup}} = 0$. Then, as $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi'_n\|_{\text{sup}} = 0$, the assertion follows from the above estimate. $\square$

Let $\bar{L}'_i$ be an integrable adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X and let ϕ' be a continuous function on X_v^{an} . Then we can easily see the following formulae:

- (i) $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) \geq 0$ if $\bar{L}_1, \dots, \bar{L}_d$ are relatively nef and $\phi \geq 0$.
- (ii) $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; a\phi + a'\phi') = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi')$ for $a, a' \in \mathbb{R}$.

(iii) Let $\{\phi_n\}_{n=1}^\infty$ be a sequence of continuous functions on X_v^{an} with

$$\lim_{n \rightarrow \infty} \|\phi_n - \phi\|_{\text{sup}} = 0.$$

Then $\lim_{n \rightarrow \infty} \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi_n) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$.

(iv) $\widehat{\deg}_v(\bar{L}_1 \cdots (a\bar{L}_i + a'\bar{L}_i') \cdots \bar{L}_d; \phi)$
 $= a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i' \cdots \bar{L}_d; \phi)$

for $a, a' \in \mathbb{R}$.

(v) $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_j \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_j \cdots \bar{L}_i \cdots \bar{L}_d; \phi)$.

In the case where $\bar{L}_1, \dots, \bar{L}_d$ are relatively nef, by the above properties (i), (ii) and (iii), the functional $C^0(X_v^{\text{an}}) \rightarrow \mathbb{R}$ given by $\phi \mapsto \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$ yields a Radon measure on X_v^{an} . This Radon measure is called the *Radon measure induced by the local intersection* $\bar{L}_1 \cdots \bar{L}_d$ and it is denoted by $\mu_{\bar{L}_1 \cdots \bar{L}_d}$.

4. ADELIC ARITHMETIC $\mathbb{R}$ -CARTIER DIVISORS

In this section, we will introduce adelic arithmetic $\mathbb{R}$ -Cartier divisors on arithmetic varieties and investigate their several basic properties.

Throughout this section, let K be a number field and let X be a d -dimensional, projective, normal and geometrically integral variety over K .

4.1. Definition and basic properties. Let O_K be the ring of integers in K . We denote the set of all maximal ideals of O_K by M_K . For $\mathfrak{p} \in M_K$, the absolute value $v_{\mathfrak{p}}$ of K at $\mathfrak{p}$ is given by

$$v_{\mathfrak{p}}(x) = \#(O_K/\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(x)}.$$

Let $K_{\mathfrak{p}}$ be the completion of K with respect to $v_{\mathfrak{p}}$ and let $X_{\mathfrak{p}} := X \times_{\text{Spec}(K)} \text{Spec}(K_{\mathfrak{p}})$, which is also a projective, normal and geometrically integral variety over $K_{\mathfrak{p}}$ (cf. Lemma 1.1.9). Let $X(\mathbb{C})$ be the set of all $\mathbb{C}$ -valued points of X , that is,

$$X(\mathbb{C}) := \{x : \text{Spec}(\mathbb{C}) \rightarrow X \mid x \text{ is a morphism of schemes}\}.$$

Note that $X(\mathbb{C})$ is a projective analytic space and $X(\mathbb{C})$ is not necessarily connected. Let $F_{\infty} : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ be the complex conjugation map, that is, for $x \in X(\mathbb{C})$, $F_{\infty}(x)$ is given by the composition of morphisms

$$\text{Spec}(\mathbb{C}) \xrightarrow{c} \text{Spec}(\mathbb{C}) \quad \text{and} \quad \text{Spec}(\mathbb{C}) \xrightarrow{x} X,$$

where $\text{Spec}(\mathbb{C}) \xrightarrow{c} \text{Spec}(\mathbb{C})$ is the morphism arising from the complex conjugation. The complex conjugation map $F_{\infty} : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ is an anti-holomorphic isomorphism. The space of F_{∞} -invariant real valued continuous functions on $X(\mathbb{C})$ is denoted by $C_{F_{\infty}}^0(X(\mathbb{C}))$, that is,

$$C_{F_{\infty}}^0(X(\mathbb{C})) := \{f \in C^0(X(\mathbb{C})) \mid f \circ F_{\infty} = f\}.$$

Definition 4.1.1. A pair $\overline{D} = (D, g)$ of an $\mathbb{R}$ -Cartier divisor D on X and a collection of Green functions $g = \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K} \cup \{g_{\infty}\}$ is called an *adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X* if the following conditions (1) and (2) hold:

- (1) For all $\mathfrak{p} \in M_K$, $g_{\mathfrak{p}}$ is a D -Green function of C^0 -type on $X_{\mathfrak{p}}^{\text{an}}$. In addition, there exist a non-empty open set U of $\text{Spec}(O_K)$, a normal model $\mathcal{X}_U$ of X over U and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}_U$ on $\mathcal{X}_U$ such that $\mathcal{D}_U \cap X = D$ and $g_{\mathfrak{p}}$ is the D -Green function induced by $(\mathcal{X}_U, \mathcal{D}_U)$ for all $\mathfrak{p} \in U \cap M_K$.
- (2) The Green function g_{∞} is a D -Green function of C^0 -type on $X(\mathbb{C})$ such that $g_{\infty} = g_{\infty} \circ F_{\infty}$ (cf. Subsection 1.4).

Moreover, a pair $\overline{D}' = (D, g')$ of an $\mathbb{R}$ -Cartier divisor D on X and a collection of Green functions $g' = \{g'_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K}$ is called a *global adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X* if the above condition (1) holds for $\{g'_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K}$, that is, $g'_{\mathfrak{p}}$ is a D -Green function of C^0 -type on $X_{\mathfrak{p}}^{\text{an}}$ for all $\mathfrak{p} \in M_K$, and there exist a non-empty open set U' of $\text{Spec}(O_K)$, a normal model $\mathcal{X}'_{U'}$ of X over U' and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}'_{U'}$ on $\mathcal{X}'_{U'}$ such that $\mathcal{D}'_{U'} \cap X = D$ and $g'_{\mathfrak{p}}$ is the D -Green function induced by $(\mathcal{X}'_{U'}, \mathcal{D}'_{U'})$ for all $\mathfrak{p} \in U' \cap M_K$.

The pair $(\mathcal{X}_U, \mathcal{D}_U)$ in condition (1) is called a *defining model of $\overline{D}$ over U* . If we forget the Green function g_{∞} on $X(\mathbb{C})$ from $\overline{D}$, we have a global adelic $\mathbb{R}$ -Cartier divisor on X , which is denoted by $\overline{D}^{\tau}$ and is called the *truncation of $\overline{D}$* . For simplicity, a collection of Green functions $g = \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K} \cup \{g_{\infty}\}$ is often denoted by the following symbol:

$$g = \sum_{\mathfrak{p} \in M_K} g_{\mathfrak{p}}[\mathfrak{p}] + g_{\infty}[\infty].$$

Let $\text{Rat}(X)$ be the rational function field of X . For $\varphi \in \text{Rat}(X)^{\times}$, we define $\widehat{(\varphi)}$ to be

$$\widehat{(\varphi)} := \left((\varphi), \sum_{\mathfrak{p} \in M_K} (-\log |\varphi_{\mathfrak{p}}|^2)[\mathfrak{p}] + (-\log |\varphi_{\infty}|^2)[\infty] \right),$$

where $\varphi_{\mathfrak{p}}$ and φ_{∞} are the rational functions on $X_{\mathfrak{p}}$ and $X(\mathbb{C})$ induced by φ , respectively. The adelic arithmetic divisor $\widehat{(\varphi)}$ is called an *adelic arithmetic principal divisor*. Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . For $a_1, a_2 \in \mathbb{R}$, we define $a_1 \overline{D}_1 + a_2 \overline{D}_2$ to be

$$a_1 \overline{D}_1 + a_2 \overline{D}_2 := (a_1 D_1 + a_2 D_2, a_1 g_1 + a_2 g_2),$$

where

$$a_1 g_1 + a_2 g_2 = \sum_{\mathfrak{p} \in M_K} (a_1 (g_1)_{\mathfrak{p}} + a_2 (g_2)_{\mathfrak{p}})[\mathfrak{p}] + (a_1 (g_1)_{\infty} + a_2 (g_2)_{\infty})[\infty].$$

The space of all adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X is denoted by $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$, which forms an $\mathbb{R}$ -vector space by the previous definition. Moreover, we define $\overline{D}_1 \leq \overline{D}_2$ by the following conditions:

- (a) $D_1 \leq D_2$.
- (b) $(g_1)_{\varphi} \leq (g_2)_{\varphi}$ for all $\varphi \in M_K \cup \{\infty\}$.

Similarly, for global adelic $\mathbb{R}$ -Cartier divisors

$$(D_1, \{(g_1)_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K}) \quad \text{and} \quad (D_2, \{(g_2)_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K}),$$

$$(D_1, \{(g_1)_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K}) \leq (D_2, \{(g_2)_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K})$$

is defined by $D_1 \leq D_2$ and $(g_1)_{\mathfrak{p}} \leq (g_2)_{\mathfrak{p}}$ for all $\mathfrak{p} \in M_K$. In general, for $\varphi = \varphi_1^{a_1} \cdots \varphi_r^{a_r} \in \text{Rat}(X)_{\mathbb{R}}^{\times} := \text{Rat}(X)^{\times} \otimes \mathbb{R}$ ($\varphi_1, \dots, \varphi_r \in \text{Rat}(X)^{\times}$, $a_1, \dots, a_r \in \mathbb{R}$), we define $\widehat{(\varphi)}$ to be

$$\widehat{(\varphi)} := a_1 \widehat{(\varphi_1)} + \cdots + a_r \widehat{(\varphi_r)},$$

which is well-defined because the map $\text{Rat}(X)^\times \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_\mathbb{R}$ given by $\varphi \mapsto \widehat{(\varphi)}$ is a homomorphism.

Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(O_K)$ and let $\overline{\mathcal{D}} = (\mathcal{D}, g_\infty)$ be an arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{X}$ (cf. [30, Section 5]). The pair $(\mathcal{X}, \overline{\mathcal{D}})$ gives rise to an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X , that is,

$$\left(\mathcal{D} \cap X, \sum_{\mathfrak{p} \in M_K} g_{(\mathcal{X}_{(\mathfrak{p})}, \mathcal{D}_{(\mathfrak{p})})}[\mathfrak{p}] + g_\infty[\infty] \right),$$

where $\mathcal{X}_{(\mathfrak{p})}$ is the localization of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ at $\mathfrak{p}$ and $\mathcal{D}_{(\mathfrak{p})}$ is the restriction of $\mathcal{D}$ to $\mathcal{X}_{(\mathfrak{p})}$. We use the symbol $\mathcal{X}_{(\mathfrak{p})}$ to distinguish it from $X_{\mathfrak{p}}$ at the beginning of this subsection. We denote it by $\overline{\mathcal{D}}^a$ and it is called the *associated adelic arithmetic $\mathbb{R}$ -Cartier divisor with $\overline{\mathcal{D}}$* . Note that $\widehat{(\varphi)} = \left(\widehat{(\varphi)}_{\mathcal{X}} \right)^a$ for $\varphi \in \text{Rat}(\mathcal{X})_\mathbb{R}^\times$, where $\widehat{(\varphi)}_{\mathcal{X}}$ is the arithmetic principal divisor of φ on $\mathcal{X}$. Similarly, the *associated global adelic $\mathbb{R}$ -Cartier divisor $\mathcal{D}^a$ with $\mathcal{D}$* is defined by

$$\mathcal{D}^a := \left(\mathcal{D} \cap X, \{ g_{(\mathcal{X}_{(\mathfrak{p})}, \mathcal{D}_{(\mathfrak{p})})} \}_{\mathfrak{p} \in M_K} \right).$$

By abuse of notation, we often use the notations $\overline{\mathcal{D}} \leq \overline{D}_2$ and $\overline{D}_1 \leq \overline{\mathcal{D}}$ instead of $\overline{\mathcal{D}} \leq \overline{D}_2$ and $\overline{D}_1 \leq \overline{\mathcal{D}}^a$, respectively. The following proposition is the arithmetic version of Proposition 2.2.1 and it follows from Proposition 2.2.1.

Proposition 4.1.2. *Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(O_K)$ and let $\widehat{\text{Div}}_{C^0}(\mathcal{X})_\mathbb{R}$ be the space of arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on $\mathcal{X}$. Let*

$$\iota : \widehat{\text{Div}}_{C^0}(\mathcal{X})_\mathbb{R} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_\mathbb{R}$$

be the map given by $\overline{\mathcal{D}} \mapsto \overline{\mathcal{D}}^a$. Then we have the following:

- (1) *The map $\iota : \widehat{\text{Div}}_{C^0}(\mathcal{X})_\mathbb{R} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_\mathbb{R}$ is an injective homomorphism of $\mathbb{R}$ -vector spaces.*
- (2) $\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}}_2 \iff \overline{\mathcal{D}}_1^a \leq \overline{\mathcal{D}}_2^a$.

Theorem 4.1.3 (Approximation theorem of adelic arithmetic $\mathbb{R}$ -divisors). *Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X and let $(\mathcal{X}_U, \mathcal{D}_U)$ be a defining model of $\overline{D}$ over a non-empty open set $U \subseteq \text{Spec}(O_K)$. For any positive number $\epsilon > 0$, there exist a normal model $\mathcal{X}_\epsilon$ over $\text{Spec}(O_K)$, and $\mathbb{R}$ -Cartier divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{X}_\epsilon$ with the following properties:*

- (1) $\mathcal{X}_\epsilon|_U = \mathcal{X}_U$, $\mathcal{D}_1|_U = \mathcal{D}_U$ and $\mathcal{D}_2|_U = \mathcal{D}_U$.
- (2) *If we set $S = M_K \setminus U$, $\overline{\mathcal{D}}_1 = (\mathcal{D}_1, g_\infty)$ and $\overline{\mathcal{D}}_2 = (\mathcal{D}_2, g_\infty)$, then*

$$\overline{D} - \left(0, \sum_{\mathfrak{p} \in S} \epsilon[\mathfrak{p}] \right) \leq \overline{\mathcal{D}}_1^a \leq \overline{D} \leq \overline{\mathcal{D}}_2^a \leq \overline{D} + \left(0, \sum_{\mathfrak{p} \in S} \epsilon[\mathfrak{p}] \right).$$

Proof. This is a consequence of Theorem 3.3.7. For $\mathfrak{p} \in \text{Spec}(O_K) \setminus \{0\}$, let $(O_K)_{(\mathfrak{p})}$ denote the localization of O_K at $\mathfrak{p}$. By Theorem 3.3.7, for each $\mathfrak{p} \in S$, there are a normal model $\mathcal{X}_{\epsilon, \mathfrak{p}}$ of X over $\text{Spec}((O_K)_{(\mathfrak{p})})$ and $\mathbb{R}$ -Cartier divisors $\mathcal{D}_{1, \mathfrak{p}}$ and $\mathcal{D}_{2, \mathfrak{p}}$ on $\mathcal{X}_{\epsilon, \mathfrak{p}}$ such that

$$\mathcal{D}_{1, \mathfrak{p}} \cap X = \mathcal{D}_{2, \mathfrak{p}} \cap X = D \quad \text{and} \quad (D, g_{\mathfrak{p}}) - (0, \epsilon) \leq \mathcal{D}_{1, \mathfrak{p}} \leq (D, g_{\mathfrak{p}}) \leq \mathcal{D}_{2, \mathfrak{p}} \leq (D, g_{\mathfrak{p}}) + (0, \epsilon).$$

As S is finite, we can construct a normal $\mathcal{X}_\epsilon$ over $\text{Spec}(O_K)$ and $\mathbb{R}$ -Cartier divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{X}_\epsilon$ such that

$$\mathcal{X}_\epsilon|_U = \mathcal{X}_U, \quad \mathcal{D}_i|_U = \mathcal{D}_U, \quad (\mathcal{X}_\epsilon)_{(\mathfrak{p})} = \mathcal{X}_{\epsilon,\mathfrak{p}} \quad \text{and} \quad \mathcal{D}_i \cap \mathcal{X}_{\epsilon,\mathfrak{p}} = \mathcal{D}_{i,\mathfrak{p}}$$

for $\mathfrak{p} \in S$ and $i = 1, 2$, where $(\mathcal{X}_\epsilon)_{(\mathfrak{p})}$ is the localization of $\mathcal{X}_\epsilon$ at $\mathfrak{p}$, so that we have the assertion of the theorem. $\square$

4.2. Global degree. Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let x be a closed point of X . First we assume that $x \notin \text{Supp}_{\mathbb{R}}(D)$. For $\mathfrak{p} \in M_K$, the local degree of $\overline{D}$ over the absolute value $\nu_{\mathfrak{p}}$ is denoted by $\widehat{\deg}_{\mathfrak{p}}(\overline{D}|_x)$ (cf. Subsection 2.3). Moreover, $\widehat{\deg}_{\infty}(\overline{D}|_x)$ is defined by

$$\widehat{\deg}_{\infty}(\overline{D}|_x) := \frac{1}{2} \sum_{\sigma: K(x) \hookrightarrow \mathbb{C}} g_{\infty}(x_{\sigma}),$$

where $K(x)$ is the residue field at x and x_{σ} is the $\mathbb{C}$ -valued point given by

$$\mathcal{O}_{X,x} \rightarrow K(x) \xrightarrow{\sigma} \mathbb{C}.$$

Let U be a non-empty Zariski open set of $\text{Spec}(O_K)$ such that $\overline{D}$ has a defining model $(\mathcal{X}_U, \mathcal{D}_U)$ over U . Let Δ_x be the closure of x in $\mathcal{X}_U$. Shrinking U if necessary, we may assume that $\Delta_x \cap \text{Supp}_{\mathbb{R}}(\mathcal{D}_U) = \emptyset$, which implies that $\widehat{\deg}_{\mathfrak{p}}(\overline{D}|_x) = 0$ for $\mathfrak{p} \in U$. Therefore, we can define $\widehat{\deg}(\overline{D}|_x)$ to be

$$\widehat{\deg}(\overline{D}|_x) = \sum_{\mathfrak{p} \in M_K} \widehat{\deg}_{\mathfrak{p}}(\overline{D}|_x) + \widehat{\deg}_{\infty}(\overline{D}|_x).$$

Note that

$$(4.2.1) \quad \widehat{\deg}((\widehat{\varphi})|_x) = 0$$

for $\varphi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ with $x \notin \text{Supp}((\varphi))$. In general, we can find $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ such that $x \notin \text{Supp}_{\mathbb{R}}(D + (\phi))$ (cf. [30, Lemma 5.2.3]). By using (4.2.1), we can see that the quantity $\widehat{\deg}(\overline{D} + (\widehat{\phi})|_x)$ does not depend on the choice of ϕ , so that it is denoted by $\widehat{\deg}(\overline{D}|_x)$ and is called the *global degree of $\overline{D}$ along x* . The equation (4.2.1) can be generalized as follows:

$$(4.2.2) \quad \widehat{\deg}((\widehat{\psi})|_x) = 0$$

for all $\psi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$. It is easy to see that

$$\widehat{\deg}(a\overline{D} + a'\overline{D}'|_x) = a\widehat{\deg}(\overline{D}|_x) + a'\widehat{\deg}(\overline{D}'|_x)$$

for $\overline{D}, \overline{D}' \in \widehat{\text{Div}}_{C^0}^{\text{an}}(X)_{\mathbb{R}}$ and $a, a' \in \mathbb{R}$.

Lemma 4.2.3. *Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $D_1 = D_2$ and $g_1 \leq g_2$, then $\widehat{\deg}(\overline{D}_1|_x) \leq \widehat{\deg}(\overline{D}_2|_x)$ for all closed points x of X .*

Proof. As $D_1 = D_2$ and $g_1 \leq g_2$, there are non-negative continuous functions $\phi_{\mathfrak{p}}$ on $X_{\mathfrak{p}}^{\text{an}}$ and ϕ_{∞} on $X(\mathbb{C})$ such that $(g_2)_{\mathfrak{p}} = (g_1)_{\mathfrak{p}} + \phi_{\mathfrak{p}}$ and $(g_2)_{\infty} = (g_1)_{\infty} + \phi_{\infty}$, respectively. We set $\phi = \sum_{\mathfrak{p} \in M_K} \phi_{\mathfrak{p}}[\mathfrak{p}] + \phi_{\infty}[\infty]$. Then, as $\widehat{\deg}((0, \phi)|_x) \geq 0$,

$$\widehat{\deg}(\overline{D}_2|_x) = \widehat{\deg}(\overline{D}_1|_x) + \widehat{\deg}((0, \phi)|_x) \geq \widehat{\deg}(\overline{D}_1|_x),$$

as required. $\square$

4.3. Volume of adelic arithmetic $\mathbb{R}$ -Cartier divisors. Let D be an $\mathbb{R}$ -Cartier divisor on X , $\overline{D}' = (D, g')$ a global adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X , and $\overline{D} = (D, g)$ an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . We define $H^0(X, D)$ to be

$$H^0(X, D) := \{\varphi \in \text{Rat}(X)^\times \mid D + (\varphi) \geq 0\} \cup \{0\}.$$

For $\wp \in M_K \cup \{\infty\}$ and $\phi \in H^0(X, D)$, it is known that $|\phi|_{g_\wp} := |\phi| \exp(-g_\wp/2)$ extends to a continuous function (cf. Proposition 1.4.2 and Proposition 2.1.3), so that its supremum is denoted by $\|\phi\|_{g_\wp}$. Using the above norms, we define $\hat{H}^0(X, \overline{D}')$ and $\hat{H}^0(X, \overline{D})$ to be

$$\begin{cases} \hat{H}^0(X, \overline{D}') := \{\varphi \in H^0(X, D) \mid \|\varphi\|_{g'_\wp} \leq 1 \text{ for all } \wp \in M_K\}, \\ \hat{H}^0(X, \overline{D}) := \{\varphi \in H^0(X, D) \mid \|\varphi\|_{g_\wp} \leq 1 \text{ for all } \wp \in M_K \cup \{\infty\}\}. \end{cases}$$

Note that $\hat{H}^0(X, \overline{D}')$ is a submodule of $H^0(X, D)$ by using Proposition 2.1.3. Let us check the following proposition:

Proposition 4.3.1. (1) $\hat{H}^0(X, \overline{D}')$ and $\hat{H}^0(X, \overline{D})$ are given in the following ways:

$$\begin{cases} \hat{H}^0(X, \overline{D}') = \{\varphi \in \text{Rat}(X)^\times \mid \overline{D}' + (\widehat{\varphi})^\tau \geq 0\} \cup \{0\}, \\ \hat{H}^0(X, \overline{D}) = \{\varphi \in \text{Rat}(X)^\times \mid \overline{D} + (\widehat{\varphi}) \geq 0\} \cup \{0\}. \end{cases}$$

(2) We assume that $\overline{D}^\tau = \overline{D}'$. If there are a normal model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{D} \cap X = D$ and $g_\wp$ is the Green function arising from $(\mathcal{X}, \mathcal{D})$ for each $\wp \in M_K$, then

$$\hat{H}^0(X, \overline{D}') = H^0(\mathcal{X}, \mathcal{D}) \quad \text{and} \quad \hat{H}^0(X, \overline{D}) = \hat{H}^0(\mathcal{X}, (\mathcal{D}, g_\infty)).$$

(3) $\hat{H}^0(X, \overline{D}')$ is a finitely generated free $\mathbb{Z}$ -module and $\hat{H}^0(X, \overline{D})$ is a finite set. We denote $\log \#(\hat{H}^0(X, \overline{D}))$ by $\hat{h}^0(X, \overline{D})$.

(4) Let $\mu : Y \rightarrow X$ be a birational morphism of projective, normal and geometrically integral variety over K . Then, for $\wp \in M_K \cup \{\infty\}$, and $\phi \in H^0(X, D)$, we have $\|\phi\|_{g_\wp} = \|\mu^*(\phi)\|_{\mu^*(g_\wp)}$. In particular,

$$\hat{H}^0(X, \overline{D}') \rightarrow \hat{H}^0(Y, \mu^*(\overline{D}')) \quad \text{and} \quad \hat{H}^0(X, \overline{D}) \rightarrow \hat{H}^0(Y, \mu^*(\overline{D}))$$

are bijective.

Proof. (1) Note that

$$\|\varphi\|_{g_\infty} \leq 1 \iff g_\infty - \log |\varphi|^2 \geq 0 \text{ on } X(\mathbb{C})$$

and

$$\|\varphi\|_{g_\wp} \leq 1 \iff g_\wp - \log |\varphi|^2 \geq 0 \text{ on } X_p^{\text{an}}$$

for $\wp \in M_K$. Thus (1) follows.

(2) The assertions of (2) follow from (1) and Proposition 4.1.2.

(3) Clearly we may assume that $\overline{D}^\tau = \overline{D}'$. We can find a normal model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}} = (\mathcal{D}, h)$ of C^0 -type on $\mathcal{X}$ such that $\overline{D} \leq \overline{\mathcal{D}}^a$. Thus

$$\hat{H}^0(X, \overline{D}') \subseteq \hat{H}^0(X, \overline{\mathcal{D}}^a) = H^0(\mathcal{X}, \mathcal{D})$$

by (2). Note that $H^0(\mathcal{X}, \mathcal{D})$ is a finitely generated free $\mathbb{Z}$ -module, so that $\hat{H}^0(X, \overline{D}')$ is also a finitely generated free $\mathbb{Z}$ -module. Since $\hat{H}^0(X, \overline{D}) \subseteq \hat{H}^0(\mathcal{X}, \overline{\mathcal{D}})$, the last assertion is obvious.

(4) The assertion $\|\phi\|_{g_\varphi} = \|\mu^*(\phi)\|_{\mu^*(g_\varphi)}$ is obvious. For the last assertion, note that $H^0(X, D) \rightarrow H^0(Y, \mu^*(D))$ is an isomorphism. $\square$

Definition 4.3.2. Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . The quantity $\hat{\chi}(X, \overline{D})$ is defined by

$$\hat{\chi}(X, \overline{D}) := \hat{\chi}(\hat{H}^0(X, \overline{D}^\tau), \|\cdot\|_{g_\infty})$$

(cf. Conventions and terminology 0.5.3). Note that

$$\hat{h}^0(X, \overline{D}) = \hat{h}^0(\hat{H}^0(X, \overline{D}^\tau), \|\cdot\|_{g_\infty}).$$

Moreover, we define the *volume* $\widehat{\text{vol}}(\overline{D})$ of $\overline{D}$ and the $\hat{\chi}$ -*volume* $\widehat{\text{vol}}_\chi(\overline{D})$ of $\overline{D}$ to be

$$\widehat{\text{vol}}(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(X, n\overline{D})}{n^{d+1}/(d+1)!}$$

and

$$\widehat{\text{vol}}_\chi(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\hat{\chi}(X, n\overline{D})}{n^{d+1}/(d+1)!},$$

respectively, where $d = \dim X$. By Minkowski's theorem, $\widehat{\text{vol}}_\chi(\overline{D}) \leq \widehat{\text{vol}}(\overline{D})$. Let $\overline{L} = (L, h)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Clearly, if $\overline{L} \leq \overline{D}$, then $\hat{h}^0(X, \overline{L}) \leq \hat{h}^0(X, \overline{D})$ and $\widehat{\text{vol}}(\overline{L}) \leq \widehat{\text{vol}}(\overline{D})$. Further, by (1.1.3.1), if $\overline{L} \leq \overline{D}$ and $L = D$, then $\hat{\chi}(X, \overline{L}) \leq \hat{\chi}(X, \overline{D})$ and $\widehat{\text{vol}}_\chi(\overline{L}) \leq \widehat{\text{vol}}_\chi(\overline{D})$. Let $\mu : Y \rightarrow X$ be a birational morphism of projective, normal and geometrically integral variety over K . Then, by (4) in Proposition 4.3.1,

$$(4.3.3) \quad \widehat{\text{vol}}(\mu^*(\overline{D})) = \widehat{\text{vol}}(\overline{D}) \quad \text{and} \quad \widehat{\text{vol}}_\chi(\mu^*(\overline{D})) = \widehat{\text{vol}}_\chi(\overline{D}).$$

For the symbol $\natural$, $\widehat{\text{vol}}_\natural(\overline{D})$ stands for either $\widehat{\text{vol}}(\overline{D})$ or $\widehat{\text{vol}}_\chi(\overline{D})$, that is,

$$\widehat{\text{vol}}_\natural(\overline{D}) = \begin{cases} \widehat{\text{vol}}(\overline{D}) & \text{if } \natural \text{ is blank,} \\ \widehat{\text{vol}}_\chi(\overline{D}) & \text{if } \natural \text{ is } \chi. \end{cases}$$

4.4. Positivity of adelic arithmetic $\mathbb{R}$ -Cartier divisors. Here let us introduce several kinds of the positivity of adelic arithmetic divisors.

Definition 4.4.1. Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X .

- **Big:** We say that $\overline{D}$ is *big* if $\widehat{\text{vol}}(\overline{D}) > 0$. According to [25], we can give an alternative definition, that is, for any adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{L}$ of C^0 -type on X , $\hat{H}^0(X, n\overline{D} + \overline{L}) \neq \{0\}$ for some positive integer n . Actually the two definitions are equivalent by the continuity of the volume function.

- **Pseudo-effective:** $\overline{D}$ is said to be *pseudo-effective* if $\overline{D} + \overline{A}$ is big for any big adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{A}$ of C^0 -type on X .

- **Relatively nef:** $\overline{D}$ is said to be *relatively nef* if g_φ is of $(C^0 \cap \text{PSH})$ -type for all $\varphi \in M_K \cup \{\infty\}$ (cf. Subsection 1.4 and Definition 2.1.6). If $(\mathcal{X}_U, \mathcal{D}_U)$ is a defining model of $\overline{D}$, then $\mathcal{D}_U$ is relatively nef with respect to $\mathcal{X}_U \rightarrow U$ by Proposition 2.1.7.

- **Nef:** We say that $\overline{D}$ is *nef* if $\overline{D}$ is relatively nef and $\widehat{\deg}(\overline{D}|_x) \geq 0$ for all closed points x of X . For example, if $\phi \in \text{Rat}(X)^\times$, then the adelic arithmetic principal divisor $(\widehat{\phi})$ of ϕ is nef.

Let us see the following proposition.

Proposition 4.4.2. *Let $\overline{D} = (D, \{g_p\}_{p \in M_K} \cup \{g_\infty\})$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then we have the following:*

- (1) *Let U be a non-empty open set of $\text{Spec}(O_K)$ such that there is a defining model $(\mathcal{X}_U, \mathcal{D}_U)$ of $\overline{D}$ over U . If $\overline{D}$ is relatively nef, then there are sequences $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ and $\{(\mathcal{X}'_n, \mathcal{D}'_n)\}_{n=1}^\infty$ with the following properties:*
 - (1.1) *For every $n \geq 1$, $\mathcal{X}_n$ is a normal model of X over $\text{Spec}(O_K)$ such that $\mathcal{X}_n|_U = \mathcal{X}_U$.*
 - (1.2) *For every $n \geq 1$, $\mathcal{D}_n$ and $\mathcal{D}'_n$ are relatively nef $\mathbb{R}$ -Cartier divisors on $\mathcal{X}_n$ such that $\mathcal{D}_n|_U = \mathcal{D}'_n|_U = \mathcal{D}_U$.*
 - (1.3) *$(\mathcal{D}'_n, g_\infty)^a \leq \overline{D} \leq (\mathcal{D}_n, g_\infty)^a$ for all $n \geq 1$.*
 - (1.4) *If we set*

$$\phi_{n,p} := g_p - g_{((\mathcal{X}_n)_{(p)}, (\mathcal{D}_n)_{(p)})} \quad \text{and} \quad \phi'_{n,p} := g_p - g_{((\mathcal{X}_n)_{(p)}, (\mathcal{D}'_n)_{(p)})},$$

then

$$\lim_{n \rightarrow \infty} \|\phi_{n,p}\|_{\text{sup}} = \lim_{n \rightarrow \infty} \|\phi'_{n,p}\|_{\text{sup}} = 0,$$

where $(\mathcal{X}_n)_{(p)}$ is the localization of $\mathcal{X}_n \rightarrow \text{Spec}(O_K)$ at p and $(\mathcal{D}_n)_{(p)}$ and $(\mathcal{D}'_n)_{(p)}$ are the restrictions of $\mathcal{D}_n$ and $\mathcal{D}'_n$ to $(\mathcal{X}_n)_{(p)}$, respectively.

- (2) *If $\overline{D}$ is nef, then $\overline{D}$ is pseudo-effective.*
- (3) *If D is big on X and $\overline{D}$ is pseudo-effective, then $\overline{D} + (0, \epsilon[\infty])$ is big for any positive number ϵ .*

Proof. (1) By Proposition 2.1.7, $\mathcal{D}_U$ is relatively nef with respect to $\mathcal{X}_U \rightarrow U$. Thus the assertion follows from Proposition 2.1.8 in the same way as the proof of Theorem 4.1.3.

(2) Clearly we may assume that X is smooth over K . Let us choose a non-empty open set U of $\text{Spec}(O_K)$ such that $\overline{D}$ has a defining model over U . Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(O_K)$ and let $\overline{\mathcal{A}}$ be an arithmetic Cartier divisor of C^∞ -type on $\mathcal{X}$ such that

$$\overline{\mathcal{A}} - \left(\sum_{p \in M_K \setminus U} F_p, 0 \right)$$

is ample, where F_p is the fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ over p . It is sufficient to show that $\overline{D} + \epsilon \overline{\mathcal{A}}^a$ is big for all $\epsilon \in \mathbb{R}_{>0}$. By (1), we can choose a normal model $\mathcal{X}'$ of X over $\text{Spec}(O_K)$ and a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}'$ such that

$$(\mathcal{D}, g_\infty)^a - \left(0, \sum_{p \in M_K \setminus U} \epsilon[p] \right) \leq \overline{D} \leq (\mathcal{D}, g_\infty)^a.$$

We may assume that there is a birational morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}$. Then $(\mathcal{D}, g_\infty)$ is nef by Lemma 4.2.3 and

$$\left((\mathcal{D}, g_\infty) + \epsilon \mu^* \left(\overline{\mathcal{A}} - \left(\sum_{p \in M_K \setminus U} F_p, 0 \right) \right) \right)^a \leq \overline{D} + \epsilon \overline{\mathcal{A}}^a.$$

Note that $(\mathcal{D}, g_\infty) + \epsilon \mu^* \left(\overline{\mathcal{A}} - \left(\sum_{p \in M_K \setminus U} F_p, 0 \right) \right)$ is nef and big by [30, Proposition 6.2.2], as required.

(3) Let $\overline{A}$ be a big adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let us see the following claim:

Claim 4.4.2.1. *There are $m \in \mathbb{Z}_{>0}$, $\phi \in \text{Rat}(X)^\times$ and $\lambda \in \mathbb{R}$ such that*

$$m\overline{D} - \overline{A} + (\widehat{\phi}) + (0, \lambda[\infty]) \geq 0.$$

Proof. Since D is big on X , there are a positive integer m and a non-zero rational function ψ on X such that $mD - A + (\psi)$ is effective. We set $(L, h) := m\overline{D} - \overline{A} + (\widehat{\psi})$. Let U be a non-empty open set of $\text{Spec}(O_K)$ such that $\overline{L}$ has a defining model $\mathcal{L}_U$ over U . As $L \geq 0$, shrinking U if necessary, we may assume that $\mathcal{L}_U$ is effective. In particular, $h_{\mathfrak{p}} \geq 0$ for $\mathfrak{p} \in M_K \cap U$. Thus there is $\lambda' \in \mathbb{R}$ such that

$$(L, h) \geq \left(0, \sum_{\mathfrak{p} \in M_K \setminus U} (-\lambda')[\mathfrak{p}] + (-\lambda')[\infty]\right).$$

We choose $N \in \mathbb{Z}_{>0}$ such that $-\lambda' + 2 \text{ord}_{\mathfrak{p}}(N) \#(O_K/\mathfrak{p}) \geq 0$ for all $\mathfrak{p} \in M_K \setminus U$. If we set $\Delta = m\overline{D} - \overline{A} + (\widehat{N\psi})$, then

$$\begin{aligned} \Delta &= (L, h) + (\widehat{N}) \\ &\geq (L, h) + \left(0, \sum_{\mathfrak{p} \in M_K \setminus U} 2 \text{ord}_{\mathfrak{p}}(N) \#(O_K/\mathfrak{p}) [\mathfrak{p}] - \log N^2 [\infty]\right) \\ &\geq \left(0, \sum_{\mathfrak{p} \in M_K \setminus U} (-\lambda' + 2 \text{ord}_{\mathfrak{p}}(N) \#(O_K/\mathfrak{p})) [\mathfrak{p}] - (\lambda' + \log N^2) [\infty]\right) \\ &\geq (0, -(\lambda' + \log N^2) [\infty]), \end{aligned}$$

as required. $\square$

Let n be a positive integer such that $\lambda/(n+m) \leq \epsilon$. Then

$$\begin{aligned} \overline{D} + (1/n)(\overline{A} - (\widehat{\phi})) &\leq \overline{D} + (1/n)(m\overline{D} + (0, \lambda[\infty])) \\ &= ((n+m)/n)(\overline{D} + (0, \lambda/(n+m)[\infty])) \\ &\leq ((n+m)/n)(\overline{D} + (0, \epsilon[\infty])), \end{aligned}$$

so that we have the assertion. $\square$

In addition to the above positivity, an adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{D}$ of C^0 -type on X is said to be *integrable* if there are relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{D}'$ and $\overline{D}''$ of C^0 -type on X such that $\overline{D} = \overline{D}' - \overline{D}''$. The set of all integrable adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X is denoted by $\widehat{\text{Div}}_{\text{int}}^a(X)_{\mathbb{R}}$. Note that $\widehat{\text{Div}}_{\text{int}}^a(X)_{\mathbb{R}}$ forms a subspace of $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$ over $\mathbb{R}$.

Remark 4.4.3. Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(O_K)$. We recall that an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on $\mathcal{X}$ is said to be *integrable* if there are relatively nef arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}'$ and $\overline{\mathcal{D}}''$ of C^0 -type on $\mathcal{X}$ such that $\overline{\mathcal{D}} = \overline{\mathcal{D}}' - \overline{\mathcal{D}}''$ (cf. Definition 1.4.4 and [31, Subsection 2.1]). Note that $\overline{\mathcal{D}}$ is integrable if and only if $\overline{\mathcal{D}}^a$ is integrable.

Finally let us introduce the relative nefness of a global adelic $\mathbb{R}$ -Cartier divisor.

Definition 4.4.4. Let $\overline{D} = (D, \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K})$ be a global adelic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . We say that $\overline{D}$ is *relatively nef* if $g_{\mathfrak{p}}$ is of $(C^0 \cap \text{PSH})$ -type for all $\mathfrak{p} \in M_K$.

4.5. Global intersection number. The purpose of this subsection is to construct the intersection pairing

$$\left(\widehat{\text{Div}}_{\text{int}}^a(X)_{\mathbb{R}}\right)^{d+1} \rightarrow \mathbb{R} \quad \left((\overline{D}_1, \dots, \overline{D}_{d+1}) \mapsto \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})\right)$$

by using the local intersection number (cf. Subsection 2.4). For this, let us begin with the following lemma.

Lemma 4.5.1. *Let $\overline{D} = (D, g)$ be an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then there are a normal model $\mathcal{X}$ of X , an integrable arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on $\mathcal{X}$, a finite subset $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ of M_K and integrable continuous functions $\phi_1, \dots, \phi_r$ on $X_{\mathfrak{p}_1}^{\text{an}}, \dots, X_{\mathfrak{p}_r}^{\text{an}}$, respectively such that*

$$\overline{D} = \overline{\mathcal{D}}^a + \left(0, \sum_{i=1}^r \phi_i[\mathfrak{p}_i]\right).$$

Proof. By definition, there are relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{L}_1$ and $\overline{L}_2$ of C^0 -type on X such that $\overline{D} = \overline{L}_1 - \overline{L}_2$, so that, by using Proposition 4.4.2, we can find a normal model $\mathcal{X}$ of X , relatively nef arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ of C^0 -type on $\mathcal{X}$ and a finite subset $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ of M_K such that

$$\overline{L}_1 = \overline{\mathcal{L}}_1^a + \left(0, \sum_{i=1}^r \varphi_i[\mathfrak{p}_i]\right) \quad \text{and} \quad \overline{L}_2 = \overline{\mathcal{L}}_2^a + \left(0, \sum_{i=1}^r \psi_i[\mathfrak{p}_i]\right),$$

where φ_i and ψ_i are continuous functions on $X_{\mathfrak{p}_i}^{\text{an}}$. Note that φ_i and ψ_i are integrable. Thus, if we set $\overline{\mathcal{D}} = \overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2$ and $\phi_i = \varphi_i - \psi_i$, then we have the assertion. $\square$

Let $\overline{D}_1 = (D_1, g_1), \dots, \overline{D}_{d+1} = (D_{d+1}, g_{d+1})$ be integrable adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then, by Lemma 4.5.1, there are a normal model $\mathcal{X}$ of X , integrable arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}$ of C^0 -type on $\mathcal{X}$, and a finite subset S of M_K such that

$$\overline{D}_i = \overline{\mathcal{D}}_i^a + \left(0, \sum_{\mathfrak{p} \in S} \phi_{i,\mathfrak{p}}[\mathfrak{p}]\right),$$

where $\phi_{i,\mathfrak{p}}$'s are integrable continuous functions $X_{\mathfrak{p}}^{\text{an}}$.

First we assume that X is smooth over K . We would like to define the intersection number $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$ to be

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1}) &:= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) \\ &+ \sum_{\mathfrak{p} \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}} \left(\prod_{i \in I} (0, \phi_{i,\mathfrak{p}}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(\mathfrak{p})} \right), \end{aligned}$$

where $\widehat{\deg}_{\mathfrak{p}}$ is the local intersection number at $\mathfrak{p}$ (cf. Subsection 2.4) and $(\mathcal{D}_j)_{(\mathfrak{p})}$ means the restriction of $\mathcal{D}_j$ to the localization of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ at $\mathfrak{p}$. For this purpose, we need to see that the above formula does not depend on the choice of $\mathcal{X}$, $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}$ and S . We denote the right hand side of the above by $\Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S)$. Let $\mathcal{X}'$, $\overline{\mathcal{D}}'_1, \dots, \overline{\mathcal{D}}'_{d+1}$ and S' be another choice. In order to check

$$\Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S) = \Delta(\mathcal{X}', \overline{\mathcal{D}}'_1, \dots, \overline{\mathcal{D}}'_{d+1}, S'),$$

we may assume that $\mathcal{X}' = \mathcal{X}$ and $S' = S$. Note that there are vertical $\mathbb{R}$ -Cartier divisors $\mathcal{E}_1, \dots, \mathcal{E}_{d+1}$ on $\mathcal{X}$ such that $\text{Supp}_{\mathbb{R}}(\mathcal{E}_1), \dots, \text{Supp}_{\mathbb{R}}(\mathcal{E}_{d+1}) \subseteq \sum_{\mathfrak{p} \in S} F_{\mathfrak{p}}$ and $\mathcal{D}'_i = \mathcal{D}_i + \mathcal{E}_i$ for $i = 1, \dots, d+1$, where $F_{\mathfrak{p}}$ is the fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ over $\mathfrak{p}$. Thus it is sufficient to show that

$$\Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S) = \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \dots, \overline{\mathcal{D}}_{d+1} + (\mathcal{E}_{d+1}, 0), S)$$

for all vertical $\mathbb{R}$ -Cartier divisors $\mathcal{E}_1, \dots, \mathcal{E}_{d+1}$ on $\mathcal{X}$ with

$$\text{Supp}_{\mathbb{R}}(\mathcal{E}_1), \dots, \text{Supp}_{\mathbb{R}}(\mathcal{E}_{d+1}) \subseteq \sum_{\mathfrak{p} \in S} F_{\mathfrak{p}}.$$

If we can show

$$(4.5.2) \quad \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_l, \dots, \overline{\mathcal{D}}_{d+1}, S) = \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_l + (\mathcal{E}, 0), \dots, \overline{\mathcal{D}}_{d+1}, S)$$

for a vertical $\mathbb{R}$ -Cartier divisor $\mathcal{E}$ on $\mathcal{X}$ with $\text{Supp}_{\mathbb{R}}(\mathcal{E}) \subseteq \sum_{\mathfrak{p} \in S} F_{\mathfrak{p}}$, then

$$\begin{aligned} \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S) &= \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \overline{\mathcal{D}}_2, \dots, \overline{\mathcal{D}}_{d+1}, S) \\ &= \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \overline{\mathcal{D}}_2 + (\mathcal{E}_2, 0), \overline{\mathcal{D}}_3, \dots, \overline{\mathcal{D}}_{d+1}, S) \\ &= \dots = \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \dots, \overline{\mathcal{D}}_{d+1} + (\mathcal{E}_{d+1}, 0), S). \end{aligned}$$

Therefore, it suffices to check (4.5.2). We set $e_{\mathfrak{p}} = g_{(\mathcal{X}_{(\mathfrak{p})}, \mathcal{E}_{(\mathfrak{p})})}$. Then

$$\overline{D}_l = (\overline{\mathcal{D}}_l + (\mathcal{E}, 0))^a + \left(0, \sum_{\mathfrak{p} \in S} (\phi_{l,\mathfrak{p}} - e_{\mathfrak{p}})[\mathfrak{p}] \right),$$

so that

$$\begin{aligned} \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_l + (\mathcal{E}, 0), \dots, \overline{\mathcal{D}}_{d+1}, S) &= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots (\overline{\mathcal{D}}_l + (\mathcal{E}, 0)) \cdots \overline{\mathcal{D}}_{d+1}) + \\ &\quad \sum_{\mathfrak{p} \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}} \left(\prod_{i \in I} (0, \phi_{i,\mathfrak{p}} - \delta_{il} e_{\mathfrak{p}}) \cdot \prod_{j \notin I} (\mathcal{D}_j + \delta_{jl} \mathcal{E})_{(\mathfrak{p})} \right). \end{aligned}$$

Note that

$$\begin{aligned} \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots (\overline{\mathcal{D}}_l + (\mathcal{E}, 0)) \cdots \overline{\mathcal{D}}_{d+1}) &= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) \\ &\quad + \sum_{\mathfrak{p} \in S} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}}((\mathcal{D}_1)_{(\mathfrak{p})} \cdots (\mathcal{D}_{l-1})_{(\mathfrak{p})} \cdot (\mathcal{D}_{l+1})_{(\mathfrak{p})} \cdots (\mathcal{D}_{d+1})_{(\mathfrak{p})} \cdot \mathcal{E}_{(\mathfrak{p})}). \end{aligned}$$

Moreover,

$$\widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p} - \delta_{il} e_p) \cdot \prod_{j \notin I} (\mathcal{D}_j + \delta_{jl} \mathcal{E})_{(p)} \right) = \begin{cases} \widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(p)} \right) \\ \quad - \widehat{\deg}_p \left(\prod_{i \in I \setminus \{l\}} (0, \phi_{i,p}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(p)} \cdot e_p \right) & \text{if } l \in I, \\ \widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(p)} \right) \\ \quad + \widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p}) \cdot \prod_{j \notin I \cup \{l\}} (\mathcal{D}_j)_{(p)} \cdot e_p \right) & \text{if } l \notin I, \end{cases}$$

and hence

$$\begin{aligned} \sum_{p \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/p) \widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p} - \delta_{il} e_p) \cdot \prod_{j \notin I} (\mathcal{D}_j + \delta_{jl} \mathcal{E})_{(p)} \right) \\ = \sum_{p \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/p) \widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(p)} \right) \\ - \sum_{p \in S} \log \#(O_K/p) \widehat{\deg}_p ((\mathcal{D}_1)_{(p)} \cdots (\mathcal{D}_{l-1})_{(p)} \cdot (\mathcal{D}_{l+1})_{(p)} \cdots (\mathcal{D}_{d+1})_{(p)} \cdot e_p). \end{aligned}$$

Therefore, we have (4.5.2).

In general, let us take a desingularization $\mu : Y \rightarrow X$ of X . By virtue of (4.3.3) together with [31, Subsection 2.1] and (4) in Proposition 2.4.11, we can see that the quantity

$$\widehat{\deg}(\mu^*(\overline{D}_1) \cdots \mu^*(\overline{D}_{d+1}))$$

does not depend on the choice of μ , so that we define $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$ to be

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1}) := \widehat{\deg}(\mu^*(\overline{D}_1) \cdots \mu^*(\overline{D}_{d+1})).$$

By our construction, it is easy to see that the map

$$(\overline{D}_1, \dots, \overline{D}_{d+1}) \mapsto \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$$

is multi-linear and symmetric (cf. (2.4.7), (2.4.8) and Proposition 2.4.9).

Let $\overline{D}_1, \dots, \overline{D}_{d+1}, \overline{D}'_1, \dots, \overline{D}'_{d+1}$ be integrable adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Let T be a finite set of M_K and let $\varphi_{1,p}, \dots, \varphi_{d,p}$ be integrable continuous functions on X_p^{an} for $p \in T$. By using Lemma 1.1.5, we can see that if

$$\overline{D}'_i = \overline{D}_i + \left(0, \sum_{p \in T} \varphi_{i,p} [p] \right)$$

for $i = 1, \dots, d+1$, then

$$(4.5.3) \quad \widehat{\deg}(\overline{D}'_1 \cdots \overline{D}'_{d+1}) = \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1}) \\ + \sum_{\substack{\mathfrak{p} \in T \\ I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}} \left(\prod_{i \in I} (0, \varphi_{i, \mathfrak{p}}) \cdot \prod_{j \notin I} (D_j, (g_j)_{\mathfrak{p}}) \right),$$

where $\overline{D}_j = (D_j, \sum_{\mathfrak{p}} (g_j)_{\mathfrak{p}}[\mathfrak{p}] + (g_j)_{\infty}[\infty])$ for $j = 1, \dots, d+1$.

Proposition 4.5.4. *Let $\overline{D}_1, \dots, \overline{D}_d, \overline{D}_{d+1}$ be integrable adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then we have the following:*

- (1) *If $\nu: X' \rightarrow X$ is a birational morphism of normal, projective and geometrically integral varieties over K , then*

$$\widehat{\deg}(\nu^*(\overline{D}_1) \cdots \nu^*(\overline{D}_{d+1})) = \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1}).$$

- (2) *For $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \widehat{(\phi)}) = 0$.*

- (3) *If $\overline{D}_1, \dots, \overline{D}_d$ are nef and $\overline{D}_{d+1}$ is pseudo-effective, then*

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) \geq 0.$$

Proof. (1) is obvious. For (2) and (3), we may assume that X is smooth over K .

(2) Let us begin with the following claim:

Claim 4.5.4.1. *Let $\mathcal{X}$ be a normal model of X and let $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ be integrable arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on $\mathcal{X}$. Then $\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}) = 0$.*

Proof. Step 1 (the case where $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are of C^{∞} -type) : If $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are arithmetic Cartier divisors of C^{∞} -type on $\mathcal{X}$ and $\phi \in \text{Rat}(X)^{\times}$, then the assertion is well-known. On the other hand, by [30, Proposition 2.4.2], we can find arithmetic Cartier divisors $\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_r$ of C^{∞} -type on $\mathcal{X}$, $\phi_1, \dots, \phi_l \in \text{Rat}(X)^{\times}$, $a_{ij} \in \mathbb{R}$ ($i = 1, \dots, d$, $j = 1, \dots, r$) and $b_1, \dots, b_l \in \mathbb{R}$ such that $\overline{\mathcal{D}}_i = \sum_{j=1}^r a_{ij} \overline{\mathcal{E}}_j$ and $\phi = \phi_1^{b_1} \cdots \phi_l^{b_l}$. Thus, using the linearity of the intersection pairing, we have the assertion.

Step 2 (the case where $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are relatively nef) : Let $\overline{\mathcal{A}}$ be an ample arithmetic Cartier divisor of C^{∞} -type on $\mathcal{X}$. As

$$\lim_{n \rightarrow \infty} \widehat{\deg}((\overline{\mathcal{D}}_1 + (1/n)\overline{\mathcal{A}}) \cdots (\overline{\mathcal{D}}_d + (1/n)\overline{\mathcal{A}}) \cdot \widehat{(\phi)}) = \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}),$$

we may assume that $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are ample on X . Then, by [30, Theorem 4.6], there are sequences $\{f_{1,n}\}_{n=1}^{\infty}, \dots, \{f_{d,n}\}_{n=1}^{\infty}$ of F_{∞} -invariant continuous functions on $X(\mathbb{C})$ such that

$$\lim_{n \rightarrow \infty} \|f_{i,n}\|_{\sup} = 0$$

and $\overline{\mathcal{D}}_i + (0, f_{i,n})$ is relatively nef and of C^{∞} -type for $i = 1, \dots, d$ and $n \geq 1$. Therefore, by using [31, Lemma 1.2.1] together with Step 1, we have

$$\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}) = \lim_{n \rightarrow \infty} \widehat{\deg}((\overline{\mathcal{D}}_1 + (0, f_{1,n})) \cdots (\overline{\mathcal{D}}_d + (0, f_{d,n})) \cdot \widehat{(\phi)}) = 0.$$

Step 3 (general case) : Since $\overline{\mathcal{D}}_i$ is integrable, there are relatively nef arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{L}}_i$ and $\overline{\mathcal{M}}_i$ of C^0 -type on $\mathcal{X}$ such that $\overline{\mathcal{D}}_i = \overline{\mathcal{L}}_i - \overline{\mathcal{M}}_i$. Thus the assertion follows from Step 2. $\square$

Let us start the proof of (2). By Lemma 4.5.1, we can find a normal model $\mathcal{X}$ of X , integrable arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ of C^0 -type on $\mathcal{X}$, and a finite subset S of M_K such that

$$\overline{D}_i = \overline{\mathcal{D}}_i^a + \left(0, \sum_{p \in S} \phi_{i,p}[p] \right),$$

where $\phi_{i,p}$'s are integrable continuous functions X_p^{an} . Then, by (4.5.3),

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \widehat{(\phi)}) &= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}) \\ &+ \sum_{p \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/p) \widehat{\deg}_p \left(\prod_{i \in I} (0, \phi_{i,p}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(p)} \cdot (\phi)_{(p)} \right). \end{aligned}$$

Therefore, (2) follows from Claim 4.5.4.1 and (1) in Proposition 2.4.11.

(3) First let us see the following claim:

Claim 4.5.4.2. *Let $\mathcal{X}$ be a normal model of X and $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d, \overline{\mathcal{D}}_{d+1}$ be integrable arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on $\mathcal{X}$.*

(a) *If $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are nef and $\overline{\mathcal{D}}_{d+1}$ is effective, then*

$$\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \overline{\mathcal{D}}_{d+1}) \geq 0.$$

(b) *Let $\overline{E}$ be an effective and integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . If $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are nef, then*

$$\widehat{\deg}(\overline{\mathcal{D}}_1^a \cdots \overline{\mathcal{D}}_d^a \cdot \overline{E}) \geq 0.$$

Proof. (a) Let $\overline{\mathcal{A}}$ be an ample arithmetic Cartier divisor of C^∞ -type on $\mathcal{X}$. It is sufficient to show that

$$\widehat{\deg}((\overline{\mathcal{D}}_1 + \epsilon \overline{\mathcal{A}}) \cdots (\overline{\mathcal{D}}_d + \epsilon \overline{\mathcal{A}}) \cdot \overline{\mathcal{D}}_{d+1}) \geq 0$$

for $\epsilon > 0$. First we assume that $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are of C^∞ -type. Then, by [30, Proposition 6.2.2], $\overline{\mathcal{D}}_i + \epsilon \overline{\mathcal{A}}$ is ample for every i , that is, there are ample arithmetic Cartier divisors $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_r$ of C^∞ -type on $\mathcal{X}$ such that $\overline{\mathcal{D}}_i + \epsilon \overline{\mathcal{A}} = \sum_{j=1}^r a_{ij} \overline{\mathcal{A}}_j$ for some $a_{ij} \in \mathbb{R}_{\geq 0}$. On the other hand, by [30, Proposition 2.4.2], there are effective arithmetic Cartier divisors $\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_r$ of C^∞ -type on $\mathcal{X}$ such that $\overline{\mathcal{D}}_{d+1} = b_1 \overline{\mathcal{E}}_1 + \cdots + b_l \overline{\mathcal{E}}_l$ for some $b_1, \dots, b_l \in \mathbb{R}_{\geq 0}$, and hence the assertion follows from [25, Proposition 2.3].

In general, as before, by [30, Theorem 4.6], there are sequences

$$\{f_{1,n}\}_{n=1}^\infty, \dots, \{f_{d,n}\}_{n=1}^\infty$$

of F_∞ -invariant non-negative continuous functions on $X(\mathbb{C})$ such that

$$\lim_{n \rightarrow \infty} \|f_{i,n}\|_{\text{sup}} = 0$$

and $\overline{\mathcal{D}}_i + \epsilon \overline{\mathcal{A}} + (0, f_{i,n})$ is nef and of C^∞ -type for $i = 1, \dots, d$ and $n \geq 1$. Therefore, by [31, Lemma 1.2.1], we have

$$\begin{aligned} \widehat{\deg}((\overline{\mathcal{D}}_1 + \epsilon \overline{\mathcal{A}}) \cdots (\overline{\mathcal{D}}_d + \epsilon \overline{\mathcal{A}}) \cdot \overline{\mathcal{D}}_{d+1}) \\ = \lim_{n \rightarrow \infty} \widehat{\deg}((\overline{\mathcal{D}}_1 + \epsilon \overline{\mathcal{A}} + (0, f_{1,n})) \cdots (\overline{\mathcal{D}}_d + \epsilon \overline{\mathcal{A}} + (0, f_{d,n})) \cdot \overline{\mathcal{D}}_{d+1}) \geq 0, \end{aligned}$$

as required.

(b) By Theorem 4.1.3, there is a finite subset S of M_K with the following property: for any $n \in \mathbb{Z}_{>0}$, there are a normal model $\mathcal{X}_n$ of X together with a birational morphism $\mu_n : \mathcal{X}_n \rightarrow \mathcal{X}$, and an integrable arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{E}}_n$ of C^0 -type on $\mathcal{X}_n$ such that

$$\overline{E} \leq \overline{\mathcal{E}}_n^a \leq \overline{E} + \left(0, \sum_{\mathfrak{p} \in S} (1/n)[\mathfrak{p}]\right).$$

Then $\overline{\mathcal{E}}_n$ is effective by (2) in Proposition 4.1.2 and, if we set

$$\overline{\mathcal{E}}_n^a = \overline{E} + \left(0, \sum_{\mathfrak{p} \in S} \psi_{n,\mathfrak{p}}[\mathfrak{p}]\right),$$

then $\psi_{n,\mathfrak{p}}$ is a continuous function on $X_{\mathfrak{p}}^{\text{an}}$ with $0 \leq \psi_{n,\mathfrak{p}} \leq 1/n$ for every n and $\mathfrak{p} \in S$. Therefore, by (a),

$$\begin{aligned} \widehat{\deg}(\overline{\mathcal{D}}_1^a \cdots \overline{\mathcal{D}}_d^a \cdot \overline{E}) &= \widehat{\deg}(\mu_n^*(\overline{\mathcal{D}}_1) \cdots \mu_n^*(\overline{\mathcal{D}}_d) \cdot \overline{\mathcal{E}}_n) \\ &= \sum_{\mathfrak{p} \in S} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}}(\mathcal{D}_1 \cdots \mathcal{D}_d; \psi_{n,\mathfrak{p}}) \\ &\geq - \sum_{\mathfrak{p} \in S} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}}(\mathcal{D}_1 \cdots \mathcal{D}_d; \psi_{n,\mathfrak{p}}) \end{aligned}$$

On the other hand, by (3) in Proposition 2.4.11,

$$\left| \widehat{\deg}_{\mathfrak{p}}(\mathcal{D}_1 \cdots \mathcal{D}_d; \psi_{n,\mathfrak{p}}) \right| \leq \frac{1}{-2n \log v(\varpi)} \deg((\mathcal{D}_1 \cap X) \cdots (\mathcal{D}_d \cap X)).$$

Thus we have the assertion of (b). $\square$

Claim 4.5.4.3. *If $\overline{D}_1, \dots, \overline{D}_d$ are nef and $\overline{D}_{d+1}$ is effective, then*

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) \geq 0.$$

Proof. By (1) in Proposition 4.4.2, as before, we can find a finite subset T of M_K with the following property: for any $n \in \mathbb{Z}_{>0}$, there are a normal model $\mathcal{X}_n$ and relatively nef arithmetic $\mathbb{R}$ -Cartier divisors

$$\overline{\mathcal{D}}_{1,n}, \dots, \overline{\mathcal{D}}_{d,n}$$

of C^0 -type on $\mathcal{X}_n$ such that

$$\overline{D}_i \leq \overline{\mathcal{D}}_{i,n}^a \leq \overline{D}_i + \left(0, \sum_{\mathfrak{p} \in T} (1/n)[\mathfrak{p}]\right)$$

for $i = 1, \dots, d$. Note that $\overline{\mathcal{D}}_{i,n}$ is nef for every i and n by Lemma 4.2.3, and if we set

$$\overline{\mathcal{D}}_{i,n}^a = \overline{D}_i + \left(0, \sum_{\mathfrak{p} \in T} \varphi_{i,n,\mathfrak{p}}[\mathfrak{p}]\right),$$

then $\varphi_{i,n,\mathfrak{p}}$ is a continuous function on $X_{\mathfrak{p}}^{\text{an}}$ with $0 \leq \varphi_{i,n,\mathfrak{p}} \leq 1/n$ for every n and $\mathfrak{p} \in T$. Then, by (4.5.3),

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) &= \widehat{\deg}(\overline{\mathcal{D}}_{1,n}^a \cdots \overline{\mathcal{D}}_{d,n}^a \cdot \overline{D}_{d+1}) + \sum_{\mathfrak{p} \in T} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \\ &\quad \log \#(O_K/\mathfrak{p}) \widehat{\deg}_{\mathfrak{p}} \left(\prod_{i \in I} (0, -\varphi_{i,n,\mathfrak{p}}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(\mathfrak{p})} \cdot (D_{d+1}, (g_{d+1})_{\mathfrak{p}}) \right), \end{aligned}$$

where $\overline{D}_{d+1} = (D_{d+1}, \sum_{\mathfrak{p}} (g_{d+1})_{\mathfrak{p}}[\mathfrak{p}] + (g_{d+1})_{\infty}[\infty])$. By using (3) in Proposition 2.4.11, it is easy to see that

$$\lim_{n \rightarrow \infty} \widehat{\deg}_{\mathfrak{p}} \left(\prod_{i \in I} (0, -\varphi_{i,n,\mathfrak{p}}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(\mathfrak{p})} \cdot (D_{d+1}, (g_{d+1})_{\mathfrak{p}}) \right) = 0$$

for all $\mathfrak{p} \in T$ and $I \subseteq \{1, \dots, d\}$ with $I \neq \emptyset$ (cf. the proof of (b) in Claim 4.5.4.2). Therefore, the assertion follows from (b) in Claim 4.5.4.2. $\square$

Let $\overline{B}$ be a nef and big adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . For $\epsilon > 0$, as $\overline{D}_{d+1} + \epsilon \overline{B}$ is big, there are $n \in \mathbb{Z}_{>0}$ and $\psi \in \text{Rat}(X)^{\times}$ such that

$$\overline{D}_{d+1} + \epsilon \overline{B} + (1/n)(\widehat{\psi}) \geq 0,$$

so that, by using (2) and Claim 4.5.4.3,

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) + \epsilon \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{B}) \\ = \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot (\overline{D}_{d+1} + \epsilon \overline{B} + (1/n)(\widehat{\psi}))) \geq 0. \end{aligned}$$

Thus the assertion follows. $\square$

5. CONTINUITY OF THE VOLUME FUNCTION

The purpose of this section is to consider the continuity of the volume function and its applications. The continuity of the volume function is a very fundamental result in the theory of birational Arakelov geometry. It actually has a lot of applications by using perturbation methods. The generalized Hodge index theorem is one of the significant examples, which is a generalization of results due to Faltings-Gillet-Soulé-Zhang (cf. [10], [11] and [37]).

Throughout this section, let K be a number field and let X be a d -dimensional, projective, normal and geometrically integral variety over K .

5.1. Basic properties of the volume. In this subsection, we investigate several basic properties of the volume function. First of all, let us begin with the finiteness of $\widehat{\text{vol}}$, the limit theorem and the positive homogeneity of $\widehat{\text{vol}}$.

Theorem 5.1.1. (1) (Finiteness) $\widehat{\text{vol}}(\overline{D}) < \infty$.

(2) (Limit theorem) *The “lim sup” in the definition of $\widehat{\text{vol}}$ (cf. Definition 4.3.2) can be replaced by “lim”, that is,*

$$\widehat{\text{vol}}(\overline{D}) = \lim_{t \rightarrow \infty} \frac{\hat{h}^0(X, t\overline{D})}{t^{d+1}/(d+1)!},$$

where t is a positive real number.

(3) (Positive homogeneity) $\widehat{\text{vol}}(a\overline{D}) = a^{d+1}\widehat{\text{vol}}(\overline{D})$ for $a \in \mathbb{R}_{\geq 0}$.

Proof. In order to prove (1), (2) and (3), we may assume that X is smooth over K by (4) in Proposition 4.3.1 and (4.3.3).

(1) is obvious because we can find a normal model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on $\mathcal{X}$ such that $\overline{D} \leq \overline{\mathcal{D}}^a$.

(2) Let $(\mathcal{X}_U, \mathcal{D}_U)$ be a defining model of $\overline{D}$ over a non-empty open set U of $\text{Spec}(O_K)$. By Theorem 4.1.3, for any $\epsilon > 0$, there are a normal model $\mathcal{X}_\epsilon$ of X and an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}_\epsilon$ of C^0 -type on $\mathcal{X}_\epsilon$ such that $\mathcal{X}_\epsilon|_U = \mathcal{X}_U$, $\mathcal{D}_\epsilon|_U = \mathcal{D}_U$ and

$$\overline{\mathcal{D}}_\epsilon^a \leq \overline{D} \leq \overline{\mathcal{D}}_\epsilon^a + \left(0, \sum_{\mathfrak{p} \in S} 2\epsilon \log \#(O_K/\mathfrak{p})[\mathfrak{p}]\right).$$

where $S = M_K \setminus U$. Note that the fiber $F_{\mathfrak{p}}$ of $\mathcal{X}_\epsilon \rightarrow \text{Spec}(O_K)$ over $\mathfrak{p}$ yields a constant function $2 \log \#(O_K/\mathfrak{p})$ on $X_{\mathfrak{p}}^{\text{an}}$. Thus the above inequalities mean that

$$\overline{\mathcal{D}}_\epsilon^a \leq \overline{D} \leq \left(\overline{\mathcal{D}}_\epsilon + \epsilon \left(\sum_{\mathfrak{p} \in S} F_{\mathfrak{p}}, 0\right)\right)^a.$$

As $\widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon)$ and $\widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon + \epsilon(\sum_{\mathfrak{p}} F_{\mathfrak{p}}, 0))$ can be expressed by “lim” (cf. [27, Theorem 5.1] and [30, Theorem 5.2.2]), if we set

$$\Delta = \limsup_{t \rightarrow \infty} \frac{\hat{h}^0(X, t\overline{D})}{t^{d+1}/(d+1)!} - \liminf_{t \rightarrow \infty} \frac{\hat{h}^0(X, t\overline{D})}{t^{d+1}/(d+1)!},$$

then

$$0 \leq \Delta \leq \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon \left(\sum_{\mathfrak{p} \in S} F_{\mathfrak{p}}, 0\right)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon).$$

Let us choose a positive number N such that $N \in \mathfrak{p}$ for all $\mathfrak{p} \in S$. Then $\sum_{\mathfrak{p} \in S} F_{\mathfrak{p}} \leq (N)$. Thus, by using [27, Proposition 4.6] and [30, Theorem 5.2.2],

$$\begin{aligned} \Delta &\leq \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon \left(\sum_{\mathfrak{p}} F_{\mathfrak{p}}, 0\right)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &\leq \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon((N), 0)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \epsilon^{d+1} \widehat{\text{vol}}\left((1/\epsilon)\overline{\mathcal{D}}_\epsilon + ((N), 0)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \epsilon^{d+1} \widehat{\text{vol}}\left((1/\epsilon)\overline{\mathcal{D}}_\epsilon + ((N), 0) - \widehat{(N)}\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon((N), 0) - \epsilon\widehat{(N)}\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + (0, 2\epsilon \log N)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &\leq \epsilon \log(N)(d+1)[K : \mathbb{Q}] \text{vol}(X, D). \end{aligned}$$

Note that $\log(N)(d+1)[K : \mathbb{Q}] \text{vol}(X, D)$ does not depend on ϵ , so that $\Delta = 0$.

(3) If $a = 0$, then the assertion is obvious. Otherwise, by using (2),

$$\begin{aligned} \widehat{\text{vol}}(a\overline{D}) &= \lim_{t \rightarrow \infty} \frac{\hat{h}^0(X, ta\overline{D})}{t^{d+1}/(d+1)!} = a^{d+1} \lim_{t \rightarrow \infty} \frac{\hat{h}^0(X, ta\overline{D})}{(ta)^{d+1}/(d+1)!} \\ &= a^{d+1} \widehat{\text{vol}}(\overline{D}), \end{aligned}$$

as desired. □

Next let us consider the following estimate for the proof of the continuity of $\widehat{\text{vol}}$ and $\widehat{\text{vol}}_\chi$. Let $\widehat{\text{vol}}_{\mathfrak{p}}$ be either $\widehat{\text{vol}}$ or $\widehat{\text{vol}}_\chi$.

Proposition 5.1.2. *Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor on X . Let $\varphi_1 \in C^0(X_{\mathfrak{p}_1}^{\text{an}}), \dots, \varphi_r \in C^0(X_{\mathfrak{p}_r}^{\text{an}}), \varphi_\infty \in C_{F_\infty}^0(X(\mathbb{C}))$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_r \in M_K$. Then*

$$\left| \widehat{\text{vol}}_{\mathfrak{q}} \left(\overline{D} + \left(0, \sum_{i=1}^r \varphi_i[\mathfrak{p}_i] + \varphi_\infty[\infty] \right) \right) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \right| \leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2} \left(\sum_{i=1}^r \|\varphi_i\|_{\text{sup}} + \|\varphi_\infty\|_{\text{sup}} \right),$$

where $\widehat{\text{vol}}_{\mathfrak{q}}$ is either $\widehat{\text{vol}}$ or $\widehat{\text{vol}}_\chi$ (see Definition 4.3.2).

Let us begin with the following lemma:

Lemma 5.1.3. *Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X and $\mathfrak{p} \in M_K$. Then we have the following:*

- (1) $\widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, a[\mathfrak{p}])) \leq \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) + \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2[K_{\mathfrak{p}}:\mathbb{Q}_p]} a$ for $a \in \mathbb{R}_{\geq 0}$, where p is the prime number with $p\mathbb{Z} = \mathbb{Z} \cap \mathfrak{p}$.
- (2) $\widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, a[\infty])) \leq \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) + \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2} a$ for $a \in \mathbb{R}_{\geq 0}$.

Proof. (1) For each $n \in \mathbb{Z}_{>0}$, let a_n be the smallest integer that is greater than or equal to $\frac{na}{2\text{ord}_{\mathfrak{p}}(p)\log\#(O_K/\mathfrak{p})}$, that is,

$$a_n = \left\lceil \frac{na}{2\text{ord}_{\mathfrak{p}}(p)\log\#(O_K/\mathfrak{p})} \right\rceil := \min \left\{ x \in \mathbb{Z} \mid \frac{na}{2\text{ord}_{\mathfrak{p}}(p)\log\#(O_K/\mathfrak{p})} \leq x \right\}.$$

Then we have the following claim:

Claim 5.1.3.1. $p^{a_n} \hat{H}^0(X, n(\overline{D}^\tau + (0, a[\mathfrak{p}]))) \subseteq \hat{H}^0(X, n\overline{D}^\tau)$.

Proof. Let $\phi \in \hat{H}^0(X, n(\overline{D}^\tau + (0, a[\mathfrak{p}])))$. Let $\mathfrak{q} \in \text{Spec}(O_K)$. If $\mathfrak{q} \neq \mathfrak{p}$, then

$$\|p^{a_n} \phi\|_{ng_{\mathfrak{q}}} = v_{\mathfrak{q}}(p)^{a_n} \|\phi\|_{ng_{\mathfrak{q}}} \leq 1.$$

Otherwise (i.e. $\mathfrak{q} = \mathfrak{p}$),

$$\begin{aligned} \|p^{a_n} \phi\|_{ng_{\mathfrak{p}}} &= v_{\mathfrak{p}}(p)^{a_n} \|\phi\|_{ng_{\mathfrak{p}}} = \exp\left(-a_n \text{ord}_{\mathfrak{p}}(p) \log\#(O_K/\mathfrak{p}) + \frac{na}{2}\right) \|\phi\|_{n(g_{\mathfrak{p}}+a)} \\ &\leq \exp\left(\frac{-na}{2\text{ord}_{\mathfrak{p}}(p)\log\#(O_K/\mathfrak{p})} \text{ord}_{\mathfrak{p}}(p) \log\#(O_K/\mathfrak{p}) + \frac{na}{2}\right) \|\phi\|_{n(g_{\mathfrak{p}}+a)} \\ &= \|\phi\|_{n(g_{\mathfrak{p}}+a)} \leq 1, \end{aligned}$$

as required. □

We set

$$\begin{cases} Q_n = \text{Coker}(\hat{H}^0(X, n\overline{D}^\tau) \rightarrow \hat{H}^0(X, n(\overline{D}^\tau + (0, a[\mathfrak{p}])))), \\ r_n = \dim_{\mathbb{Q}} H^0(X, nD) = [K:\mathbb{Q}] \dim_K H^0(X, nD). \end{cases}$$

Then, by (1.1.3.4) and (1.1.3.5),

$$\begin{cases} \hat{h}^0(X, n(\overline{D} + (0, a[\mathfrak{p}]))) \leq \hat{h}^0(X, n\overline{D}) + \log\#(Q_n) + \log(6)r_n, \\ \hat{\chi}(X, n(\overline{D} + (0, a[\mathfrak{p}]))) = \hat{\chi}(X, n\overline{D}) + \log\#(Q_n). \end{cases}$$

By the above claim,

$$\begin{aligned} \log \#(Q_n) &\leq \log \#(\hat{H}^0(X, n(\overline{D}^\tau + (0, a[\mathfrak{p}]))) / p^{a_n} \hat{H}^0(X, n(\overline{D}^\tau + (0, a[\mathfrak{p}]))) \\ &= a_n r_n \log(p). \end{aligned}$$

Therefore, (1) follows because

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n r_n \log(p)}{n^{d+1}/(d+1)!} &= \frac{a(d+1)[K:\mathbb{Q}]\log(p)}{2 \operatorname{ord}_{\mathfrak{p}}(p) \log \#(O_K/\mathfrak{p})} \lim_{n \rightarrow \infty} \frac{\dim_K H^0(X, nD)}{n^d/d!} \\ &= \frac{a(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D)}{2 \operatorname{ord}_{\mathfrak{p}}(p)[O_K/\mathfrak{p}:\mathbb{Z}/p\mathbb{Z}]} = \frac{a(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D)}{2[K_{\mathfrak{p}}:\mathbb{Q}_p]}. \end{aligned}$$

(2) By (1.1.3.2) and (1.1.3.3),

$$\begin{cases} \hat{h}^0(X, n(\overline{D} + (0, a[\infty]))) \leq \hat{h}^0(X, n\overline{D}) + (na/2)r_n + \log(3)r_n, \\ \hat{\chi}(X, n(\overline{D} + (0, a[\infty]))) = \hat{\chi}(X, n\overline{D}) + (na/2)r_n. \end{cases}$$

Therefore, (2) follows. $\square$

Proof of Proposition 5.1.2. Let us start the proof of Proposition 5.1.2. First we check the following special cases:

Claim 5.1.4.1. *The following inequalities hold:*

$$\begin{aligned} (1) \quad & \left| \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_1[\mathfrak{p}_1])) - \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) \right| \leq \frac{(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D)}{2} \|\varphi_1\|_{\sup}. \\ (2) \quad & \left| \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_{\infty}[\infty])) - \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) \right| \leq \frac{(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D)}{2} \|\varphi_{\infty}\|_{\sup}. \end{aligned}$$

Proof. (1) By using (1) in Lemma 5.1.3,

$$\begin{aligned} \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_1[\mathfrak{p}_1])) - \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) &\leq \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} + (0, \|\varphi_1\|_{\sup}[\mathfrak{p}_1])) - \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) \\ &\leq \frac{(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D)}{2} \|\varphi_1\|_{\sup}. \end{aligned}$$

Applying (1) in Lemma 5.1.3 to the case where $\overline{D}$ is $\overline{D} - (0, a[\mathfrak{p}_1])$, we have

$$\widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) \leq \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} - (0, a[\mathfrak{p}_1])) + (a/2)(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D).$$

Therefore,

$$\begin{aligned} \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) - \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_1[\mathfrak{p}_1])) &\leq \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D}) - \widehat{\operatorname{vol}}_{\mathfrak{q}}(\overline{D} - (0, \|\varphi_1\|_{\sup}[\mathfrak{p}_1])) \\ &\leq \frac{(d+1)[K:\mathbb{Q}]\operatorname{vol}(X, D)}{2} \|\varphi_1\|_{\sup}. \end{aligned}$$

Thus we have (1).

(2) can be shown in the same way as (1) by using (2) in Lemma 5.1.3. $\square$

In general, we set

$$\overline{D}_j = \begin{cases} \overline{D} + (0, \varphi_{\infty}[\infty]) & \text{if } j = 0, \\ \overline{D} + (0, \sum_{i=1}^j \varphi_i[\mathfrak{p}_i] + \varphi_{\infty}[\infty]) & \text{if } j \geq 1. \end{cases}$$

Then, as

$$\begin{aligned} & \left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\overline{D} + \left(0, \sum_{i=1}^r \varphi_i[\mathfrak{p}_i] + \varphi_{\infty}[\infty] \right) \right) - \widehat{\text{vol}}_{\mathfrak{h}}(\overline{D}) \right| \\ & \leq \sum_{j=1}^r \left| \widehat{\text{vol}}_{\mathfrak{h}}(\overline{D}_j) - \widehat{\text{vol}}_{\mathfrak{h}}(\overline{D}_{j-1}) \right| + \left| \widehat{\text{vol}}_{\mathfrak{h}}(\overline{D}_0) - \widehat{\text{vol}}_{\mathfrak{h}}(\overline{D}) \right|, \end{aligned}$$

using the previous claim, we have the assertion. $\square$

As a consequence of the above estimate, we have the following proposition, so that we can deduce Fujita's approximation theorem for adelic arithmetic $\mathbb{R}$ -Cartier divisors (cf. Theorem 5.1.6).

Proposition 5.1.5. *Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then, for a positive number ϵ , there are a normal model of $\mathcal{X}$ over $\text{Spec}(O_K)$ and arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}'$ of C^0 -type on $\mathcal{X}$ such that*

$$\overline{\mathcal{D}}^a \leq \overline{D} \leq \overline{\mathcal{D}}'^a, \quad 0 \leq \widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}(\overline{\mathcal{D}}) \leq \epsilon \quad \text{and} \quad 0 \leq \widehat{\text{vol}}(\overline{\mathcal{D}}') - \widehat{\text{vol}}(\overline{D}) \leq \epsilon.$$

Proof. Let U be a non-empty open set of $\text{Spec}(O_K)$ such that $\overline{D}$ has defining models $\mathcal{D}_U$ over U . We set $S = M_K \setminus U$. We choose a positive number ϵ' such that

$$\epsilon'(d+1)[K:\mathbb{Q}]\widehat{\text{vol}}(X, D)\#(S) \leq 2\epsilon.$$

By Theorem 4.1.3, there are a normal model of $\mathcal{X}$ and arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}'$ of C^0 -type on $\mathcal{X}$ such that

$$\overline{D} - \left(0, \sum_{\mathfrak{p} \in S} \epsilon'[\mathfrak{p}] \right) \leq \overline{\mathcal{D}} \leq \overline{D} \leq \overline{\mathcal{D}}' \leq \overline{D} + \left(0, \sum_{\mathfrak{p} \in S} \epsilon'[\mathfrak{p}] \right).$$

Then

$$0 \leq \widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}(\overline{\mathcal{D}}) \leq \widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}\left(\overline{D} - \left(0, \sum_{\mathfrak{p} \in S} \epsilon'[\mathfrak{p}] \right)\right)$$

and

$$0 \leq \widehat{\text{vol}}(\overline{\mathcal{D}}') - \widehat{\text{vol}}(\overline{D}) \leq \widehat{\text{vol}}\left(\overline{D} + \left(0, \sum_{\mathfrak{p} \in S} \epsilon'[\mathfrak{p}] \right)\right) - \widehat{\text{vol}}(\overline{D}).$$

On the other hand, by Proposition 5.1.2,

$$\widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}\left(\overline{D} - \left(0, \sum_{\mathfrak{p} \in S} \epsilon'[\mathfrak{p}] \right)\right) \leq (\epsilon'/2)(d+1)[K:\mathbb{Q}]\widehat{\text{vol}}(X, D)\#(S)$$

and

$$\widehat{\text{vol}}\left(\overline{D} + \left(0, \sum_{\mathfrak{p} \in S} \epsilon'[\mathfrak{p}] \right)\right) - \widehat{\text{vol}}(\overline{D}) \leq (\epsilon'/2)(d+1)[K:\mathbb{Q}]\widehat{\text{vol}}(X, D)\#(S).$$

Thus the assertion follows. $\square$

The following theorem is the adelic version of arithmetic Fujita's approximation theorem. It has been already established by Boucksom and H. Chen [5]. Here we give another proof of it and generalize it to $\mathbb{R}$ -divisors.

Theorem 5.1.6 (Fujita's approximation theorem for adelic arithmetic divisors). *Let $\overline{D}$ be a big adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then, for any positive number ϵ , there are a birational morphism $\mu : Y \rightarrow X$ of normal, projective and geometrically integral varieties over K and a nef adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{P}$ of C^0 -type on Y such that $\overline{P} \leq \mu^*(\overline{D})$ and $\widehat{\text{vol}}(\overline{P}) \geq \widehat{\text{vol}}(\overline{D}) - \epsilon$ (cf. Theorem 0.4.3 in Introduction).*

Proof. Clearly we may assume that X is smooth over K . By Proposition 5.1.5, we can find a normal model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on $\mathcal{X}$ such that

$$\overline{\mathcal{D}} \leq \overline{D} \quad \text{and} \quad \widehat{\text{vol}}(\overline{\mathcal{D}}) \geq \widehat{\text{vol}}(\overline{D}) - \epsilon/2.$$

Moreover, by virtue of Fujita's approximation theorem for arithmetic $\mathbb{R}$ -Cartier divisors due to Chen-Yuan (cf. [8], [35] and [30, Theorem 5.2.2]), there are a birational morphism $\tilde{\mu} : \mathcal{Y} \rightarrow \mathcal{X}$ of generically smooth, normal and projective arithmetic varieties and a nef arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{P}}$ of C^0 -type on $\mathcal{Y}$ such that

$$\overline{\mathcal{P}} \leq \tilde{\mu}^*(\overline{\mathcal{D}}) \quad \text{and} \quad \widehat{\text{vol}}(\overline{\mathcal{P}}) \geq \widehat{\text{vol}}(\overline{\mathcal{D}}) - \epsilon/2.$$

Thus if we set $Y = \mathcal{Y} \times_{\text{Spec}(O_K)} \text{Spec}(K)$, $\mu = \tilde{\mu}|_Y$ and $\overline{P} = \overline{\mathcal{P}}^a$, then we have the assertion. $\square$

In addition, we have the following variant of Fujita's approximation theorem (cf. [28, Theorem 2.4]).

Theorem 5.1.7. *Let $\overline{D}$ be a big adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . For any positive number ϵ , there is $n(\epsilon) \in \mathbb{Z}_{>0}$ such that, for all $n \geq n(\epsilon)$,*

$$\liminf_{m \rightarrow \infty} \frac{\#(K_{m,n}(\overline{D}))}{m^{d+1}n^{d+1}} \geq \frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \epsilon,$$

where $K_{m,n}(\overline{D})$ is the convex lattice hull of

$$\{\phi_1 \cdots \phi_m \mid \phi_1, \dots, \phi_m \in \hat{H}^0(X, n\overline{D})\}$$

in $H^0(X, mn\overline{D})$ (for the definition of the convex lattice hull, see Conventions and terminology 0.5.9).

Proof. Let us begin with the following claim:

Claim 5.1.7.1. *There is $n_0 \in \mathbb{Z}_{>0}$ such that $\hat{H}^0(X, n\overline{D}) \neq \{0\}$ for all $n \geq n_0$.*

Proof. We choose an effective adelic $\mathbb{R}$ -Cartier divisor $\overline{A}$ of C^0 -type on X such that $\hat{H}^0(X, \overline{D} + \overline{A}) \neq \{0\}$. We also choose $a \in \mathbb{Z}_{>0}$ such that $\hat{H}^0(X, a\overline{D} - \overline{A}) \neq \{0\}$. Then, as $a\overline{D} = (a\overline{D} - \overline{A}) + \overline{A}$ and $(a+1)\overline{D} = (a\overline{D} - \overline{A}) + (\overline{D} + \overline{A})$, we have $\hat{H}^0(X, a\overline{D}) \neq \{0\}$ and $\hat{H}^0(X, (a+1)\overline{D}) \neq \{0\}$. Let n be an integer with $n \geq a^2 + a$. If we set $n = aq + r$ ($0 \leq r < a$), then $q > a$, so that $q = b + r$ for some $b \in \mathbb{Z}_{>0}$, and hence $n = ba + r(a+1)$. Therefore, $\hat{H}^0(X, n\overline{D}) \neq \{0\}$. $\square$

Step 1: We suppose that there exist a model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{Q}$ -Cartier divisor $\overline{\mathcal{D}}$ on $\mathcal{X}$ such that $\overline{\mathcal{D}}^a = \overline{D}$. Clearly we may assume that

$$\epsilon < \frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!}.$$

We choose $\lambda_0 \in \mathbb{R}$ such that $0 < \lambda_0 < 1$ and

$$\epsilon' := \frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \lambda_0^{-1} \left(\frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \epsilon \right) > 0.$$

Then

$$\lambda_0 \left(\frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \epsilon' \right) = \frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \epsilon$$

Let l be a positive integer such that $l\overline{\mathcal{D}}$ is an arithmetic Cartier divisor on $\mathcal{X}$. Then, by [28, Theorem 2.4], there is a positive integer t_0 such that, for all $t \geq t_0$,

$$\liminf_{m \rightarrow \infty} \frac{\#(K_{m,t}(l\overline{\mathcal{D}}))}{m^{d+1}t^{d+1}} \geq \frac{\widehat{\text{vol}}(l\overline{\mathcal{D}})}{(d+1)!} - l^{d+1}\epsilon',$$

where $K_{m,t}(l\overline{\mathcal{D}})$ is the convex lattice hull of

$$\{\phi_1 \cdots \phi_m \mid \phi_1, \dots, \phi_m \in \hat{H}^0(X, t\overline{\mathcal{D}})\}$$

in $H^0(\mathcal{X}, mtl\overline{\mathcal{D}})$. We choose $t_1 \in \mathbb{Z}_{>0}$ with $t_1 l \geq n_0$. By the above claim, there are non-zero rational functions $\psi_0, \dots, \psi_{l-1}$ such that $\psi_r \in \hat{H}^0(X, (t_1 l + r)\overline{\mathcal{D}})$ for $r = 0, \dots, l-1$. Let n be an integer with

$$n \geq \max \left\{ \frac{l(t_1 + 1)}{1 - \lambda_0^{1/(d+1)}}, l(t_0 + t_1 + 1) \right\}.$$

We set $n = l(t + t_1) + r$ ($0 \leq r < l$). Then, as $t \geq t_0$ and $K_{m,t}(l\overline{\mathcal{D}})\psi_r^m \subseteq K_{m,n}(\overline{D})$, we have

$$\liminf_{m \rightarrow \infty} \frac{\#(K_{m,n}(\overline{D}))}{m^{d+1}t^{d+1}} \geq \frac{\widehat{\text{vol}}(l\overline{\mathcal{D}})}{(d+1)!} - l^{d+1}\epsilon' = l^{d+1} \left(\frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} - \epsilon' \right).$$

Therefore,

$$\begin{aligned} \liminf_{m \rightarrow \infty} \frac{\#(K_{m,n}(\overline{D}))}{m^{d+1}n^{d+1}} &= \left(\frac{t}{n} \right)^{d+1} \liminf_{m \rightarrow \infty} \frac{\#(K_{m,n}(\overline{D}))}{m^{d+1}t^{d+1}} \geq \left(\frac{lt}{n} \right)^{d+1} \left(\frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} - \epsilon' \right) \\ &\geq \left(\frac{n - l(t_1 + 1)}{n} \right)^{d+1} \left(\frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} - \epsilon' \right) \\ &\geq \lambda_0 \left(\frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} - \epsilon' \right) = \frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \epsilon, \end{aligned}$$

as required.

Step 2: We suppose that there exist a model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ on $\mathcal{X}$ such that $\overline{D} = \overline{\mathcal{D}}^a$. By using [30, Lemma 5.2.3, Lemma 5.2.4], there are effective arithmetic $\mathbb{Q}$ -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r$ of C^0 -type on $\mathcal{X}$ and real numbers $a_1, \dots, a_r$ such that $\overline{\mathcal{D}} = a_1 \overline{\mathcal{D}}_1 + \dots + a_r \overline{\mathcal{D}}_r$. Thus, by virtue of the continuity of the volume function, there are rational numbers $b_1, \dots, b_r$ such that

$$\overline{\mathcal{D}}' := b_1 \overline{\mathcal{D}}_1 + \dots + b_r \overline{\mathcal{D}}_r \leq \overline{\mathcal{D}} \quad \text{and} \quad \frac{\widehat{\text{vol}}(\overline{\mathcal{D}}')}{(d+1)!} \geq \frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} - \epsilon/2.$$

By Step 1, there is a positive integer n_0 such that, for all $n \geq n_0$,

$$\liminf_{m \rightarrow \infty} \frac{\log \#(K_{m,n}(\overline{\mathcal{D}}'))}{m^{d+1}n^{d+1}} \geq \frac{\widehat{\text{vol}}(\overline{\mathcal{D}}')}{(d+1)!} - \epsilon/2,$$

where $K_{m,n}(\overline{\mathcal{D}}')$ is the convex lattice hull of $\{\varphi_1 \cdots \varphi_m \mid \varphi_1, \dots, \varphi_m \in \hat{H}^0(\mathcal{X}, n\overline{\mathcal{D}}')\}$. Note that $K_{m,n}(\overline{\mathcal{D}}') \subseteq K_{m,n}(\overline{D})$, so that

$$\liminf_{m \rightarrow \infty} \frac{\log \#(K_{m,n}(\overline{D}))}{m^{d+1} n^{d+1}} \geq \frac{\widehat{\text{vol}}(\overline{\mathcal{D}}')}{(d+1)!} - \epsilon/2 \geq \frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} - \epsilon.$$

Step 3: By Proposition 5.1.5, we can find a normal model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{R}$ -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on $\mathcal{X}$ such that

$$\overline{\mathcal{D}} \leq \overline{D} \quad \text{and} \quad \frac{\widehat{\text{vol}}(\overline{\mathcal{D}})}{(d+1)!} \geq \frac{\widehat{\text{vol}}(\overline{D})}{(d+1)!} - \epsilon/2.$$

Thus, in the same way as Step 2, the assertion follows from Step 2. $\square$

5.2. Proof of the continuity of the volume function. The purpose of this subsection is to prove the continuity of the volume functions $\widehat{\text{vol}}$ and $\widehat{\text{vol}}_\chi$ for adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type. Namely we have the following theorem:

Theorem 5.2.1 (Continuity of the volume functions for adelic arithmetic divisors). *Let $\widehat{\text{vol}}_{\mathfrak{h}}$ be either $\widehat{\text{vol}}$ or $\widehat{\text{vol}}_\chi$. The volume function $\widehat{\text{vol}}_{\mathfrak{h}} : \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \dots, \overline{D}_r, \overline{A}_1, \dots, \overline{A}_{r'}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be a finite subset of M_K . For a compact subset B in $\mathbb{R}^r$ and a positive number ϵ , there are positive numbers δ and δ' such that*

$$\left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{\mathfrak{p}_l} [\mathfrak{p}_l] + \varphi_\infty [\infty] \right) \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon$$

holds for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{\mathfrak{p}_1} \in C^0(X_{\mathfrak{p}_1}^{\text{an}}), \dots, \varphi_{\mathfrak{p}_s} \in C^0(X_{\mathfrak{p}_s}^{\text{an}})$ and $\varphi_\infty \in C_{F_\infty}^0(X(\mathbb{C}))$ with $(a_1, \dots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{\mathfrak{p}_l}\|_{\text{sup}} + \|\varphi_\infty\|_{\text{sup}} \leq \delta'$ (cf. Theorem 0.4.1 in Introduction).

Proof. Clearly we may assume that X is smooth over K by (4.3.3). Let us choose a non-empty open set U of $\text{Spec}(O_K)$ such that $\overline{D}_i$ ($i = 1, \dots, r$) and $\overline{A}_j$ ($j = 1, \dots, r'$) have defining models $\mathcal{D}_{i,U}$ and $\mathcal{A}_{j,U}$ over U , respectively. We set $T = M_K \setminus U$ and

$$C = \max \left\{ \text{vol} \left(X, \sum_{i=1}^r a_i \mathcal{D}_i + \sum_{j=1}^{r'} \delta_j \mathcal{A}_j \right) \left(\sum_{i=1}^r |a_i| + 1 \right) \left| \begin{array}{l} (a_1, \dots, a_r) \in B \\ \sum_{j=1}^{r'} |\delta_j| \leq 1 \end{array} \right. \right\}.$$

We choose a positive number ϵ' such that

$$\epsilon'(d+1)[K:\mathbb{Q}]\#(T)C \leq \epsilon/3.$$

By Theorem 4.1.3, there are a normal model of $\mathcal{X}$ and arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r, \overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_{r'}$ of C^0 -type on $\mathcal{X}$ such that

$$\overline{\mathcal{D}}_i^a \leq \overline{D}_i \leq \overline{\mathcal{D}}_i^a + \left(0, \sum_{\mathfrak{p} \in T} \epsilon' [\mathfrak{p}] \right) \quad \text{and} \quad \overline{\mathcal{A}}_j^a \leq \overline{A}_j \leq \overline{\mathcal{A}}_j^a + \left(0, \sum_{\mathfrak{p} \in T} \epsilon' [\mathfrak{p}] \right)$$

for all $i = 1, \dots, r$ and $j = 1, \dots, r'$. Then,

$$\begin{aligned} & \sum_{i=1}^r a_i \overline{\mathcal{D}}_i^a + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j^a - \left(0, \sum_{\mathfrak{p} \in T} \epsilon' \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right) [\mathfrak{p}] \right) \\ & \leq \sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \\ & \leq \sum_{i=1}^r a_i \overline{\mathcal{D}}_i^a + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j^a + \left(0, \sum_{\mathfrak{p} \in T} \epsilon' \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right) [\mathfrak{p}] \right). \end{aligned}$$

Therefore, by using Proposition 5.1.2,

$$\begin{aligned} (5.2.1.1) \quad & \left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) \right| \\ & \leq \frac{\epsilon'(d+1)[K:\mathbb{Q}]\#(T)}{2} \text{vol}_{\mathfrak{h}} \left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j \right) \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right) \end{aligned}$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$. Moreover, by [30, Theorem 5.2.2] and [18, Corollary 3.4.4] together with the Heine-Cantor theorem, there is a positive number δ such that $\delta \leq 1$ and

$$\left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \leq \epsilon/6$$

for all $(a_1, \dots, a_r) \in B$ and $\delta_1, \dots, \delta_{r'} \in \mathbb{R}$ with

$$|\delta_1| + \dots + |\delta_{r'}| \leq \delta.$$

Therefore, if we set

$$\Delta = \left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right|,$$

then

$$\begin{aligned} \Delta & \leq \left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \\ & \quad + \left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \\ & \quad + \left| \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) - \widehat{\text{vol}}_{\mathfrak{h}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) \right|. \end{aligned}$$

Thus, by using (5.2.1.1), for $(a_1, \dots, a_r) \in B$ and $\delta_1, \dots, \delta_{r'} \in \mathbb{R}$ with

$$|\delta_1| + \dots + |\delta_{r'}| \leq \delta,$$

we have

$$(5.2.1.2) \quad \Delta \leq \epsilon/6 + \epsilon/6 + \epsilon/6 = \epsilon/2.$$

On the other hand, by Proposition 5.1.2 again,

$$\begin{aligned} & \left| \widehat{\text{vol}}_{\mathfrak{q}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{\mathfrak{p}_l}[\mathfrak{p}_l] + \varphi_{\infty}[\infty] \right) \right) \right. \\ & \quad \left. - \widehat{\text{vol}}_{\mathfrak{q}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) \right| \\ & \leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}\left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j\right)}{2} \left(\sum_{l=1}^s \|\varphi_{\mathfrak{p}_l}\|_{\text{sup}} + \|\varphi_{\infty}\|_{\text{sup}} \right). \end{aligned}$$

Here we set

$$C' = \max \left\{ \text{vol} \left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j \right) \mid (a_1, \dots, a_r) \in B, \sum_{j=1}^{r'} |\delta_j| \leq \delta \right\}$$

and choose a positive number δ' such that

$$(d+1)[K:\mathbb{Q}]C'\delta' \leq \epsilon.$$

Then

$$\begin{aligned} & \left| \widehat{\text{vol}}_{\mathfrak{q}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{\mathfrak{p}_l}[\mathfrak{p}_l] + \varphi_{\infty}[\infty] \right) \right) \right. \\ & \quad \left. - \widehat{\text{vol}}_{\mathfrak{q}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) \right| \leq \epsilon/2 \end{aligned}$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{\mathfrak{p}_1} \in C^0(X_{\mathfrak{p}_1}^{\text{an}}), \dots, \varphi_{\mathfrak{p}_s} \in C^0(X_{\mathfrak{p}_s}^{\text{an}})$ and $\varphi_{\infty} \in C_{F_{\infty}}^0(X(\mathbb{C}))$ with $(a_1, \dots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{\mathfrak{p}_l}\|_{\text{sup}} + \|\varphi_{\infty}\|_{\text{sup}} \leq \delta'$. Thus, by the above estimate together with (5.2.1.2), we have the assertion. $\square$

5.3. Applications. Here we would like to give two applications of the continuity of the volume function, that is, the log-concavity of $\widehat{\text{vol}}$ and the generalized Hodge index theorem for adelic arithmetic divisors.

Theorem 5.3.1. *Let $\overline{D}_1$ and $\overline{D}_2$ be pseudo-effective adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then*

$$\widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2)^{1/(d+1)} \geq \widehat{\text{vol}}(\overline{D}_1)^{1/(d+1)} + \widehat{\text{vol}}(\overline{D}_2)^{1/(d+1)}.$$

Proof. Let U be a non-empty open set of $\text{Spec}(O_K)$ such that $\overline{D}_1$ and $\overline{D}_2$ have defining models $\mathcal{D}_{1,U}$ and $\mathcal{D}_{2,U}$ over U . We set $T = M_K \setminus U$. For a positive number ϵ , by Theorem 4.1.3, there are a normal model $\mathcal{X}$ of X and arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ of C^0 -type on $\mathcal{X}$ such that

$$\overline{D}_1 \leq \overline{\mathcal{D}}_1 \leq \overline{D}_1 + \left(0, \sum_{\mathfrak{p} \in T} \epsilon[\mathfrak{p}], 0 \right) \quad \text{and} \quad \overline{D}_2 \leq \overline{\mathcal{D}}_2 \leq \overline{D}_2 + \left(0, \sum_{\mathfrak{p} \in T} \epsilon[\mathfrak{p}], 0 \right).$$

As $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ are pseudo-effective, by [35, Theorem B] or [30, Theorem 5.2.2], we have

$$\widehat{\text{vol}}(\overline{\mathcal{D}}_1 + \overline{\mathcal{D}}_2)^{1/(d+1)} \geq \widehat{\text{vol}}(\overline{\mathcal{D}}_1)^{1/(d+1)} + \widehat{\text{vol}}(\overline{\mathcal{D}}_2)^{1/(d+1)},$$

and hence

$$\widehat{\text{vol}}\left(\overline{D}_1 + \overline{D}_2 + 2\epsilon\left(0, \sum_{\mathfrak{p} \in T} [\mathfrak{p}], 0\right)\right)^{1/(d+1)} \geq \widehat{\text{vol}}(\overline{D}_1)^{1/(d+1)} + \widehat{\text{vol}}(\overline{D}_2)^{1/(d+1)}.$$

Thus the assertion follows from the continuity of $\widehat{\text{vol}}$ on $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$ (cf. Theorem 5.2.1). $\square$

Theorem 5.3.2 (Generalized Hodge index theorem for adelic arithmetic divisors). *Let $\overline{D} = (D, g)$ be a relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then $\widehat{\deg}(\overline{D}^{d+1}) = \widehat{\text{vol}}_{\chi}(\overline{D})$. In particular, $\widehat{\deg}(\overline{D}^{d+1}) \leq \widehat{\text{vol}}(\overline{D})$. Moreover, if $\overline{D}$ is nef, then $\widehat{\deg}(\overline{D}^{d+1}) = \widehat{\text{vol}}(\overline{D})$ (cf. Theorem 0.4.2 in Introduction).*

Proof. Clearly we may assume that X is smooth over K by (4.3.3) and (1) in Proposition 4.5.4. Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(O_K)$ and $\overline{\mathcal{D}} = (\mathcal{D}, g_{\infty})$ an arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{X}$. First let us see the following claim:

Claim 5.3.2.1. *If $\overline{\mathcal{D}}$ is relatively nef, then $\widehat{\deg}(\overline{\mathcal{D}}^{d+1}) = \widehat{\text{vol}}_{\chi}(\overline{\mathcal{D}})$.*

Proof. We divide the proof into four steps:

Step 1 (the case where $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{Q}$ -Cartier divisor of C^{∞} -type, $\mathcal{D}$ is ample on $\mathcal{X}$ and $c_1(\overline{\mathcal{D}})$ is a positive form) : This is a classic case. For example, it follows from the arithmetic Riemann-Roch theorem due to Gillet-Soulé (cf. [11]).

Step 2 (the case where $\overline{\mathcal{D}}$ is of C^{∞} -type, $\mathcal{D}$ is relatively nef, $c_1(\overline{\mathcal{D}})$ is a semi-positive form) : As any arithmetic Cartier divisor of C^{∞} -type can be written as a difference of ample arithmetic Cartier divisors of C^{∞} -type, we can find ample arithmetic Cartier divisors $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_l$ of C^{∞} -type and real numbers $a_1, \dots, a_l$ such that

$$\overline{\mathcal{D}} = a_1 \overline{\mathcal{A}}_1 + \dots + a_l \overline{\mathcal{A}}_l.$$

Then, for any rational numbers $b_1, \dots, b_l$ with $a_i < b_i$ for all i , $b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l$ is ample and $c_1(b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l)$ is positive because

$$b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l = \overline{\mathcal{D}} + (b_1 - a_1) \overline{\mathcal{A}}_1 + \dots + (b_l - a_l) \overline{\mathcal{A}}_l.$$

Thus, by Step 1,

$$\widehat{\deg}((b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l)^{d+1}) = \widehat{\text{vol}}_{\chi}(b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l).$$

Therefore, the assertion follows from Theorem 5.2.1.

Step 3 (the case where $\mathcal{D} \cap X$ is ample on X) : Let h be an F_{∞} -invariant $\mathcal{D}$ -Green function of C^{∞} -type on $X(\mathbb{C})$ such that $c_1(\mathcal{D}, h)$ is a positive form. Then there is a continuous function ϕ on $X(\mathbb{C})$ such that $g_{\infty} = h + \phi$, and hence $c_1(\mathcal{D}, h) + dd^c([\phi]) \geq 0$. Thus, by [30, Lemma 4.2], there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ of F_{∞} -invariant C^{∞} -functions on $X(\mathbb{C})$ with the following properties:

- (a) $\lim_{n \rightarrow \infty} \|\phi_n - \phi\|_{\text{sup}} = 0$.
- (b) If we set $\overline{\mathcal{E}}_n = (\mathcal{D}, h + \phi_n)$, then $c_1(\overline{\mathcal{E}}_n)$ is a semipositive form.

Then, by Step 2, $\widehat{\deg}(\overline{\mathcal{E}}_n^{d+1}) = \widehat{\text{vol}}_{\chi}(\overline{\mathcal{E}}_n)$ for all n . As $\overline{\mathcal{E}}_n = \overline{\mathcal{D}} + (0, \phi_n - \phi)$, by Theorem 5.2.1,

$$\lim_{n \rightarrow \infty} \widehat{\text{vol}}_{\chi}(\overline{\mathcal{E}}_n) = \widehat{\text{vol}}_{\chi}(\overline{\mathcal{D}}).$$

Moreover, by using [31, Lemma 1.2.1],

$$\lim_{n \rightarrow \infty} \widehat{\deg}(\overline{\mathcal{E}}_n^{d+1}) = \widehat{\deg}(\overline{\mathcal{D}}^{d+1}),$$

as required.

Step 4 (general case) : Finally we prove the assertion of the claim. Let $\overline{\mathcal{A}}$ be an ample arithmetic Cartier divisor of C^∞ -type on $\mathcal{X}$. Then, since $\mathcal{D} + \epsilon \overline{\mathcal{A}}$ is ample on X for any positive number ϵ , we have $\widehat{\deg}((\overline{\mathcal{D}} + \epsilon \overline{\mathcal{A}})^{d+1}) = \widehat{\text{vol}}_\chi(\overline{\mathcal{D}} + \epsilon \overline{\mathcal{A}})$ by Step 3. Thus, the assertion follows from Theorem 5.2.1. $\square$

We assume that $\overline{D}$ is relatively nef. Let us choose a non-empty open set U of $\text{Spec}(O_K)$, a normal model $\mathcal{X}_U$ over U and a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{D}_U$ on $\mathcal{X}_U$ such that $\mathcal{D}_U \cap X = D$ and g_p is the Green function arising from $(\mathcal{X}_U, \mathcal{D}_U)$ for all $p \in U \cap M_K$. Moreover, by Proposition 4.4.2, there is a sequence $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ with the following properties:

- (1) $\mathcal{X}_n$ is a normal model of X over $\text{Spec}(O_K)$ such that $\mathcal{X}_n|_U = \mathcal{X}_U$.
- (2) $\mathcal{D}_n$ is relatively nef $\mathbb{R}$ -Cartier divisor on $\mathcal{X}_n$ and $\mathcal{D}_n|_U = \mathcal{D}_U$.
- (3) $\overline{D} \leq (\mathcal{D}_n, g_\infty)^a$.
- (4) If we set $\phi_{n,p} = g_{((\mathcal{X}_n)_{(p)}, (\mathcal{D}_n)_{(p)})} - g_p$ for $p \in M_K \setminus U$, then

$$\lim_{n \rightarrow \infty} \|\phi_{n,p}\|_{\text{sup}} = 0.$$

As $(\mathcal{D}_n, g_\infty)^a = \overline{D} + (0, \sum_{p \in M_K \setminus U} \phi_{n,p}[\mathfrak{p}])$, by Theorem 5.2.1,

$$(5.3.2.2) \quad \lim_{n \rightarrow \infty} \widehat{\text{vol}}((\mathcal{D}_n, g_\infty)) = \widehat{\text{vol}}(\overline{D}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \widehat{\text{vol}}_\chi((\mathcal{D}_n, g_\infty)) = \widehat{\text{vol}}_\chi(\overline{D}).$$

Here let us see

$$(5.3.2.3) \quad \lim_{n \rightarrow \infty} \widehat{\deg}((\mathcal{D}_n, g_\infty)^{d+1}) = \widehat{\deg}(\overline{D}^{d+1}).$$

Indeed, we set

$$\psi_p = g_p - g_{((\mathcal{X}_1)_{(p)}, (\mathcal{D}_1)_{(p)})} \quad \text{and} \quad \psi_{n,p} = g_{((\mathcal{X}_n)_{(p)}, (\mathcal{D}_n)_{(p)})} - g_{((\mathcal{X}_1)_{(p)}, (\mathcal{D}_1)_{(p)})}$$

for $p \in M_K \setminus U$. Note that $\phi_{n,p} = \psi_p - \psi_{n,p}$. Then, by using Lemma 1.1.4, we can see

$$\begin{aligned} \widehat{\deg}((\mathcal{D}_n, g_\infty)^{d+1}) &= \widehat{\deg}((\mathcal{D}_1, g_\infty)^{d+1}) \\ &\quad + \sum_{i=1}^{d+1} \sum_{p \in M_K \setminus U} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_p((\mathcal{D}_n)_{(p)}^{i-1} \cdot (\mathcal{D}_1)_{(p)}^{d+1-i}; \psi_{n,p}). \end{aligned}$$

Thus, by virtue of Proposition-Definition 2.4.3,

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{\deg}((\mathcal{D}_n, g_\infty)^{d+1}) &= \widehat{\deg}((\mathcal{D}_1, g_\infty)^{d+1}) \\ &\quad + \sum_{i=1}^{d+1} \sum_{p \in M_K \setminus U} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_p((D, g_p)^{i-1} \cdot (\mathcal{D}_1)_{(p)}^{d+1-i}; \psi_p) = \widehat{\deg}(\overline{D}^{d+1}), \end{aligned}$$

as desired.

By (5.3.2.2) and (5.3.2.3) together with the above claim, we have the first assertion. If $\overline{D}$ is nef, then $(\mathcal{D}_n, g_\infty)$ is also nef by property (3) and Lemma 4.2.3, and hence the second assertion follows from (5.3.2.2) and (5.3.2.3) by using [30, Proposition 6.4.2] and [31, Proposition 2.1.1]. $\square$

Corollary 5.3.3. *Let $\overline{D}_1$ and $\overline{D}_2$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then, for $s, t \in \mathbb{R}_{\geq 0}$, we have the following:*

- (1) *If $\overline{D}_1$ and $\overline{D}_2$ are big and $s + t > 0$, then $s\overline{D}_1 + t\overline{D}_2$ is also big.*
- (2) *If $\overline{D}_1$ and $\overline{D}_2$ are pseudo-effective, then $s\overline{D}_1 + t\overline{D}_2$ is also pseudo-effective.*

Proof. (1) By Theorem 5.3.1 together with Theorem 5.1.1,

$$\widehat{\text{vol}}(s\overline{D}_1 + t\overline{D}_2)^{1/(d+1)} \geq s\widehat{\text{vol}}(\overline{D}_1)^{1/(d+1)} + t\widehat{\text{vol}}(\overline{D}_2)^{1/(d+1)} > 0,$$

as required.

(2) Clearly we may assume that $s + t > 0$. Let $\overline{A}$ be a big adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then, as

$$s\overline{D}_1 + t\overline{D}_2 + \overline{A} = s\left(\overline{D}_1 + \frac{1}{s+t}\overline{A}\right) + t\left(\overline{D}_2 + \frac{1}{s+t}\overline{A}\right),$$

(1) implies (2). □

6. ZARISKI DECOMPOSITIONS OF ADELIC ARITHMETIC DIVISORS ON ARITHMETIC SURFACES

Let $\mathcal{X}$ be a regular projective arithmetic surface and let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{X}$. The set of all nef arithmetic $\mathbb{R}$ -Cartier divisors $\overline{\mathcal{L}}$ of C^0 -type on $\mathcal{X}$ with $\overline{\mathcal{L}} \leq \overline{\mathcal{D}}$ is denoted by $\Upsilon(\overline{\mathcal{D}})$. In [30, Theorem 9.2.1], it is shown that if $\Upsilon(\overline{\mathcal{D}}) \neq \emptyset$, then $\Upsilon(\overline{\mathcal{D}})$ has the greatest element $\overline{\mathcal{P}}$, that is, $\overline{\mathcal{P}} \in \Upsilon(\overline{\mathcal{D}})$ and $\overline{\mathcal{L}} \leq \overline{\mathcal{P}}$ for all $\overline{\mathcal{L}} \in \Upsilon(\overline{\mathcal{D}})$. If we set $\overline{\mathcal{N}} := \overline{\mathcal{D}} - \overline{\mathcal{P}}$, then $\overline{\mathcal{D}} = \overline{\mathcal{P}} + \overline{\mathcal{N}}$ yields the Zariski decomposition of $\overline{\mathcal{D}}$. For example, we can see that the natural map

$$\hat{H}^0(\mathcal{X}, n\overline{\mathcal{P}}) \rightarrow \hat{H}^0(\mathcal{X}, n\overline{\mathcal{D}})$$

is bijective for every $n \in \mathbb{Z}_{>0}$. In particular, $\widehat{\text{vol}}(\overline{\mathcal{P}}) = \widehat{\text{vol}}(\overline{\mathcal{D}})$, so that it gives rise to a refinement of Fujita's approximation theorem for arithmetic divisors. In this section, we consider a generalization of the above result to an adelic arithmetic $\mathbb{R}$ -Cartier divisor.

6.1. Local Zariski decompositions of adelic divisors on algebraic curves. Let k be a field and v a non-trivial discrete absolute value of k . We assume that k° is excellent. Let ϖ be a uniformizing parameter of k° . Let X be a projective, smooth and geometrically integral curve over k . Let k_v be the completion of k with respect to v and $X_v := X \times_{\text{Spec}(k)} \text{Spec}(k_v)$. The purpose of this subsection is to prove the following theorem:

Theorem 6.1.1 (Local Zariski decomposition). *Let $\overline{D} = (D, g)$ be an adelic $\mathbb{R}$ -Cartier divisor on X and let Q be an $\mathbb{R}$ -Cartier divisor on X with $Q \leq D$. Here we set*

$$\Sigma(\overline{D}; Q) := \left\{ \overline{L} \left| \begin{array}{l} \overline{L} \text{ is a relatively nef adelic } \mathbb{R}\text{-Cartier divisor on } X \\ \text{such that } L \leq Q \text{ and } \overline{L} \leq \overline{D} \end{array} \right. \right\}.$$

We assume that $\deg(Q) \geq 0$. Then there exists a Q -Green function q of $(C^0 \cap \text{PSH})$ -type on X_v^{an} such that $\overline{Q} := (Q, q)$ gives rise to the greatest element of $\Sigma(\overline{D}; Q)$, that is, $\overline{Q} \in \Sigma(\overline{D}; Q)$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Sigma(\overline{D}; Q)$. Moreover, we have the following:

- (1) *If $\overline{D}$ is given by an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on a regular model $\mathcal{X}$ of X over $\text{Spec}(k^\circ)$, then $\overline{Q}$ is given by a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}$ on $\mathcal{X}$.*
- (2) *If $Q = D$, then $\widehat{\deg}_v(\overline{Q}; g - q) = 0$ (cf. Proposition-Definition 2.4.3).*

Before starting the proof of Theorem 6.1.1, we need to prepare two lemmas.

Lemma 6.1.2. *Let $\mathcal{X}$ be a regular model of X over $\mathrm{Spec}(k^\circ)$. Then we have the following:*

- (1) *Let $\pi : \mathcal{X}' \rightarrow \mathcal{X}$ be a birational morphism of regular models of X over $\mathrm{Spec}(k^\circ)$, and let $\mathcal{L}'$ be a relatively nef $\mathbb{R}$ -Cartier divisor on $\mathcal{X}'$. Then $\pi_*(\mathcal{L}')$ is relatively nef and $\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}'$ is effective.*
- (2) *If $\mathcal{L}_1, \dots, \mathcal{L}_l$ are relatively nef $\mathbb{R}$ -Cartier divisors on $\mathcal{X}$, then*

$$\max\{\mathcal{L}_1, \dots, \mathcal{L}_l\}$$

is also relatively nef (for the definition of $\max\{\mathcal{L}_1, \dots, \mathcal{L}_l\}$, see Conventions and terminology 0.5.8).

- (3) *Let $\mathcal{D}$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$ and $g = g_{(\mathcal{X}, \mathcal{D})}$. Then g is of $(C^0 \cap \mathrm{PSH})$ -type if and only if $\mathcal{D}$ is relatively nef.*

Proof. (1) Let C be an irreducible component of the central fiber $\mathcal{X}_\circ$ of $\mathcal{X} \rightarrow \mathrm{Spec}(k^\circ)$. Then

$$0 \leq (\mathcal{L}' \cdot \pi^*(C)) = (\pi_*(\mathcal{L}') \cdot C).$$

Thus $\pi_*(\mathcal{L}')$ is relatively nef.

Let us consider the second assertion. By [22, Theorem 9.2.2], π can be obtained by a succession of blowing-ups at closed points. We prove it by induction on the number of blowing-ups. First we consider the case where π is a blowing-up at a closed point. Let C be the exceptional curve of π . Then

$$\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}' = aC$$

for some $a \in \mathbb{R}$. As

$$((\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}') \cdot C) = -(\mathcal{L}' \cdot C) \leq 0 \quad \text{and} \quad (C \cdot C) < 0,$$

we have $a \geq 0$, as required. In general, we decompose π into two birational morphisms $\pi_1 : \mathcal{X}' \rightarrow \mathcal{X}_1$ and $\pi_2 : \mathcal{X}_1 \rightarrow \mathcal{X}$ of regular models of X , that is, $\pi = \pi_2 \circ \pi_1$. Note that $(\pi_1)_*(\mathcal{L}')$ is relatively nef by the previous observation. Thus, by the induction hypothesis,

$$\pi_1^*(\pi_1)_*(\mathcal{L}') - \mathcal{L}' \quad \text{and} \quad \pi_2^*(\pi_2)_*((\pi_1)_*(\mathcal{L}')) - (\pi_1)_*(\mathcal{L}')$$

are effective, so that

$$\pi^*(\pi_*(\mathcal{L}')) - \pi_1^*(\pi_1)_*(\mathcal{L}') = \pi_1^*(\pi_2^*(\pi_2)_*((\pi_1)_*(\mathcal{L}')) - (\pi_1)_*(\mathcal{L}'))$$

is also effective. Therefore, as

$$\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}' = (\pi^*(\pi_*(\mathcal{L}')) - \pi_1^*(\pi_1)_*(\mathcal{L}')) + (\pi_1^*(\pi_1)_*(\mathcal{L}') - \mathcal{L}'),$$

we have the assertion.

(2) We set $\mathcal{L}'_i := \max\{\mathcal{L}_1, \dots, \mathcal{L}_l\} - \mathcal{L}_i$ for each i . Let C be an irreducible component of $\mathcal{X}_\circ$. Then there is i such that $C \not\subseteq \mathrm{Supp}_{\mathbb{R}}(\mathcal{L}'_i)$. As $\mathcal{L}'_i$ is effective, we have $\deg(\mathcal{L}'_i|_C) \geq 0$, so that

$$\deg(\max\{\mathcal{L}_1, \dots, \mathcal{L}_l\}|_C) = \deg(\mathcal{L}_i|_C) + \deg(\mathcal{L}'_i|_C) \geq 0.$$

(3) This is a special case of Proposition 2.1.7. □

Lemma 6.1.3. *Let $\mathcal{X}$ be a regular model of X and $\mathcal{D}$ an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$. Let Q be an $\mathbb{R}$ -Cartier divisor on X with $Q \leq D := \mathcal{D} \cap X$. Here we set*

$$\Sigma_{\mathcal{X}}(\mathcal{D}; Q) := \left\{ \mathcal{L} \mid \begin{array}{l} \mathcal{L} \text{ is a relatively nef } \mathbb{R}\text{-Cartier divisor on } \mathcal{X} \\ \text{such that } \mathcal{L} \cap X \leq Q \text{ and } \mathcal{L} \leq \mathcal{D} \end{array} \right\}.$$

- (1) *We assume that $\deg(Q) \geq 0$. Then there is a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}$ on $\mathcal{X}$ such that $\mathcal{Q} \cap X = Q$ and $\mathcal{Q}$ gives rise to the greatest element of $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, that is, $\mathcal{Q} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ and $\mathcal{L} \leq \mathcal{Q}$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$. Moreover, if $Q = D$, then $(\mathcal{Q} \cdot \mathcal{D} - \mathcal{Q}) = 0$, that is, $\widehat{\deg}_v(\mathcal{Q}^a; g_{(\mathcal{X}, \mathcal{D} - \mathcal{Q})}) = 0$.*
- (2) *Let $\pi : \mathcal{X}' \rightarrow \mathcal{X}$ be a birational morphism of regular models of X . If $\mathcal{Q}$ is the greatest element of $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, then $\pi^*(\mathcal{Q})$ is also the greatest element of*

$$\Sigma_{\mathcal{X}'}(\pi^*(\mathcal{D}); Q).$$

Proof. (1) Let us begin with following claim:

Claim 6.1.3.1. (i) *There is a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{P}_0$ on $\mathcal{X}$ with $\mathcal{P}_0 \cap X = Q$.*
(ii) *There is $\mathcal{P} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ with $\mathcal{P} \cap X = Q$.*

Proof. (i) First we assume that $\deg(Q) = 0$. Let $\mathcal{P}'$ be the closure of Q in $\mathcal{X}$. Let $C_1, \dots, C_r$ be irreducible components of $\mathcal{X}_\circ$. As $(\mathcal{P}' \cdot \mathcal{X}_\circ) = 0$, by Zariski's lemma, we can find $a_1, \dots, a_r \in \mathbb{R}$ such that

$$\left(\sum_{i=1}^r a_i C_i \cdot C_j \right) = (\mathcal{P}' \cdot C_j)$$

for all $j = 1, \dots, r$. Thus, if we set $\mathcal{P}_0 = \mathcal{P}' - \sum_{i=1}^r a_i C_i$, then $\mathcal{P}_0$ is relatively nef and $\mathcal{P}_0 \cap X = Q$.

Next we assume that $\deg(Q) > 0$. Then there is $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{Q}}^\times$ such that $Q + (\phi)_X \geq 0$, where $(\phi)_X$ is the $\mathbb{Q}$ -principal divisor of ϕ on X . Let $\mathcal{P}'$ be the closure of $Q + (\phi)_X$ in $\mathcal{X}$. As $Q + (\phi)_X$ is effective, $\mathcal{P}'$ is relatively nef. Here we set $\mathcal{P}_0 = \mathcal{P}' - (\phi)$ on $\mathcal{X}$. Then $\mathcal{P}_0$ is relatively nef and

$$\mathcal{P}_0 \cap X = \mathcal{P}' \cap X - (\phi) \cap X = Q + (\phi)_X - (\phi)_X = Q.$$

(ii) follows from (i) because $\mathcal{P}_0 - n\mathcal{X}_\circ \leq \mathcal{D}$ for a sufficiently large n and $\mathcal{P}_0 - n\mathcal{X}_\circ$ is relatively nef. $\square$

For a prime divisor C on $\mathcal{X}$ (that is, C is a reduced and irreducible curve on $\mathcal{X}$), we set

$$q_C := \sup \{ \text{mult}_C(\mathcal{L}) \mid \mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q) \},$$

which exists in $\mathbb{R}$ because $\text{mult}_C(\mathcal{L}) \leq \text{mult}_C(\mathcal{D})$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$. We fix $\mathcal{P} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ with $\mathcal{P} \cap X = Q$ by using Claim 6.1.3.1.

Claim 6.1.3.2. *There is a sequence $\{\mathcal{L}_n\}_{n=1}^\infty$ of $\mathbb{R}$ -Cartier divisors in $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ such that $\mathcal{P} \leq \mathcal{L}_n$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_n) = q_C$ for all prime divisors C in $\text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})$.*

Proof. For each prime divisor C in $\text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})$, there is a sequence $\{\mathcal{L}_{C,n}\}_{n=1}^\infty$ in $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ such that

$$\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_{C,n}) = q_C.$$

If we set

$$\mathcal{L}_n = \max(\{\mathcal{L}_{C,n}\}_{C \subseteq \text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})} \cup \{\mathcal{P}\}),$$

then $\mathcal{P} \leq \mathcal{L}_n$ and $\mathcal{L}_n \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by (2) in Lemma 6.1.2. Moreover, as

$$\text{mult}_C(\mathcal{L}_{C,n}) \leq \text{mult}_C(\mathcal{L}_n) \leq q_C,$$

$$\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_n) = q_C. \quad \square$$

Since $\max\{\mathcal{P}, \mathcal{L}\} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by (2) in Lemma 6.1.2, we have

$$\text{mult}_C(\mathcal{P}) \leq q_C \leq \text{mult}_C(\mathcal{D}).$$

In particular, if $C \not\subseteq \text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})$, then $q_C = 0$, so that we can set $\mathcal{Q} := \sum_C q_C C$. Clearly $\mathcal{Q} \cap X = Q$ because $\mathcal{P} \leq \mathcal{Q}$. Moreover, $\mathcal{Q} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by Claim 6.1.3.2, and $\mathcal{L} \leq \mathcal{Q}$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by our construction.

Here we assume that $Q = D$. Then $\mathcal{D} - \mathcal{Q}$ is vertical. Let C be an irreducible component of $\text{Supp}_{\mathbb{R}}(\mathcal{D} - \mathcal{Q})$. If $(\mathcal{Q} \cdot C) > 0$, then $\mathcal{Q} + \epsilon C$ is relatively nef for a sufficiently small positive number ϵ . Moreover, $\mathcal{Q} + \epsilon C \leq \mathcal{D}$. This is a contradiction, so that $(\mathcal{Q} \cdot C) = 0$. Therefore, $(\mathcal{Q} \cdot \mathcal{D} - \mathcal{Q}) = 0$.

(2) Clearly $\pi^*(\mathcal{Q}) \in \Sigma_{\mathcal{X}'}(\pi^*(\mathcal{D}); Q)$. Let $\mathcal{L}' \in \Sigma_{\mathcal{X}'}(\pi^*(\mathcal{D}); Q)$. As $\pi_*(\mathcal{L}')$ is relatively nef by (1) in Lemma 6.1.2, we have $\pi_*(\mathcal{L}') \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, and hence $\pi_*(\mathcal{L}') \leq \mathcal{Q}$. Thus, by using (1) in Lemma 6.1.2,

$$\mathcal{L}' \leq \pi^*(\pi_*(\mathcal{L}')) \leq \pi^*(\mathcal{Q}),$$

as required. $\square$

Proof of Theorem 6.1.1. Let us start the proof of Theorem 6.1.1. We fix a regular model $\mathcal{X}_0$ with the following properties:

- (a) If $\overline{D}$ is given by an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on a regular model $\mathcal{X}$, then $\mathcal{X}_0 = \mathcal{X}$.
- (b) There is a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}_0$ on $\mathcal{X}_0$ with $\mathcal{Q}_0 \cap X = Q$ (for details, see Claim 6.1.3.1).

By Theorem 3.3.7, for each $n \geq 1$, we can find a regular model $\mathcal{X}_n$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}_n$ on $\mathcal{X}_n$ such that

$$\overline{D} - \frac{1}{n(n+1)}(\mathcal{X}_n)_\circ^a \leq \mathcal{D}_n^a \leq \overline{D} + \frac{1}{n(n+1)}(\mathcal{X}_n)_\circ^a.$$

Replacing $\mathcal{X}_n$ by a suitable regular model of X if necessary, we may assume that there is a birational morphism $\pi_{n+1} : \mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$ for every $n \geq 0$. Note that if $\overline{D}$ is given by an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$, then $\mathcal{X}_n = \mathcal{X}$ and $\mathcal{D}_n = \mathcal{D}$ for all $n \geq 1$. By using (1) in Lemma 6.1.3, let $\mathcal{Q}_n$ be the greatest element of $\Sigma_{\mathcal{X}_n}(\mathcal{D}_n; Q)$. Let us check the following claim:

Claim 6.1.4. (i) *The following inequalities*

$$\mathcal{Q}_{n+1} - \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{Q}_n) \leq \mathcal{Q}_{n+1} + \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ$$

hold for all $n \geq 1$.

(ii) *Moreover,*

$$\mathcal{Q}_{n+1} - \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{Q}_n) \leq \mathcal{Q}_{n+1} + \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ$$

hold for all $n \geq 1$.

Proof. (i) The first inequality follows from the following observation:

$$\begin{aligned} \pi_{n+1}^*(\mathcal{Q}_n)^a &\geq \overline{D} - \frac{1}{n(n+1)} \pi_{n+1}^*((\mathcal{X}_n)_\circ)^a \\ &\geq \left(\mathcal{Q}_{n+1}^a - \frac{1}{(n+1)(n+2)} (\mathcal{X}_{n+1})_\circ^a \right) - \frac{1}{n(n+1)} (\mathcal{X}_{n+1})_\circ^a \\ &\geq \left(\mathcal{Q}_{n+1} - \frac{2}{n(n+1)} (\mathcal{X}_{n+1})_\circ \right)^a. \end{aligned}$$

The second inequality is similar.

(ii) Note that $\pi_{n+1}^*(\mathcal{Q}_n) - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \in \Sigma_{\mathcal{X}_{n+1}}(\mathcal{Q}_{n+1}; Q)$ because

$$\pi_{n+1}^*(\mathcal{Q}_n) - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \leq \mathcal{Q}_{n+1}$$

by (i), so that $\pi_{n+1}^*(\mathcal{Q}_n) - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \leq \mathcal{Q}_{n+1}$. Similarly

$$\mathcal{Q}_{n+1} - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \in \Sigma_{\mathcal{X}_{n+1}}(\pi_{n+1}^*(\mathcal{Q}_n); Q)$$

by using (i), and hence $\mathcal{Q}_{n+1} - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{Q}_n)$ by (2) in Lemma 6.1.3. $\square$

Let $\mathcal{E}_n := \mathcal{Q}_n - \rho_n^*(\mathcal{Q}_0)$, where $\rho_n := \pi_1 \circ \dots \circ \pi_n : \mathcal{X}_n \rightarrow \mathcal{X}_0$. Then $\mathcal{E}_n$ is vertical and

$$\mathcal{E}_{n+1} - \frac{2}{n(n+1)} (\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{E}_n) \leq \mathcal{E}_{n+1} + \frac{2}{n(n+1)} (\mathcal{X}_{n+1})_\circ$$

by (ii) of the previous claim, so that

$$\varphi_{n+1} - 4 \left(\frac{1}{n} - \frac{1}{n+1} \right) (-\log v(\varpi)) \leq \varphi_n \leq \varphi_{n+1} + 4 \left(\frac{1}{n} - \frac{1}{n+1} \right) (-\log v(\varpi)),$$

where $\varphi_n := g_{(\mathcal{X}_n, \mathcal{E}_n)}$. Therefore, if we set

$$\varphi'_n = \varphi_n - \frac{4(-\log v(\varpi))}{n} \quad \text{and} \quad \varphi''_n = \varphi_n + \frac{4(-\log v(\varpi))}{n},$$

then

$$\varphi'_1 \leq \dots \leq \varphi'_n \leq \varphi'_{n+1} \leq \dots \leq \varphi''_{n+1} \leq \varphi''_n \leq \dots \leq \varphi''_1,$$

and hence $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$ exists for each $x \in X_v^{\text{an}}$. Moreover, as

$$|\varphi_n(x) - \varphi(x)| \leq \varphi''_n(x) - \varphi'_n(x) \leq (8/n)(-\log v(\varpi)),$$

the sequence $\{\varphi_n\}_{n=1}^\infty$ converges to φ uniformly. In particular, φ is continuous on X_v^{an} . We set $q := g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi$. As $\mathcal{Q}_n$ is relatively nef, q is a Q -Green function of $(\text{PSH} \cap C^0)$ -type. Note that in the case where $\overline{D}$ is given by a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}$ on $\mathcal{X}$, then $q = g_{(\mathcal{X}, \mathcal{Q}_1)}$.

Let us see that $\overline{Q} := (Q, q)$ is the greatest element of $\Sigma(\overline{D}; Q)$. As $\{\varphi_n\}_{n=1}^\infty$ converges to φ uniformly and

$$g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi_n = g_{(\mathcal{X}_n, \mathcal{Q}_n)} \leq g_{(\mathcal{X}_n, \mathcal{Q}_n)} \leq g + \frac{-2 \log v(\varpi)}{n(n+1)},$$

we can see that $\overline{Q} \leq \overline{D}$, that is, $\overline{Q} \in \Sigma(\overline{D}; Q)$, so that we need to see that $\overline{L} \leq \overline{Q}$ for all $\overline{L} = (L, g_L) \in \Sigma(\overline{D}; Q)$.

First we assume that $\overline{L}$ is given by an $\mathbb{R}$ -Cartier divisor $\mathcal{L}$ on a regular model $\mathcal{Y}$. By (3) in Lemma 6.1.2, $\mathcal{L}$ is relatively nef. For each $n \geq 1$, we choose a regular model $\mathcal{Y}_n$

of X such that there are birational morphisms $\nu_n : \mathcal{Y}_n \rightarrow \mathcal{Y}$ and $\mu_n : \mathcal{Y}_n \rightarrow \mathcal{X}_n$. If we set $\mathcal{F}_n = (\mu_n)_*(\nu_n^*(\mathcal{L}))$, then $\mathcal{F}_n$ is relatively nef by (1) in Lemma 6.1.2. Moreover, as

$$\nu_n^*(\mathcal{L})^a \leq \overline{D} \leq \left(\mu_n^*(\mathcal{Q}_n) + \frac{1}{n(n+1)}(\mathcal{Y}_n)_\circ \right)^a,$$

we have $\mathcal{F}_n \leq \mathcal{Q}_n + (1/n(n+1))(\mathcal{X}_n)_\circ$ by using Proposition 2.2.1, and hence

$$\mathcal{F}_n \leq \mathcal{Q}_n + (1/n(n+1))(\mathcal{X}_n)_\circ$$

because $\mathcal{F}_n - (1/n(n+1))(\mathcal{X}_n)_\circ \in \Sigma_{\mathcal{X}_n}(\mathcal{Q}_n; Q)$. Therefore, by (1) in Lemma 6.1.2,

$$\nu_n^*(\mathcal{L}) \leq \mu_n^*(\mathcal{F}_n) \leq \mu_n^*(\mathcal{Q}_n) + \frac{1}{n(n+1)}(\mathcal{Y}_n)_\circ.$$

In particular,

$$g_{(\mathcal{Y}, \mathcal{L})} \leq g_{(\mathcal{X}_n, \mathcal{Q}_n)} + \frac{-2 \log v(\varpi)}{n(n+1)} = g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi_n + \frac{-2 \log v(\varpi)}{n(n+1)}.$$

Note that $\{\varphi_n\}_{n=1}^\infty$ converges to φ uniformly, so that we have $g_{(\mathcal{Y}, \mathcal{L})} \leq q$.

In general, by Proposition 2.1.8, there are a sequence $\{\mathcal{Y}_l\}_{l=1}^\infty$ of regular models of X and a sequence $\{\mathcal{L}_l\}_{l=1}^\infty$ of relatively nef $\mathbb{R}$ -Cartier divisors such that $\mathcal{L}_l$ is defined on $\mathcal{Y}_l$, $\mathcal{L}_l \cap X = L$, $(\mathcal{L}_l)^a \leq \overline{L}$ and $g_L = \lim_{l \rightarrow \infty} g_{(\mathcal{Y}_l, \mathcal{L}_l)}$ uniformly. By the previous observation, $g_{(\mathcal{Y}_l, \mathcal{L}_l)} \leq q$ for all l , so that $g_L \leq q$.

Let us see additional assertions (1) and (2) in the theorem. Assertion (1) is obvious by our construction. Let us consider (2). We assume that $Q = D$. If we set $\theta_n := g_{(\mathcal{X}_n, \mathcal{Q}_n)} - g_{(\mathcal{X}_n, \mathcal{Q}_n)}$ and $\theta = g - q$, then $\{\theta_n\}_{n=1}^\infty$ converges to θ uniformly because

$$\theta_n = (g_{(\mathcal{X}_n, \mathcal{Q}_n)} - g) + g - (g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi_n).$$

Thus, by Proposition-Definition 2.4.3,

$$\lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{Q}_n; \theta_n) = \widehat{\deg}_v(\overline{Q}; \theta).$$

On the other hand, $\widehat{\deg}_v(\mathcal{Q}_n; \theta_n) = 0$ by Lemma 6.1.3, so that assertion (2) follows. $\square$

Finally we consider the maximal of two Green functions, which will be used in the next subsection.

Proposition 6.1.5. *Let D_1 and D_2 be $\mathbb{R}$ -Cartier divisors on X and let $D_3 := \max\{D_1, D_2\}$. For $i = 1, 2$, let g_i be a D_i -Green function of C^0 -type on X_v^{an} .*

- (1) *$\max\{g_1, g_2\}$ is a D_3 -Green function of C^0 -type on X_v^{an} .*
- (2) *If g_1 and g_2 are of $(C^0 \cap \text{PSH})$ -type, then $\max\{g_1, g_2\}$ is also of $(C^0 \cap \text{PSH})$ -type.*

Proof. (1) Let $\pi : X_v \rightarrow X$ be the canonical morphism. It is easy to see that

$$\max\{\pi^*(D_1), \pi^*(D_2)\} = \pi^*(\max\{D_1, D_2\}),$$

so that the assertion follows from Proposition 2.1.5.

(2) For each $n \geq 1$, by Proposition 2.1.8, there are a regular model $\mathcal{X}'_n$ of X and relatively nef $\mathbb{R}$ -Cartier divisors $\mathcal{Q}'_{1,n}$ and $\mathcal{Q}'_{2,n}$ on $\mathcal{X}'_n$ such that

$$0 \leq g_1 - g_{(\mathcal{X}'_n, \mathcal{Q}'_{1,n})} \leq 1/n \quad \text{and} \quad 0 \leq g_2 - g_{(\mathcal{X}'_n, \mathcal{Q}'_{2,n})} \leq 1/n.$$

On the other hand, by Theorem 3.3.7, we can find a regular model $\mathcal{X}''_n$ of X and an $\mathbb{R}$ -Cartier divisor $\mathcal{E}''_n$ on $\mathcal{X}''_n$ such that $\mathcal{E}''_n \cap X = D_3$ and

$$\max\{g_1, g_2\} \leq g_{(\mathcal{X}''_n, \mathcal{E}''_n)} \leq \max\{g_1, g_2\} + 1/n.$$

We choose birational morphisms $\nu_n : \mathcal{X}_n \rightarrow \mathcal{X}'_n$ and $\mu_n : \mathcal{X}_n \rightarrow \mathcal{X}''_n$ and we set

$$\mathcal{D}_{1,n} := \nu_n^*(\mathcal{D}'_{1,n}), \mathcal{D}_{2,n} := \nu_n^*(\mathcal{D}'_{2,n}), \mathcal{D}_{3,n} := \max\{\mathcal{D}_{1,n}, \mathcal{D}_{2,n}\} \text{ and } \mathcal{E}_n := \mu_n^*(\mathcal{E}''_n).$$

As $\mathcal{D}_{1,n} \leq \mathcal{D}_{3,n}$ and $\mathcal{D}_{2,n} \leq \mathcal{D}_{3,n}$, we have

$$\max\{g_{(\mathcal{X}_n, \mathcal{D}_{1,n})}, g_{(\mathcal{X}_n, \mathcal{D}_{2,n})}\} \leq g_{(\mathcal{X}_n, \mathcal{D}_{3,n})},$$

so that

$$\begin{aligned} \max\{g_1, g_2\} - 1/n &= \max\{g_1 - 1/n, g_2 - 1/n\} \\ &\leq \max\{g_{(\mathcal{X}_n, \mathcal{D}_{1,n})}, g_{(\mathcal{X}_n, \mathcal{D}_{2,n})}\} \leq g_{(\mathcal{X}_n, \mathcal{D}_{3,n})}. \end{aligned}$$

Moreover,

$$g_{(\mathcal{X}_n, \mathcal{D}_{1,n})} \leq g_1 \leq \max\{g_1, g_2\} \leq g_{(\mathcal{X}_n, \mathcal{E}_n)}$$

and

$$g_{(\mathcal{X}_n, \mathcal{D}_{2,n})} \leq g_2 \leq \max\{g_1, g_2\} \leq g_{(\mathcal{X}_n, \mathcal{E}_n)},$$

and hence, by Proposition 2.2.1,

$$\mathcal{D}_{1,n} \leq \mathcal{E}_n \quad \text{and} \quad \mathcal{D}_{2,n} \leq \mathcal{E}_n,$$

so that $\mathcal{D}_{3,n} \leq \mathcal{E}_n$. Therefore,

$$\max\{g_1, g_2\} - 1/n \leq g_{(\mathcal{X}_n, \mathcal{D}_{3,n})} \leq \max\{g_1, g_2\} + 1/n.$$

Note that $\mathcal{D}_{3,n}$ is relatively nef by (2) in Lemma 6.1.2, and hence $\max\{g_1, g_2\}$ is of $(C^0 \cap \text{PSH})$ -type. $\square$

6.2. Proof of Zariski decompositions for adelic arithmetic divisors. In this subsection, we give the proof of Zariski decompositions for adelic arithmetic divisors. Let X be a projective, smooth and geometrically integral curve over a number field K . Let us begin with decompositions for global adelic divisors.

Theorem 6.2.1. *Let $\overline{D} = (D, \{g_p\}_{p \in M_K})$ be a global adelic $\mathbb{R}$ -Cartier divisor on X (cf. Definition 4.1.1) and let Q be an $\mathbb{R}$ -Cartier divisor on X with $Q \leq D$. Here we set*

$$\Sigma(\overline{D}; Q) := \left\{ \overline{L} = (L, \{l_p\}_{p \in M_K}) \left| \begin{array}{l} \overline{L} \text{ is a relatively nef global adelic } \mathbb{R}\text{-Cartier} \\ \text{divisor on } X \text{ such that } L \leq Q \text{ and } \overline{L} \leq \overline{D} \end{array} \right. \right\}.$$

If $\deg(Q) \geq 0$, then there exists a Q -Green function q_p of $(\text{PSH} \cap C^0)$ -type on X_p^{an} for each $p \in M_K$ such that $\overline{Q} := (Q, \{q_p\}_{p \in M_K})$ gives rise to the greatest element of $\Sigma(\overline{D}; Q)$, that is, $\overline{Q} \in \Sigma(\overline{D}; Q)$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Sigma(\overline{D}; Q)$. Moreover, if there are a non-empty Zariski open set U of $\text{Spec}(O_K)$, a regular model $\mathcal{X}_U$ of X over U and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}_U$ on $\mathcal{X}_U$ such that g_p is the Green function arising from $\mathcal{D}_U$ for all $p \in U \cap M_K$, then there is a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}_U$ on $\mathcal{X}_U$ such that q_p is given by $\mathcal{Q}_U$ for all $p \in U \cap M_K$.

Proof. Let us choose a non-empty Zariski open set U of $\text{Spec}(O_K)$, a regular model $\mathcal{X}_U$ of X over U and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}_U$ on $\mathcal{X}_U$ such that g_p is given by $\mathcal{D}_U$ for all $p \in U \cap M_K$. Moreover, we set

$$U' = \{p \in U \mid \mathcal{X}_U \rightarrow U \text{ is smooth over } p \text{ and } \mathcal{D}_U \text{ is horizontal over } p\}.$$

By Theorem 6.1.1, for each $\mathfrak{p} \in M_K$, we can find a Q -Green function $q_{\mathfrak{p}}$ of $(C^0 \cap \text{PSH})$ -type on $X_{\mathfrak{p}}^{\text{an}}$ such that $(Q, q_{\mathfrak{p}})$ yields the greatest element of

$$\left\{ (L, l_{\mathfrak{p}}) \mid \begin{array}{l} L \text{ is a nef } \mathbb{R}\text{-Cartier divisor on } X, l_{\mathfrak{p}} \text{ is an } L\text{-Green function} \\ \text{of } (C^0 \cap \text{PSH})\text{-type on } X_{\mathfrak{p}}^{\text{an}}, L \leq Q \text{ and } (L, l_{\mathfrak{p}}) \leq (D, g_{\mathfrak{p}}) \end{array} \right\}.$$

For each $\mathfrak{p} \in U$, we set $(\mathcal{X}_U)_{(\mathfrak{p})} := \mathcal{X}_U \times_U \text{Spec}((O_K)_{\mathfrak{p}})$. Then, by Theorem 6.1.1 again, $q_{\mathfrak{p}}$ is obtained by a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}_{(\mathfrak{p})}$ on $(\mathcal{X}_U)_{(\mathfrak{p})}$. Note that if $\mathfrak{p} \in U'$, then $\mathcal{Q}_{(\mathfrak{p})}$ is actually given by the Zariski closure of Q in $(\mathcal{X}_U)_{(\mathfrak{p})}$. Therefore, we can find a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{Q}_U$ on $\mathcal{X}_U$ such that $\mathcal{Q}_U \cap (\mathcal{X}_U)_{(\mathfrak{p})} = \mathcal{Q}_{(\mathfrak{p})}$. Therefore, $\overline{Q} := (Q, \{q_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K})$ forms a global adelic $\mathbb{R}$ -Cartier divisor on X . By our construction, it is obvious that $\overline{Q}$ is the greatest element of $\Sigma(\overline{D}; Q)$. Further, the second assertion of the theorem is also obvious by our construction. $\square$

As a corollary, we have the relative version of Corollary 6.2.7.

Corollary 6.2.2. *Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let $\Upsilon_{\text{rel}}(\overline{D})$ be the set of all relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{L}$ of C^0 -type on X with $\overline{L} \leq \overline{D}$. If $\deg(D) \geq 0$, then there is the greatest element $\overline{Q} = (Q, q)$ of $\Upsilon_{\text{rel}}(\overline{D})$, that is, $\overline{Q} \in \Upsilon_{\text{rel}}(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon_{\text{rel}}(\overline{D})$. Moreover, we have the following properties:*

- (1) $\overline{D} - \overline{Q}$ is vertical, that is, $D = Q$.
- (2) For every $a \in \mathbb{R}_{>0}$, the natural homomorphism $\hat{H}^0(X, a\overline{Q}^{\tau}) \rightarrow \hat{H}^0(X, a\overline{D}^{\tau})$ is bijective. Further, $\|\phi\|_{aq_{\infty}} = \|\phi\|_{ag_{\infty}}$ for all $\phi \in H^0(X(\mathbb{C}), aD)$. In particular,

$$\hat{\chi}(X, a\overline{Q}) = \hat{\chi}(X, a\overline{D}) \quad \text{and} \quad \widehat{\text{vol}}_{\chi}(\overline{Q}) = \widehat{\text{vol}}_{\chi}(\overline{D}).$$

- (3) $\overline{Q}$ is perpendicular to $\overline{D} - \overline{Q}$, that is, $\widehat{\deg}(\overline{Q} \cdot \overline{D} - \overline{Q}) = 0$.

Proof. Applying Theorem 6.2.1 to the case $Q = D$, we have the greatest element

$$\left(D, \sum_{\mathfrak{p} \in M_K} q_{\mathfrak{p}}[\mathfrak{p}] \right)$$

of $\Sigma(\overline{D}^{\tau}; D)$. Moreover, by using [30, Theorem 4.6], there is a D -Green function q_{∞} of $(C^0 \cap \text{PSH})$ -type on $X(\mathbb{C})$ such that q_{∞} yields the greatest element of

$$\{h_{\infty} \mid h_{\infty} \text{ is a } D\text{-Green function of } (C^0 \cap \text{PSH})\text{-type on } X(\mathbb{C}) \text{ and } h_{\infty} \leq g_{\infty}\}.$$

Thus $(D, \sum_{\mathfrak{p} \in M_K} q_{\mathfrak{p}}[\mathfrak{p}] + q_{\infty}[\infty])$ is our desired adelic arithmetic $\mathbb{R}$ -Cartier divisor.

Property (1) is obvious. For (2), we suppose $\phi \in \hat{H}^0(X, a\overline{D}^{\tau})$, that is,

$$-(1/a)(\phi)^a \leq \overline{D}^{\tau}$$

by Proposition 4.3.1. Note that $-(1/a)(\phi)^a$ is relatively nef, so that $-(1/a)(\phi)^a \leq \overline{Q}^{\tau}$. Therefore, $\phi \in \hat{H}^0(X, a\overline{Q}^{\tau})$ by Proposition 4.3.1. The assertion

$$\|\cdot\|_{aq_{\infty}} = \|\cdot\|_{ag_{\infty}}$$

on $H^0(X(\mathbb{C}), aD)$ follows from [32, Lemma 1.3]. Further, (3) is a consequence of (2) in Theorem 6.1.1 and [32, Lemma 1.3]. $\square$

The following theorem is one of the main results of this article.

Theorem 6.2.3. *Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let R be an $\mathbb{R}$ -Cartier divisor on X with $R \leq D$. Let $\Upsilon(\overline{D}; R)$ be the set of all nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{L} = (L, l)$ of C^0 -type on X with $L \leq R$ and $\overline{L} \leq \overline{D}$. If $\Upsilon(\overline{D}; R) \neq \emptyset$, then there is the greatest element $\overline{Q}$ of $\Upsilon(\overline{D}; R)$, that is, $\overline{Q} \in \Upsilon(\overline{D}; R)$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon(\overline{D}; R)$.*

First let us consider two lemmas.

Lemma 6.2.4. *Let $\overline{M}$ and $\overline{Q}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X and let $\{\overline{L}_n\}_{n=1}^\infty$ be a sequence of adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X with the following properties:*

- (1) $\overline{L}_n$ is nef for all $n \geq 1$.
- (2) $\overline{M} \leq \overline{L}_n \leq \overline{Q}$ for all $n \geq 1$.
- (3) For all closed points x of X , $\lim_{n \rightarrow \infty} \text{mult}_x(L_n) = \text{mult}_x(Q)$.

Then $\widehat{\deg}(\overline{Q}|_x) \geq 0$ for all closed points $x \in X$.

Proof. We set $\overline{M} = (M, m)$, $\overline{Q} = (Q, q)$ and $\overline{L}_n = (L_n, g_n)$. First we assume that there are a regular model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{Q}$ on $\mathcal{X}$ such that $\overline{Q} = (\mathcal{Q}, q_\infty)^a$. Let us see the following claim:

Claim 6.2.4.1. (i) *For all $n \geq 1$, there is an $\mathbb{R}$ -Cartier divisor $\mathcal{L}_n$ on $\mathcal{X}$ such that $\mathcal{L}_n \cap X = L_n$, $(\mathcal{L}_n, (g_n)_\infty)^a$ is nef and $\overline{L}_n \leq (\mathcal{L}_n, (g_n)_\infty)^a \leq (\mathcal{Q}, q_\infty)^a$.*
(ii) *There is an $\mathbb{R}$ -Cartier divisor $\mathcal{M}$ on $\mathcal{X}$ such that $\mathcal{M} \cap X = M$ and $(\mathcal{M}, m_\infty)^a \leq \overline{M}$.*

Proof. (i) For each $n \geq 1$, we consider the following set:

$$\{\overline{D} = (D, \{g_p\}_{p \in M_K}) \mid \overline{D} \text{ is relatively nef, } D \leq L_n \text{ and } \overline{D} \leq \mathcal{Q}^a\}.$$

Then, by using Theorem 6.2.1, there is a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{L}_n$ on $\mathcal{X}$ such that $(\mathcal{L}_n)^a$ gives rise to the greatest element of the above set. As $\overline{L}_n^\tau$ belongs to the above set, we can see that $\overline{L}_n \leq (\mathcal{L}_n, (g_n)_\infty)^a$. Moreover, as $L_n = \mathcal{L}_n \cap X$, $(\mathcal{L}_n, (g_n)_\infty)^a$ is nef by Lemma 4.2.3, so that (i) follows.

(ii) There are an $\mathbb{R}$ -Cartier divisor $\mathcal{M}'$ on $\mathcal{X}$ and a non-empty open set U of $\text{Spec}(O_K)$ such that m_p is defined by $\mathcal{M}'$ for all $p \in U \cap M_K$. For each $p \in M_K \setminus U$, let m'_p be the M -Green function arising from $\mathcal{M}'$. As $m_p - m'_p$ is a continuous function on X_p , there is a constant θ_p such that $m_p \geq m'_p + \theta_p$ for all $p \in M_K \setminus U$. Let F_p be the fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ over p . If we set

$$\overline{\mathcal{M}} = \left(\mathcal{M}' + \sum_{p \in M_K \setminus U} \frac{\theta_p}{2 \log \#(O_K/p)} F_p, m_\infty \right),$$

then $\overline{M} \geq \overline{\mathcal{M}}^a$, as required. $\square$

By the above claim, $(\mathcal{M}, m_\infty) \leq (\mathcal{L}_n, (g_n)_\infty) \leq (\mathcal{Q}, q_\infty)$ and $\mathcal{L}_n \cap X = L_n$ for all $n \geq 1$. Thus $\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_n)$ exists for all prime divisors C on $\mathcal{X}$ except finitely many fiber components, so that if we choose a subsequence $\{(\mathcal{L}_{n_i}, (g_{n_i})_\infty)\}_{i=1}^\infty$ of $\{(\mathcal{L}_n, (g_n)_\infty)\}_{n=1}^\infty$, then $\lim_{i \rightarrow \infty} \text{mult}_C(\mathcal{L}_{n_i})$ exists for all prime divisors C on $\mathcal{X}$. Therefore, by using [30, Theorem 7.1], there are an $\mathbb{R}$ -Cartier divisor $\mathcal{L}$ on $\mathcal{X}$ and a Q -Green function g_∞ of $\text{PSH}_\mathbb{R}$ -type on $X(\mathbb{C})$ with the following properties:

- (a) $\text{mult}_C(\mathcal{L}) = \lim_{i \rightarrow \infty} \text{mult}_C(\mathcal{L}_{n_i})$ for all prime divisors C on $\mathcal{X}$. In particular, $\mathcal{L} \cap X = Q$.

(b) $\widehat{\deg}((\mathcal{L}, g_\infty)|_C) \geq \limsup_{i \rightarrow \infty} \widehat{\deg}((\mathcal{L}_{n_i}, (g_{n_i})_\infty)|_C)$ for all prime divisors C on $\mathcal{X}$.

(c) $(\mathcal{L}, g_\infty) \leq (\mathcal{Q}, q_\infty)$.

Let x be a closed point of X and Δ_x the closure of x in $\mathcal{X}$. Then, as $\mathcal{L} \cap X = \mathcal{Q} \cap X$ and $(\mathcal{L}, g_\infty) \leq (\mathcal{Q}, q_\infty)$,

$$\begin{aligned} \widehat{\deg}(\overline{Q}|_x) &= \widehat{\deg}((\mathcal{Q}, q_\infty)|_{\Delta_x}) \geq \widehat{\deg}((\mathcal{L}, g_\infty)|_{\Delta_x}) \\ &\geq \limsup_{i \rightarrow \infty} \widehat{\deg}((\mathcal{L}_{n_i}, (g_{n_i})_\infty)|_{\Delta_x}) \geq 0. \end{aligned}$$

In general, let U be a non-empty open set of $\text{Spec}(O_K)$ such that $\overline{Q}$ has a defining model over U . For a positive number ϵ , by Theorem 4.1.3, there are a regular model $\mathcal{X}_\epsilon$ over $\text{Spec}(O_K)$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{Q}_\epsilon$ on $\mathcal{X}_\epsilon$ such that

$$\overline{Q} \leq (\mathcal{Q}_\epsilon, q_\infty)^a \leq \overline{Q} + \left(0, \sum_{\mathfrak{p} \in M_K \setminus U} \epsilon[\mathfrak{p}]\right).$$

Then, by the previous observation, $\widehat{\deg}((\mathcal{Q}_\epsilon, q_\infty)^a|_x) \geq 0$, so that, by Lemma 4.2.3,

$$\widehat{\deg}\left(\overline{Q} + \left(0, \sum_{\mathfrak{p} \in M_K \setminus U} \epsilon[\mathfrak{p}]\right)\Big|_x\right) \geq 0,$$

and hence

$$\widehat{\deg}(\overline{Q}|_x) \geq -\epsilon[K(x) : K]\#(M_K \setminus U),$$

where $K(x)$ is the residue field at x . Thus the assertion follows. $\square$

Lemma 6.2.5. (1) *Let*

$$\overline{L}_1 = (L_1, \{(l_1)_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K}) \quad \text{and} \quad \overline{L}_2 = (L_2, \{(l_2)_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K})$$

be global adelic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $\overline{L}_1$ and $\overline{L}_2$ are relatively nef, then

$$\max\{\overline{L}_1, \overline{L}_2\} := \left(\max\{L_1, L_2\}, \left\{\max\{(l_1)_{\mathfrak{p}}, (l_2)_{\mathfrak{p}}\}\right\}_{\mathfrak{p} \in M_K}\right)$$

is also relatively nef.

(2) *Let $\overline{Q}_1 = (Q_1, q_1)$ and $\overline{Q}_2 = (Q_2, q_2)$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $\overline{Q}_1$ and $\overline{Q}_2$ are nef, then*

$$\begin{aligned} \max\{\overline{Q}_1, \overline{Q}_2\} &:= \left(\max\{Q_1, Q_2\}, \left\{\max\{(q_1)_{\mathfrak{p}}, (q_2)_{\mathfrak{p}}\}\right\}_{\mathfrak{p} \in M_K}\right) \\ &\quad \cup \left\{\max\{(q_1)_\infty, (q_2)_\infty\}\right\} \end{aligned}$$

is also nef.

Proof. (1) We set $L_1 = a_{11}x_1 + \cdots + a_{1r}x_r$ and $L_2 = a_{21}x_1 + \cdots + a_{2r}x_r$, where $x_1, \dots, x_r$ are closed points on X and $a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2r} \in \mathbb{R}$. Let us choose a non-empty Zariski open set U of $\text{Spec}(O_K)$, a regular model $\mathcal{X}_U$ over U and nef $\mathbb{R}$ -Cartier divisors $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}_U$ such that l_1 and l_2 are given by $\mathcal{L}_1$ and $\mathcal{L}_2$ over U , respectively. For $i = 1, \dots, r$, let C_i be the Zariski closure of x_i in $\mathcal{X}_U$. Shrinking U if necessary, we may assume the following:

- (a) $\mathcal{L}_1 = a_{11}C_1 + \cdots + a_{1r}C_r$ and $\mathcal{L}_2 = a_{21}C_1 + \cdots + a_{2r}C_r$.
- (b) $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Then, by properties (a) and (b), for $p \in U$, it is easy to see that $\max\{(l_1)_p, (l_2)_p\}$ is given by $g_{((\mathcal{X}_U)_{(p)}, \max\{\mathcal{L}_1, \mathcal{L}_2\}_{(p)})}$, where $(\mathcal{X}_U)_{(p)}$ is the localization of $\mathcal{X}_U \rightarrow U$ at p . Note that $\max\{\mathcal{L}_1, \mathcal{L}_2\}$ is relatively nef by (2) in Lemma 6.1.2. Moreover, for $p \in M_K \setminus U$, by (2) in Proposition 6.1.5, $\max\{(l_1)_p, (l_2)_p\}$ is of $(C^0 \cap \text{PSH})$ -type. Thus the assertion follows.

(2) By [30, Lemma 9.1.1], $\max\{(q_1)_\infty, (q_2)_\infty\}$ is of $(C^0 \cap \text{PSH})$ -type, so that, by virtue of (1), it is sufficient to show that $\widehat{\deg}(\max\{\bar{Q}_1, \bar{Q}_2\}|_x) \geq 0$ for all closed points x of X . As

$$\text{Supp}_{\mathbb{R}}(\max\{Q_1, Q_2\} - Q_1) \cap \text{Supp}_{\mathbb{R}}(\max\{Q_1, Q_2\} - Q_2) = \emptyset,$$

we may assume that $x \notin \text{Supp}_{\mathbb{R}}(\max\{Q_1, Q_2\} - Q_1)$. If we set

$$\bar{Q} := \max\{\bar{Q}_1, \bar{Q}_2\} - \bar{Q}_1,$$

then $\bar{Q}$ is effective and $x \notin \text{Supp}_{\mathbb{R}}(Q)$, so that $\widehat{\deg}(\bar{Q}|_x) \geq 0$. Therefore,

$$\widehat{\deg}(\max\{\bar{Q}_1, \bar{Q}_2\}|_x) = \widehat{\deg}(\bar{Q}_1|_x) + \widehat{\deg}(\bar{Q}|_x) \geq 0,$$

as required. $\square$

Proof of Theorem 6.2.3. Let us start the proof of Theorem 6.2.3. We choose

$$\bar{M} = (M, m) \in \Upsilon(\bar{D}; R).$$

Let $\Upsilon([\bar{M}, \bar{D}]; R)$ be the set of all nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\bar{L} = (L, l)$ of C^0 -type on X with $L \leq R$ and $\bar{M} \leq \bar{L} \leq \bar{D}$. For each closed point $x \in X$, we set

$$a_x = \sup \{ \text{mult}_x(L) \mid (L, l) \in \Upsilon([\bar{M}, \bar{D}]; R) \}.$$

Note that, if $x \notin \text{Supp}_{\mathbb{R}}(D) \cup \text{Supp}_{\mathbb{R}}(M)$, then $\text{mult}_x(L) = 0$ for all $\bar{L} = (L, l) \in \Upsilon([\bar{M}, \bar{D}]; R)$. In particular, $a_x = 0$, so that we set $Q = \sum_x a_x x$, which is an $\mathbb{R}$ -Cartier divisor on X with $Q \leq R$. By using Theorem 6.2.1, let $(Q, \{q_p\}_{p \in M_K})$ be the greatest element of

$$A := \left\{ \bar{L} = (L, \{l_p\}_{p \in M_K}) \mid \begin{array}{l} \bar{L} \text{ is a relatively nef global adelic } \mathbb{R}\text{-Cartier} \\ \text{divisor on } X \text{ such that } L \leq Q \text{ and } \bar{L} \leq \bar{D}^\tau \end{array} \right\}.$$

Note that $(Q, \{q_p\}_{p \in M_K}) \geq \bar{M}^\tau$ because $\bar{M}^\tau \in A$. Moreover, by [30, Theorem 4.6], there is a Q -Green function q_∞ of $(C^0 \cap \text{PSH})$ -type on $X(\mathbb{C})$ such that q_∞ yields the greatest element of

$$B := \left\{ h_\infty \mid \begin{array}{l} h_\infty \text{ is a } Q\text{-Green function of } (C^0 \cap \text{PSH})\text{-type} \\ \text{on } X(\mathbb{C}) \text{ and } h_\infty \leq g_\infty \end{array} \right\}.$$

As $\max\{m_\infty, q_\infty\}$ is an element of B (cf. [30, Lemma 9.1.1]), we have $m_\infty \leq q_\infty$. Here we set

$$\bar{Q} := (Q, \{q_p\}_{p \in M_K} \cup \{q_\infty\}).$$

Clearly $\bar{M} \leq \bar{Q} \leq \bar{D}$ and $Q \leq R$.

Claim 6.2.6. *There is a sequence $\{\bar{L}_n = (L_n, l_n)\}_{n=1}^\infty$ in $\Upsilon([\bar{M}, \bar{D}]; R)$ such that*

$$a_x = \lim_{n \rightarrow \infty} \text{mult}_x(L_n)$$

for all closed points $x \in X$.

Proof. Let $\{x_1, \dots, x_N\} := \text{Supp}_{\mathbb{R}}(D) \cup \text{Supp}_{\mathbb{R}}(M)$. For each $i = 1, \dots, N$, we can find a sequence $\{\bar{L}_{i,n} = (L_{i,n}, l_{i,n})\}_{n=1}^{\infty}$ in $\Upsilon([\bar{M}, \bar{D}]; R)$ such that

$$a_{x_i} = \lim_{n \rightarrow \infty} \text{mult}_{x_i}(L_{i,n}).$$

We set $\bar{L}_n = \max\{\bar{L}_{1,n}, \dots, \bar{L}_{N,n}\}$ for $n \geq 1$. Then, by (2) in Lemma 6.2.5, $\bar{L}_n \in \Upsilon([\bar{M}, \bar{D}]; R)$, and hence

$$a_{x_i} = \lim_{n \rightarrow \infty} \text{mult}_{x_i}(L_n)$$

for $i = 1, \dots, N$. If $x \notin \{x_1, \dots, x_N\}$, then $a_x = 0$ and $\text{mult}_x(L_n) = 0$ for all n . Thus we have the claim. $\square$

As $\max\{q_{\infty}, (l_n)_{\infty}\} \in B$ by [30, Lemma 9.1.1], we have $(l_n)_{\infty} \leq q_{\infty}$. Moreover, $\bar{L}_n^{\tau} \leq \bar{Q}^{\tau}$ because $\bar{L}_n^{\tau} \in A$. Therefore, $\bar{L}_n \leq \bar{Q}$, so that, by Lemma 6.2.4, $\bar{Q}$ is nef. In particular, $\bar{Q} \in \Upsilon(\bar{D}; R)$. We need to check that $\bar{Q}$ is the greatest element of $\Upsilon(\bar{D}; R)$. Indeed, let $\bar{L} = (L, l) \in \Upsilon(\bar{D}; R)$. Then $\bar{L}^{\tau} \in A$ and $l_{\infty} \in B$, and hence $\bar{L}^{\tau} \leq \bar{Q}^{\tau}$ and $l_{\infty} \leq q_{\infty}$, as required. $\square$

Corollary 6.2.7 (Zariski decomposition for adelic arithmetic divisors). *Let $\bar{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let $\Upsilon(\bar{D})$ be the set of all nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\bar{L}$ of C^0 -type on X with $\bar{L} \leq \bar{D}$. If $\Upsilon(\bar{D}) \neq \emptyset$, then there is the greatest element $\bar{P}$ of $\Upsilon(\bar{D})$, that is, $\bar{P} \in \Upsilon(\bar{D})$ and $\bar{L} \leq \bar{P}$ for all $\bar{L} \in \Upsilon(\bar{D})$. Moreover, the natural map $\hat{H}^0(X, a\bar{P}) \rightarrow \hat{H}^0(X, a\bar{D})$ is bijective for $a \in \mathbb{R}_{>0}$. In particular, $\widehat{\text{vol}}(\bar{P}) = \widehat{\text{vol}}(\bar{D})$ (cf. Theorem 0.4.4 in Introduction).*

Proof. Applying Theorem 6.2.3 to the case where $R = D$, we have the first assertion. Let us see the second assertion. Let $\phi \in \hat{H}^0(X, a\bar{D})$, that is, $a\bar{D} + \widehat{(\phi)} \geq 0$ by Proposition 4.3.1, so that $\bar{D} \geq (1/a)\widehat{(\phi^{-1})}$. Note that $(1/a)\widehat{(\phi^{-1})}$ is nef, and hence $(1/a)\widehat{(\phi^{-1})} \in \Upsilon(\bar{D})$. Therefore, $\bar{P} \geq (1/a)\widehat{(\phi^{-1})}$, that is, $a\bar{P} + \widehat{(\phi)} \geq 0$, which means that $\phi \in \hat{H}^0(X, a\bar{P})$ by Proposition 4.3.1. $\square$

Remark 6.2.8. There are several conditions to guarantee $\Upsilon(\bar{D}) \neq \emptyset$. For example, if $\hat{H}^0(X, a\bar{D}) \neq \{0\}$ for some $a \in \mathbb{R}_{>0}$, then $\Upsilon(\bar{D}) \neq \emptyset$. Indeed, $\hat{H}^0(X, a\bar{D}) \neq \{0\}$ implies that $a\bar{D} + \widehat{(\phi)} \geq 0$ for some $\phi \in \text{Rat}(X)^{\times}$, so that $(1/a)\widehat{(\phi^{-1})} \leq \bar{D}$. Note that $(1/a)\widehat{(\phi^{-1})}$ is nef, and hence $(1/a)\widehat{(\phi^{-1})} \in \Upsilon(\bar{D})$. In particular, if $\bar{D}$ is big, then $\Upsilon(\bar{D}) \neq \emptyset$. As a conjecture, we expect that if $\bar{D}$ is pseudo-effective, then $\Upsilon(\bar{D}) \neq \emptyset$.

7. CHARACTERIZATION OF NEF ADELIC ARITHMETIC DIVISORS ON ARITHMETIC SURFACES

In this section, we consider a generalization of the numerical characterization of nef arithmetic divisors proved in [32] to adelic arithmetic divisors. Namely, we will prove that an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor $\bar{D}$ of C^0 -type on a projective smooth curve over a number field is nef if and only if $\bar{D}$ is pseudo-effective and $\widehat{\deg}(\bar{D}^2) = \widehat{\text{vol}}(\bar{D})$. Throughout this section, let X be a projective, smooth and geometrically integral variety over a number field K .

7.1. Hodge index theorem for adelic arithmetic divisors. We assume that $\dim X = 1$. Let us start with a refinement of the generalized Hodge index theorem on an arithmetic surface.

Theorem 7.1.1. *Let $\overline{D} = (D, g)$ be an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . If $\deg(D) \geq 0$, then $\deg(\overline{D}^2) \leq \widehat{\text{vol}}_\chi(\overline{D})$. Moreover, the equality holds if and only if $\overline{D}$ is relatively nef.*

Proof. Let $\Upsilon_{\text{rel}}(\overline{D})$ be the set of all relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{L}$ of C^0 -type on X with $\overline{L} \leq \overline{D}$ (cf. Corollary 6.2.2). Then, by Corollary 6.2.2, $\Upsilon_{\text{rel}}(\overline{D})$ has the greatest element $\overline{Q} = (Q, q)$, that is, $\overline{Q} \in \Upsilon_{\text{rel}}(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon_{\text{rel}}(\overline{D})$. Further, we have the following:

- (1) $\widehat{\text{vol}}_\chi(\overline{Q}) = \widehat{\text{vol}}_\chi(\overline{D})$.
- (2) If we set $\overline{N} := \overline{D} - \overline{Q}$, then $\overline{N} = (0, \phi)$ and $\widehat{\deg}(\overline{Q} \cdot \overline{N}) = 0$, where $\phi = \{\phi_p\}_{p \in M_K} \cup \{\phi_\infty\}$ is a collection of integrable continuous functions.

By Theorem 5.3.2,

$$\widehat{\deg}(\overline{Q}^2) = \widehat{\text{vol}}_\chi(\overline{Q}) = \widehat{\text{vol}}_\chi(\overline{D}).$$

Note that

$$\widehat{\deg}(\overline{N}^2) = \sum_{p \in M_K} \log \#(O_K/\mathfrak{p}) \widehat{\deg}_p((0, \phi_p); \phi_p) + \frac{1}{2} \int_{X(\mathbb{C})} \phi_\infty d d^c(\phi_\infty),$$

so that, by Lemma 3.4.1 and [31, Proposition 1.2.3], we have

$$\widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{Q}^2) + \widehat{\deg}(\overline{N}^2) \leq \widehat{\deg}(\overline{Q}^2).$$

Thus the first assertion follows. In addition, by using the equality conditions in Lemma 3.4.1 and [31, Proposition 1.2.3],

$$\begin{aligned} \widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}_\chi(\overline{D}) &\iff \deg(\overline{N}^2) = 0 \\ &\iff \phi_p \ (\forall p \in M_K) \text{ and } \phi_\infty \text{ are constant functions} \\ &\iff \overline{D} \text{ is relatively nef,} \end{aligned}$$

as required. $\square$

As a consequence of the above theorem, we have the Hodge index theorem for adelic arithmetic divisors.

Corollary 7.1.2. *Let $\overline{D} = (D, g)$ be an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . If $\deg(D) = 0$, then $\deg(\overline{D}^2) \leq 0$. Moreover, the equality holds if and only if $\overline{D} = (\widehat{\psi})_{\mathbb{R}} + (0, \lambda[\infty])$ for some $\psi \in \text{Rat}(X)_{\mathbb{R}}^\times$ and $\lambda \in \mathbb{R}$.*

Proof. By Theorem 7.1.1, $\deg(\overline{D}^2) \leq \widehat{\text{vol}}_\chi(\overline{D}) \leq \widehat{\text{vol}}(\overline{D})$, so that it is sufficient to show that $\widehat{\text{vol}}(\overline{D}) = 0$ for the first assertion. Indeed, if $\widehat{\text{vol}}(\overline{D}) > 0$, then, by the continuity of the volume function (cf. Theorem 5.2.1), $\widehat{\text{vol}}(\overline{D} - (1/n) \cdot \overline{\mathcal{A}}^a) > 0$ for a sufficiently large n , where $\overline{\mathcal{A}}$ is an ample arithmetic Cartier divisor on some regular model of X . In particular, $\deg(D) \geq (1/n) \deg(\mathcal{A} \cap X) > 0$, which is a contradiction.

Next we consider the equality condition. Clearly if $\overline{D} = (\widehat{\psi})_{\mathbb{R}} + (0, \lambda[\infty])$ for some $\psi \in \text{Rat}(X)_{\mathbb{R}}^\times$ and $\lambda \in \mathbb{R}$, then $\widehat{\deg}(\overline{D}^2) = 0$, so that we assume that $\deg(\overline{D}^2) = 0$. Let $\mathcal{X}$ be a regular model of X over $\text{Spec}(O_K)$.

Claim 7.1.2.1. *There is an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{D} \cap X = D$ and $(\mathcal{D} \cdot C) = 0$ for any vertical curves C on $\mathcal{X}$.*

Proof. Let U be a non-empty Zariski open set of $\text{Spec}(O_K)$ such that $\mathcal{X} \rightarrow \text{Spec}(O_K)$ is smooth over U . Let $\mathcal{D}_0$ be the Zariski closure of D in $\mathcal{X}$. Then the degree of $\mathcal{D}_0$ along every smooth fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ is zero. Moreover, using Zariski's lemma, for each $\mathfrak{p} \in M_K \setminus U$, there is a vertical $\mathbb{R}$ -Cartier divisor $Z_{\mathfrak{p}}$ in the fiber $F_{\mathfrak{p}}$ over $\mathfrak{p}$ such that the degree of $\mathcal{D}_0 + Z_{\mathfrak{p}}$ along every irreducible component of $F_{\mathfrak{p}}$ is zero. Therefore, if we set $\mathcal{D} = \mathcal{D}_0 + \sum_{\mathfrak{p} \in M_K \setminus U} Z_{\mathfrak{p}}$, we have our desired $\mathbb{R}$ -Cartier divisor. $\square$

We set $\overline{D}' := (\mathcal{D}, g_{\infty})^a$, $\phi_{\mathfrak{p}} := g_{\mathfrak{p}} - g_{(\mathcal{X}_{(\mathfrak{p})}, \mathcal{D}_{(\mathfrak{p})})}$ and $\phi := \sum_{\mathfrak{p} \in M_K} \phi_{\mathfrak{p}}[\mathfrak{p}]$, where $\mathcal{X}_{(\mathfrak{p})}$ is the localization of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ at $\mathfrak{p}$ and $\mathcal{D}_{(\mathfrak{p})}$ is the restriction of $\mathcal{D}$ to $\mathcal{X}_{(\mathfrak{p})}$. Then $\overline{D} = \overline{D}' + (0, \phi)$. Note that $\widehat{\deg}(\overline{D}' \cdot (0, \phi)) = 0$ because $(\mathcal{D} \cdot C) = 0$ for any vertical curve C in $\mathcal{X}$, so that

$$0 = \widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{D}'^2) + \widehat{\deg}((0, \phi)^2).$$

Therefore,

$$\widehat{\deg}(\overline{D}'^2) = \widehat{\deg}((0, \phi)^2) = 0$$

because $\widehat{\deg}(\overline{D}'^2) \leq 0$ and $\widehat{\deg}((0, \phi)^2) \leq 0$ by the first assertion. As

$$\widehat{\deg}(\overline{D}'^2) = \widehat{\deg}((\mathcal{D}, g_{\infty})^2) = 0,$$

by virtue of [32, Lemma 4.1],

$$\overline{D}' = (\widehat{\psi'})_{\mathbb{R}} + (0, \eta'[\infty])$$

for some $\psi' \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and an F_{∞} -invariant locally constant function η' on $X(\mathbb{C})$. Moreover, by using the fact $\widehat{\deg}((0, \phi)^2) = 0$ together with Lemma 3.4.1, $\phi_{\mathfrak{p}}$ is a constant for all $\mathfrak{p} \in M_K$. Note that $\phi_{\mathfrak{p}} = 0$ except finitely many $\mathfrak{p} \in M_K$. In addition, for each $\mathfrak{p} \in M_K$, there is $f_{\mathfrak{p}} \in K^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ such that $(f_{\mathfrak{p}}) = \mathfrak{p}$ on $\text{Spec}(O_K)$. Therefore,

$$\overline{D} = (\widehat{\psi''})_{\mathbb{R}} + (0, \eta''[\infty])$$

for some $\psi'' \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and an F_{∞} -invariant locally constant function η'' on $X(\mathbb{C})$. Thus, by using Dirichlet's unit theorem, we have the second assertion of the corollary (cf. [32, Proof of Lemma 4.1]). $\square$

7.2. Arithmetic asymptotic multiplicity. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let

$$\text{Rat}(X)_{\mathbb{K}}^{\times} := \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K},$$

and let

$$(\cdot)_{\mathbb{K}} : \text{Rat}(X)_{\mathbb{K}}^{\times} \rightarrow \text{Div}(X)_{\mathbb{K}} \quad \text{and} \quad (\widehat{\cdot})_{\mathbb{K}} : \text{Rat}(X)_{\mathbb{K}}^{\times} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$$

be the natural extensions of the homomorphisms

$$\text{Rat}(X)^{\times} \rightarrow \text{Div}(X) \quad \text{and} \quad \text{Rat}(X)^{\times} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$$

given by $\phi \mapsto (\phi)$ and $\phi \mapsto (\widehat{\phi})$, respectively. Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type. We define $\widehat{\Gamma}_{\mathbb{K}}^{\times}(X, \overline{D})$ to be

$$\widehat{\Gamma}_{\mathbb{K}}^{\times}(X, \overline{D}) := \left\{ \phi \in \text{Rat}(X)_{\mathbb{K}}^{\times} \mid \overline{D} + (\widehat{\phi})_{\mathbb{K}} \geq (0, 0) \right\}.$$

For $\xi \in X$, the $\mathbb{K}$ -asymptotic multiplicity of $\overline{D}$ at ξ is defined to be

$$\mu_{\mathbb{K},\xi}(\overline{D}) := \begin{cases} \inf\{\text{mult}_\xi(D + (\phi)_\mathbb{K}) \mid \phi \in \widehat{\Gamma}_\mathbb{K}^\times(X, \overline{D})\} & \text{if } \widehat{\Gamma}_\mathbb{K}^\times(X, \overline{D}) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 7.2.1. *Let $\overline{D}$ and $\overline{E}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then we have the following:*

- (1) $\mu_{\mathbb{K},\xi}(\overline{D} + \overline{E}) \leq \mu_{\mathbb{K},\xi}(\overline{D}) + \mu_{\mathbb{K},\xi}(\overline{E})$.
- (2) If $\overline{D} \leq \overline{E}$, then $\mu_{\mathbb{K},\xi}(\overline{E}) \leq \mu_{\mathbb{K},\xi}(\overline{D}) + \text{mult}_\xi(E - D)$.
- (3) $\mu_{\mathbb{K},\xi}(\overline{D} + (\widehat{\phi})_\mathbb{K}) = \mu_{\mathbb{K},\xi}(\overline{D})$ for $\phi \in \text{Rat}(X)_\mathbb{K}^\times$.
- (4) $\mu_{\mathbb{K},\xi}(a\overline{D}) = a\mu_{\mathbb{K},\xi}(\overline{D})$ for $a \in \mathbb{K}_{\geq 0}$.
- (5) $0 \leq \mu_{\mathbb{R},\xi}(\overline{D}) \leq \mu_{\mathbb{Q},\xi}(\overline{D})$.
- (6) If $\overline{D}$ is big, then $\mu_{\mathbb{R},\xi}(\overline{D}) = \mu_{\mathbb{Q},\xi}(\overline{D})$.
- (7) If $\overline{D}$ is nef and big, then $\mu_{\mathbb{K},\xi}(\overline{D}) = 0$.

Proof. (1), (2), (3), (4) and (5) can be proved in the same way as [29, Proposition 2.1]. For the proofs for (6) and (7), let us begin with the following claim:

Claim 7.2.1.1. *Let $\overline{D}_1, \dots, \overline{D}_r$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $\overline{D}$ is big, then*

$$\lim_{\substack{(x_1, \dots, x_r) \rightarrow (0, \dots, 0) \\ (x_1, \dots, x_r) \in \mathbb{K}^r}} \mu_{\mathbb{K},\xi}(\overline{D} + x_1\overline{D}_1 + \dots + x_r\overline{D}_r) = \mu_{\mathbb{K},\xi}(\overline{D}).$$

Proof. Here we define $f : \mathbb{K}^r \rightarrow \mathbb{R} \cup \{\infty\}$ to be $f(x) := \mu_{\mathbb{K},\xi}(\overline{D}_x)$ for $x = (x_1, \dots, x_r) \in \mathbb{K}^r$, where

$$\overline{D}_x = \overline{D} + x_1\overline{D}_1 + \dots + x_r\overline{D}_r.$$

Note that f is a convex function over $\mathbb{K}$, that is,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \mathbb{K}^r$ and $t \in [0, 1] \cap \mathbb{K}$. Indeed, by using (1) and (4),

$$\begin{aligned} f(tx + (1-t)y) &= \mu_{\mathbb{K},\xi}(t\overline{D}_x + (1-t)\overline{D}_y) \leq \mu_{\mathbb{K},\xi}(t\overline{D}_x) + \mu_{\mathbb{K},\xi}((1-t)\overline{D}_y) \\ &= tf(x) + (1-t)f(y). \end{aligned}$$

Moreover, by virtue of the continuity of the volume function, there is a positive rational number c such that $\overline{D}_x$ is big for all $x \in (-c, c)^r \cap \mathbb{K}^r$. Thus, if $\mathbb{K} = \mathbb{R}$, then the assertion follows from the continuity of concave functions. In the case where $\mathbb{K} = \mathbb{Q}$, by [28, Proposition 1.3.1], there is a continuous function $\tilde{f} : (-c, c)^r \rightarrow \mathbb{R}$ such that $\tilde{f} = f$ on $(-c, c)^r \cap \mathbb{Q}^r$. Thus the assertion follows. $\square$

(6) Let $\phi \in \widehat{\Gamma}_\mathbb{R}^\times(X, \overline{D})$, that is, $\phi = \phi_1^{a_1} \dots \phi_r^{a_r}$ and $\overline{D} + a_1(\widehat{\phi}_1) + \dots + a_r(\widehat{\phi}_r) \geq 0$ for some $\phi_1, \dots, \phi_r \in \text{Rat}(X)^\times$ and $a_1, \dots, a_r \in \mathbb{R}$. By using [30, Lemma 5.2.3 and Lemma 5.2.4], for each i , we can find effective adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{A}_i$ and $\overline{B}_i$ of C^0 -type on X such that $(\widehat{\phi}_i) = \overline{A}_i - \overline{B}_i$. For $n \in \mathbb{Z}_{>0}$, we choose $x_{i,n} \in \mathbb{R}$ and $x'_{i,n} \in \mathbb{Q}$ such that $a_i + x_{i,n} \in \mathbb{Q}$ and $0 \leq x_{i,n} \leq x'_{i,n} \leq 1/n$. Then

$$\begin{aligned} \overline{D} + \sum_i x'_{i,n} \overline{B}_i + \sum_i (a_i + x_{i,n})(\widehat{\phi}_i) &\geq \overline{D} + \sum_i x_{i,n} \overline{B}_i + \sum_i (a_i + x_{i,n})(\widehat{\phi}_i) \\ &= \overline{D} + \sum_i a_i(\widehat{\phi}_i) + \sum_i x_{i,n} \overline{A}_i \geq 0, \end{aligned}$$

and hence

$$\mu_{\mathbb{Q},\xi}\left(\overline{D} + \sum_i x'_{i,n} \overline{B}_i\right) \leq \text{mult}_\xi\left(D + \sum_i x'_{i,n} B_i + \sum_i (a_i + x_{i,n})(\phi_i)\right).$$

On the other hand, by using Claim 7.2.1.1,

$$\lim_{n \rightarrow \infty} \mu_{\mathbb{Q},\xi}\left(\overline{D} + \sum_i x'_{i,n} \overline{B}_i\right) = \mu_{\mathbb{Q},\xi}(\overline{D})$$

and

$$\lim_{n \rightarrow \infty} \text{mult}_\xi\left(D + \sum_i x'_{i,n} B_i + \sum_i (a_i + x_{i,n})(\phi_i)\right) = \text{mult}_\xi\left(D + \sum_i a_i(\phi_i)\right).$$

Thus

$$\mu_{\mathbb{Q},\xi}(\overline{D}) \leq \text{mult}_\xi\left(D + \sum_i a_i(\phi_i)\right),$$

which yields $\mu_{\mathbb{Q},\xi}(\overline{D}) \leq \mu_{\mathbb{R},\xi}(\overline{D})$, so that (6) follows from (5).

(7) By (5), it is sufficient to see that $\mu_{\mathbb{Q},\xi}(\overline{D}) = 0$. We set

$$\overline{D} = (D, \{g_p\}_{p \in M_K} \cup \{g_\infty\}).$$

Let U be a non-empty open set $\text{Spec}(O_K)$ such that $\overline{D}$ has a defining model over U . For each $n \in \mathbb{Z}_{>0}$, by Proposition 4.4.2, there is a normal model $\mathcal{X}$ of X and a relatively nef $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{D} \cap X = D$ and

$$(\mathcal{D}, g_\infty)^a - (1/n) \left(0, \sum_{p \in M_K \setminus U} [p]\right) \leq \overline{D} \leq (\mathcal{D}, g_\infty)^a.$$

Note that $(\mathcal{D}, g_\infty)$ is nef and big, so that, by (2) and [29, Proposition 2.1, (6)],

$$0 \leq \mu_{\mathbb{Q},\xi}\left(\overline{D} + (1/n) \left(0, \sum_{p \in M_K \setminus U} [p]\right)\right) \leq \mu_{\mathbb{Q},\xi}((\mathcal{D}, g_\infty)) = 0,$$

and hence $\mu_{\mathbb{Q},\xi}(\overline{D} + (1/n) \left(0, \sum_{p \in M_K \setminus U} [p]\right)) = 0$. Further, by Claim 7.2.1.1,

$$\mu_{\mathbb{Q},\xi}(\overline{D}) = \lim_{n \rightarrow \infty} \mu_{\mathbb{Q},\xi}\left(\overline{D} + (1/n) \left(0, \sum_{p \in M_K \setminus U} [p]\right)\right),$$

so that (7) follows. $\square$

Lemma 7.2.2. *Let $\overline{D}$ and $\overline{D}_0$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $\overline{D}_0$ is big, then*

$$\lim_{t \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0))$$

exists in $\mathbb{R}$.

Proof. We can find $t_0 \in \mathbb{R}$ such that $\overline{D} + t\overline{D}_0$ is big for all $t \geq t_0$. Here we set

$$f(t) = \mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0)$$

for $t \geq t_0$. It is sufficient to show that the function f is monotonically decreasing and bounded below on $\mathbb{R}_{\geq t_0}$, that is,

$$f(t') \leq f(t) \quad (\forall t' \geq t \geq t_0) \quad \text{and} \quad f(t) \geq M \quad (\forall t \geq t_0)$$

for some constant M .

As $\overline{D} + t'\overline{D}_0 = \overline{D} + t\overline{D}_0 + (t' - t)\overline{D}_0$, by (1) in Proposition 7.2.1,

$$\mu_{\mathbb{R},\xi}(\overline{D} + t'\overline{D}_0) \leq \mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}_0) + \mu_{\mathbb{R},\xi}((t' - t)\overline{D}_0),$$

so that the first assertion follows from (4) in Proposition 7.2.1.

We choose $a \in \mathbb{R}$ and $\psi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ with $a\overline{D}_0 + (\widehat{\psi}) - \overline{D} \geq 0$. Then, by (2) and (3) in Proposition 7.2.1,

$$\begin{aligned} \mu_{\mathbb{R},\xi}((a+t)\overline{D}_0) &= \mu_{\mathbb{R},\xi}((\overline{D} + t\overline{D}_0) + (a\overline{D}_0 + (\widehat{\psi}) - \overline{D})) \\ &\leq \mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}_0) + \mu_{\mathbb{R},\xi}(a\overline{D}_0 + (\widehat{\psi}) - \overline{D}). \end{aligned}$$

Therefore, we have

$$f(t) \geq \mu_{\mathbb{R},\xi}(a\overline{D}_0) - \mu_{\mathbb{R},\xi}(a\overline{D}_0 + (\widehat{\psi}) - \overline{D}),$$

as required. $\square$

Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . By the above lemma, we define the *extended arithmetic asymptotic multiplicity* $\nu_{\xi}(\overline{D}) \in \mathbb{R} \cup \{\infty\}$ to be

$$\nu_{\xi}(\overline{D}) := \sup_{\overline{D}_0 \in \text{Big}(X)_{\mathbb{R}}} \lim_{t \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0)),$$

where $\text{Big}(X)_{\mathbb{R}}$ is the cone consisting of big adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then we have the following:

Proposition 7.2.3. *Let $\overline{D}$ and $\overline{E}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then we have the following:*

- (1) $\nu_{\xi}(\overline{D}) \leq \mu_{\mathbb{R},\xi}(\overline{D})$. Moreover, If $\overline{D}$ is big, then $\nu_{\xi}(\overline{D}) = \mu_{\mathbb{R},\xi}(\overline{D})$.
- (2) $\nu_{\xi}(\overline{D} + \overline{E}) \leq \nu_{\xi}(\overline{D}) + \nu_{\xi}(\overline{E})$.
- (3) If $\overline{D} \leq \overline{E}$, then $\nu_{\xi}(\overline{E}) \leq \nu_{\xi}(\overline{D}) + \text{mult}_{\xi}(E - D)$.
- (4) $\nu_{\xi}(\overline{D} + (\widehat{\phi})_{\mathbb{R}}) = \nu_{\xi}(\overline{D})$ for $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$.
- (5) $\nu_{\xi}(a\overline{D}) = a \nu_{\xi}(\overline{D})$ for $a \in \mathbb{R}_{\geq 0}$.
- (6) If $\overline{D}$ is nef, then $\nu_{\xi}(\overline{D}) = 0$.

Proof. (1) For $\overline{D}_0 \in \text{Big}(X)_{\mathbb{R}}$, by (1) and (4) in Proposition 7.2.1,

$$\mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0) \leq \mu_{\mathbb{R},\xi}(\overline{D}) + \mu_{\mathbb{R},\xi}(t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0) = \mu_{\mathbb{R},\xi}(\overline{D}),$$

which implies the first assertion. If $\overline{D}$ is big, then

$$\mu_{\mathbb{R},\xi}(\overline{D}) = \lim_{t \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{D} + t\overline{D}) - \mu_{\mathbb{R},\xi}(t\overline{D})) \leq \nu_{\xi}(\overline{D}),$$

as required.

(2) Let $\overline{D}_0$ be a big adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Then, by (1) in Proposition 7.2.1,

$$\begin{aligned} \mu_{\mathbb{R},\xi}(\overline{D} + \overline{E} + t\overline{D}_0) &= \mu_{\mathbb{R},\xi}((\overline{D} + (t/2)\overline{D}_0) + (\overline{E} + (t/2)\overline{D}_0)) \\ &\leq \mu_{\mathbb{R},\xi}(\overline{D} + (t/2)\overline{D}_0) + \mu_{\mathbb{R},\xi}(\overline{E} + (t/2)\overline{D}_0), \end{aligned}$$

which implies

$$\begin{aligned} \lim_{t \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{D} + \overline{E} + t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0)) \\ \leq \lim_{t \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{D} + (t/2)\overline{D}_0) - \mu_{\mathbb{R},\xi}((t/2)\overline{D}_0)) \\ + \lim_{t \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{E} + (t/2)\overline{D}_0) - \mu_{\mathbb{R},\xi}((t/2)\overline{D}_0)) \leq \nu_{\xi}(\overline{D}) + \nu_{\xi}(\overline{E}). \end{aligned}$$

Therefore, (2) follows.

(3) By using (1) and (2),

$$\begin{aligned} \nu_\xi(\overline{E}) &= \nu_\xi(\overline{D} + (\overline{E} - \overline{D})) \leq \nu_\xi(\overline{D}) + \nu_\xi(\overline{E} - \overline{D}) \\ &\leq \nu_\xi(\overline{D}) + \mu_{\mathbb{R},\xi}(\overline{E} - \overline{D}) \leq \nu_\xi(\overline{D}) + \text{mult}_\xi(E - D). \end{aligned}$$

(4) follows from (3) in Proposition 7.2.1.

(5) Clearly we may assume that $a > 0$. Thus, for $\overline{D}_0 \in \text{Big}(X)_{\mathbb{R}}$, by (4) in Proposition 7.2.1,

$$\mu_{\mathbb{R},\xi}(a\overline{D} + t\overline{D}_0) - \mu_{\mathbb{R},\xi}(t\overline{D}_0) = a(\mu_{\mathbb{R},\xi}(\overline{D} + (t/a)\overline{D}_0) - \mu_{\mathbb{R},\xi}((t/a)\overline{D}_0)),$$

so that the assertion follows.

(6) By the following Lemma 7.2.4, $\mu_{\mathbb{R},\xi}(\overline{D} + \overline{D}_0) \leq \mu_{\mathbb{R},\xi}(\overline{D}_0)$ for all big adelic arithmetic $\mathbb{R}$ -Carrier divisors $\overline{D}_0$ of C^0 -type on X , so that $\nu_\xi(\overline{D}) \leq 0$. On the other hand, if $\overline{D}_0$ is nef and big, then $\mu_{\mathbb{R},\xi}(\overline{D} + \overline{D}_0) = \mu_{\mathbb{R},\xi}(\overline{D}_0) = 0$. Thus (6) follows. $\square$

Lemma 7.2.4. *Let $\overline{P}$ and $\overline{D}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . If $\overline{P}$ is nef and $\overline{D}$ is big, then $\mu_{\mathbb{R},\xi}(\overline{P} + \overline{D}) \leq \mu_{\mathbb{R},\xi}(\overline{D})$.*

Proof. First we suppose that there exist a model $\mathcal{X}$ of X over $\text{Spec}(O_K)$ and an arithmetic $\mathbb{Q}$ -Cartier divisor $\overline{\mathcal{D}}$ on $\mathcal{X}$ such that $\overline{\mathcal{D}}^a = \overline{D}$. By (6) in Proposition 7.2.1,

$$\mu_{\mathbb{R},\xi}(\overline{D}) = \inf \left\{ \text{mult}_\xi(D + (\phi)) \mid \phi \in \hat{\Gamma}_{\mathbb{Q}}^\times(X, \overline{D}) \right\}.$$

Fix a positive number ϵ . We choose $\phi \in \hat{\Gamma}_{\mathbb{Q}}^\times(X, \overline{D})$ such that

$$\text{mult}_\xi(D + (\phi)) \leq \mu_{\mathbb{R},\xi}(\overline{D}) + \epsilon.$$

By using Theorem 5.1.7 or [28, Theorem 2.4], there are a birational morphism $h : \mathcal{Y} \rightarrow \mathcal{X}$ of normal projective arithmetic varieties and a nef and big arithmetic $\mathbb{Q}$ -Cartier divisor $\overline{\mathcal{Q}}$ of C^0 -type on $\mathcal{Y}$ such that $\overline{\mathcal{Q}} + (\widehat{\phi}) \geq 0$ and $\overline{\mathcal{Q}} \leq \mu^*(\overline{\mathcal{D}})$. If we set $\overline{\mathcal{N}} = \mu^*(\overline{\mathcal{D}}) - \overline{\mathcal{Q}}$, then

$$\begin{aligned} \text{mult}_\xi(\overline{\mathcal{N}}) &\leq \text{mult}_\xi(\overline{\mathcal{Q}} + (\phi) + \overline{\mathcal{N}}) = \text{mult}_\xi(h^*(\overline{\mathcal{D}}) + (\phi)) \\ &= \text{mult}_\xi(D + (\phi)) \leq \mu_{\mathbb{R},\xi}(\overline{D}) + \epsilon. \end{aligned}$$

Thus, by using (1) and (7) in Proposition 7.2.1 together with the above inequality, we have

$$\begin{aligned} \mu_{\mathbb{R},\xi}(\overline{P} + \overline{D}) &= \mu_{\mathbb{R},\xi}(h^*(\overline{P}) + \overline{\mathcal{Q}}^a + \overline{\mathcal{N}}^a) \leq \mu_{\mathbb{R},\xi}(h^*(\overline{P}) + \overline{\mathcal{Q}}^a) + \mu_{\mathbb{R},\xi}(\overline{\mathcal{N}}^a) \\ &\leq \text{mult}_\xi(\overline{\mathcal{N}}) \leq \mu_{\mathbb{R},\xi}(\overline{D}) + \epsilon, \end{aligned}$$

and hence the assertion follows.

Let U be a non-empty open set $\text{Spec}(O_K)$ such that $\overline{D}$ has a defining model over U . We set $S = \text{Spec}(O_K) \setminus U$. Then, for any positive number ϵ , by Theorem 4.1.3, there is a model $(\mathcal{X}, \overline{\mathcal{D}})$ of $(X, \overline{D})$ over $\text{Spec}(O_K)$ such that

$$\overline{D} - \left(0, \sum_{\mathfrak{p} \in S} \epsilon[\mathfrak{p}] \right) \leq \overline{\mathcal{D}}^a \leq \overline{D}.$$

By virtue of [30, Lemma 5.2.3, Lemma 5.2.4], there are effective arithmetic $\mathbb{Q}$ -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r$ of C^0 -type on $\mathcal{X}$ and real numbers $a_1, \dots, a_r$ such that

$$\overline{\mathcal{D}} = a_1 \overline{\mathcal{D}}_1 + \dots + a_r \overline{\mathcal{D}}_r.$$

For each $i = 1, \dots, r$, let $\{a_{i,n}\}_{n=1}^\infty$ be a sequence of rational numbers such that $a_{i,n} \leq a_i$ and $\lim_{n \rightarrow \infty} a_{i,n} = a_i$. We set $\overline{\mathcal{D}}_n = a_{1,n}\overline{\mathcal{D}}_1 + \dots + a_{r,n}\overline{\mathcal{D}}_r$. By (1) and (2) in Proposition 7.2.1 together with the previous observation,

$$\begin{aligned} \mu_{\mathbb{R},\xi}(\overline{P} + \overline{D}) &= \mu_{\mathbb{R},\xi}(\overline{P} + \overline{\mathcal{D}}^a + \overline{D} - \overline{\mathcal{D}}^a) \leq \mu_{\mathbb{R},\xi}(\overline{P} + \overline{\mathcal{D}}^a) + \mu_{\mathbb{R},\xi}(\overline{D} - \overline{\mathcal{D}}^a) \\ &\leq \mu_{\mathbb{R},\xi}(\overline{P} + \overline{\mathcal{D}}^a) = \mu_{\mathbb{R},\xi}(\overline{P} + \overline{\mathcal{D}}_n^a + \overline{\mathcal{D}} - \overline{\mathcal{D}}_n^a) \\ &\leq \mu_{\mathbb{R},\xi}(\overline{P} + \overline{\mathcal{D}}_n^a) + \text{mult}_\xi(\overline{\mathcal{D}} - \overline{\mathcal{D}}_n) \leq \mu_{\mathbb{R},\xi}(\overline{\mathcal{D}}_n^a) + \text{mult}_\xi(\overline{\mathcal{D}} - \overline{\mathcal{D}}_n) \end{aligned}$$

In addition, by Claim 7.2.1.1,

$$\lim_{n \rightarrow \infty} (\mu_{\mathbb{R},\xi}(\overline{\mathcal{D}}_n^a) + \text{mult}_\xi(\overline{\mathcal{D}} - \overline{\mathcal{D}}_n)) = \mu_{\mathbb{R},\xi}(\overline{\mathcal{D}}^a),$$

so that we have

$$\mu_{\mathbb{R},\xi}(\overline{P} + \overline{D}) \leq \mu_{\mathbb{R},\xi}(\overline{\mathcal{D}}^a) \leq \mu_{\mathbb{R},\xi}\left(\overline{D} - \left(0, \sum_{p \in S} \epsilon[p]\right)\right).$$

Therefore, the assertion follows from Claim 7.2.1.1. $\square$

7.3. Necessary condition for the equality $\widehat{\text{vol}} = \widehat{\text{vol}}_\chi$. Let $\overline{K}$ be an algebraic closure of K , $X_{\overline{K}} := X \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$ and $d = \dim X$.

We fix a monomial order $\lesssim$ on $\mathbb{Z}_{\geq 0}^d$, that is, $\lesssim$ is a total ordering relation on $\mathbb{Z}_{\geq 0}^d$ with the following properties:

- (a) $(0, \dots, 0) \lesssim A$ for all $A \in \mathbb{Z}_{\geq 0}^d$.
- (b) If $A \lesssim B$ for $A, B \in \mathbb{Z}_{\geq 0}^d$, then $A + C \lesssim B + C$ for all $C \in \mathbb{Z}_{\geq 0}^d$.

The monomial order $\lesssim$ on $\mathbb{Z}_{\geq 0}^d$ extends uniquely to a totally ordering relation $\lesssim$ on $\mathbb{Z}^d$ such that $A + C \lesssim B + C$ holds for all $A, B, C \in \mathbb{Z}^d$ with $A \lesssim B$.

Let $z_P = (z_1, \dots, z_d)$ be a local system of parameters of $\mathcal{O}_{X_{\overline{K}},P}$ at $P \in X(\overline{K})$. Note that the completion $\widehat{\mathcal{O}}_{X_{\overline{K}},P}$ of $\mathcal{O}_{X_{\overline{K}},P}$ with respect to the maximal ideal of $\mathcal{O}_{X_{\overline{K}},P}$ is naturally isomorphic to $\overline{K}[[z_1, \dots, z_d]]$. Thus, for $f \in \mathcal{O}_{X_{\overline{K}},P}$, we can put

$$f = \sum_{(a_1, \dots, a_d) \in \mathbb{Z}_{\geq 0}^d} c_{(a_1, \dots, a_d)} z_1^{a_1} \cdots z_d^{a_d}, \quad (c_{(a_1, \dots, a_d)} \in \overline{K}).$$

We define $\text{ord}_{z_P}^{\lesssim}(f)$ to be

$$\text{ord}_{z_P}^{\lesssim}(f) := \begin{cases} \min_{\lesssim} \{(a_1, \dots, a_d) \mid c_{(a_1, \dots, a_d)} \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{otherwise,} \end{cases}$$

which gives rise to a rank d valuation, that is, the following properties are satisfied:

- (i) $\text{ord}_{z_P}^{\lesssim}(fg) = \text{ord}_{z_P}^{\lesssim}(f) + \text{ord}_{z_P}^{\lesssim}(g)$ for $f, g \in \mathcal{O}_{X_{\overline{K}},P}$.
- (ii) $\min\{\text{ord}_{z_P}^{\lesssim}(f), \text{ord}_{z_P}^{\lesssim}(g)\} \lesssim \text{ord}_{z_P}^{\lesssim}(f+g)$ for $f, g \in \mathcal{O}_{X_{\overline{K}},P}$.

By property (i), $\text{ord}_{z_P}^{\lesssim} : \mathcal{O}_{X_{\overline{K}},P} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}^d$ has the natural extension

$$\text{ord}_{z_P}^{\lesssim} : \text{Rat}(X_{\overline{K}})^\times \rightarrow \mathbb{Z}^d$$

given by $\text{ord}_{z_P}^{\lesssim}(f/g) = \text{ord}_{z_P}^{\lesssim}(f) - \text{ord}_{z_P}^{\lesssim}(g)$. Note that this extension also satisfies the same properties (i) and (ii) as before. Since $\text{ord}_{z_P}^{\lesssim}(u) = (0, \dots, 0)$ for all $u \in \mathcal{O}_{X_{\overline{K}},P}^\times$, $\text{ord}_{z_P}^{\lesssim}$

induces $\text{Rat}(X_{\bar{K}})^{\times} / \mathcal{O}_{X_{\bar{K}}, P}^{\times} \rightarrow \mathbb{Z}^d$. The composition of homomorphisms

$$\text{Div}(X_{\bar{K}}) \xrightarrow{\alpha_P} \text{Rat}^{\times}(X_{\bar{K}}) / \mathcal{O}_{X_{\bar{K}}, P}^{\times} \xrightarrow{\text{ord}_{z_P}^{\sim}} \mathbb{Z}^d$$

is denoted by $\text{mult}_{z_P}^{\sim}$, where $\alpha_P : \text{Div}(X_{\bar{K}}) \rightarrow \text{Rat}(X_{\bar{K}})^{\times} / \mathcal{O}_{X_{\bar{K}}, P}^{\times}$ is the natural homomorphism. Moreover, the homomorphism $\text{mult}_{z_P}^{\sim} : \text{Div}(X_{\bar{K}}) \rightarrow \mathbb{Z}^d$ gives rise to the natural extension $\text{Div}(X_{\bar{K}}) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}^d$ over $\mathbb{R}$. By abuse of notation, the above extension is also denoted by $\text{mult}_{z_P}^{\sim}$.

Let $\bar{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let $V_{\bullet} = \bigoplus_{m \geq 0} V_m$ be a graded subalgebra of $R(D) := \bigoplus_{m \geq 0} H^0(X, mD)$ over K . The Okounkov body $\Delta(V_{\bullet})$ of $V_{\bullet}$ is defined by the closed convex hull of

$$\bigcup_{m > 0} \left\{ \text{mult}_{z_P}^{\sim}(D_{\bar{K}} + (1/m)(\phi)) \in \mathbb{R}_{\geq 0}^d \mid \phi \in (V_m \otimes_K \bar{K}) \setminus \{0\} \right\}.$$

For $t \in \mathbb{R}$, let $V_{\bullet}^t$ be the graded subalgebra of $V_{\bullet}$ given by

$$V_{\bullet}^t := \bigoplus_{m \geq 0} \left\langle V_m \cap \hat{H}^0(X, m(\bar{D} + (0, -2t[\infty]))) \right\rangle_K,$$

where $\left\langle V_m \cap \hat{H}^0(X, m(\bar{D} + (0, -2t[\infty]))) \right\rangle_K$ means the subspace of V_m generated by

$$V_m \cap \hat{H}^0(X, m(\bar{D} + (0, -2t[\infty])))$$

over K . We define $G_{(\bar{D}; V_{\bullet})} : \Delta(V_{\bullet}) \rightarrow \mathbb{R} \cup \{-\infty\}$ to be

$$G_{(\bar{D}; V_{\bullet})}(x) := \begin{cases} \sup \{ t \in \mathbb{R} \mid x \in \Delta(V_{\bullet}^t) \} & \text{if } x \in \Delta(V_{\bullet}^t) \text{ for some } t, \\ -\infty & \text{otherwise.} \end{cases}$$

Note that $G_{(\bar{D}; V_{\bullet})}$ is an upper semicontinuous concave function (cf. [5, SubSection 1.3]).

We also define $\widehat{\text{vol}}(\bar{D}; V_{\bullet})$ and $\widehat{\text{vol}}_{\chi}(\bar{D}; V_{\bullet})$ to be

$$\begin{cases} \widehat{\text{vol}}(\bar{D}; V_{\bullet}) := \limsup_{m \rightarrow \infty} \frac{\# \log(V_m \cap \hat{H}^0(X, m\bar{D}))}{m^{d+1}/(d+1)!}, \\ \widehat{\text{vol}}_{\chi}(\bar{D}; V_{\bullet}) := \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(V_m \cap H^0(X, mD), \|\cdot\|_{m\bar{D}})}{m^{d+1}/(d+1)!}. \end{cases}$$

Let $\Theta(\bar{D}; V_{\bullet})$ be the closure of

$$\{x \in \Delta(V_{\bullet}) \mid G_{(\bar{D}; V_{\bullet})}(x) > 0\}.$$

We assume that $V_{\bullet}$ contains an ample series, that is, $V_m \neq \{0\}$ for $m \gg 1$ and there is an ample $\mathbb{Q}$ -Cartier divisor A on X with the following properties:

- $A \leq D$.
- There is a positive integer m_0 such that $H^0(X, mm_0 A) \subseteq V_{mm_0}$ for all $m \geq 1$.

Then, in the similar way as [5, Theorem 2.8], [5, Theorem 3.1] and [32, Section 3], we have the following integral formulae:

$$(7.3.1) \quad \widehat{\text{vol}}(\bar{D}; V_{\bullet}) = (d+1)! [K : \mathbb{Q}] \int_{\Theta(\bar{D}; V_{\bullet})} G_{(\bar{D}; V_{\bullet})}(x) dx$$

and

$$(7.3.2) \quad \widehat{\text{vol}}_\chi(\overline{D}; V_\bullet) = (d+1)! [K : \mathbb{Q}] \int_{\Delta(V_\bullet)} G_{(\overline{D}; V_\bullet)}(x) dx.$$

Moreover, if $\widehat{\text{vol}}(\overline{D}; V_\bullet) > 0$, then the same argument as in [32, Lemma 3.3] yields

$$(7.3.3) \quad \Theta(\overline{D}; V_\bullet) = \Delta(V_\bullet^0) = \{x \in \Delta(V_\bullet) \mid G_{(\overline{D}; V_\bullet)}(x) \geq 0\}.$$

Therefore, in the same way as [32, Section 3], we have the following theorem (cf. [32, Theorem 3.4 and Corollary 3.5]).

Theorem 7.3.4. *We assume that D is nef and big and $\widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}_\chi(\overline{D}) > 0$. Then $\mu_{\mathbb{Q}, \xi}(\overline{D}) = 0$ for all $\xi \in X$.*

Besides it, the following theorem is also obtained:

Theorem 7.3.5. *Let $\overline{Q} = (Q, q)$ be a nef adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . We assume that $\overline{D}$ is big, $\overline{Q} \leq \overline{D}$ and $\widehat{\text{vol}}(\overline{Q}) = \widehat{\text{vol}}(\overline{D})$. If we set $N = D - Q$, then $\mu_{\mathbb{Q}, \xi}(\overline{D}) = \text{mult}_\xi(N)$ for all $\xi \in X$.*

Proof. By using (2) and (7) in Proposition 7.2.1, we have

$$\mu_{\mathbb{Q}, \xi}(\overline{D}) \leq \mu_{\mathbb{Q}, \xi}(\overline{Q}) + \text{mult}_\xi(N) = \text{mult}_\xi(N).$$

Let us consider the converse inequality. Let B be the Zariski closure of $\{\xi\}$ and P a regular closed point of B . Let $z_P = (z_1, \dots, z_d)$ be a local system of parameters of $\mathcal{O}_{X, P}$ such that B is given by $z_1 = \dots = z_r = 0$. We choose a monomial order $\preceq$ of $\mathbb{Z}_{\geq 0}^d$ such that $\ell(a) \leq \ell(b)$ for all $a, b \in \mathbb{Z}_{\geq 0}^d$ with $a \preceq b$, where $\ell(x_1, \dots, x_d) = x_1 + \dots + x_d$. We set $\nu := \text{mult}_{z_P}^\preceq(N)$. For simplicity, in the case $V_\bullet = R(D)$, we denote $\Delta(V_\bullet)$, $\Delta(V_\bullet^t)$, $G_{(\overline{D}; V_\bullet)}$ and $\Theta(\overline{D}; V_\bullet)$ by Δ_D , Δ_D^t , $G_{\overline{D}}$ and $\Theta_{\overline{D}}$, respectively. Let us see the following claim:

Claim 7.3.5.1. (1) $\Delta_Q^t + \nu \subseteq \Delta_D^t$ for $t \in \mathbb{R}$.

(2) $G_{\overline{Q}}(x) \leq G_{\overline{D}}(x + \nu)$ for $x \in \Delta_Q$.

(3) $\Theta_{\overline{Q}} + \nu \subseteq \Theta_{\overline{D}}$.

(4) $\min\{\ell(x) \mid x \in \Theta_{\overline{D}}\} \leq \mu_{\mathbb{Q}, \xi}(\overline{D})$.

Proof. (1) Let $\phi \in \langle \hat{H}^0(X, m(\overline{Q} + (0, -2t[\infty]))) \rangle_{\overline{K}} \setminus \{0\}$. Then

$$\text{mult}_{z_P}^\preceq(Q + (1/m)(\phi)) + \nu = \text{mult}_{z_P}^\preceq(D + (1/m)(\phi)),$$

which shows (1).

(2) Let t be a real number with $t < G_{\overline{Q}}(x)$. Then $x \in \Delta_Q^t \subseteq \Delta_D^t - \nu$ by (1), and hence $x + \nu \in \Delta_D^t$. Thus $t \leq G_{\overline{D}}(x + \nu)$, as required.

(3) follows because $G_{\overline{Q}}(x) > 0$ implies $G_{\overline{D}}(x + \nu) > 0$ by (2).

(4) Let $\phi \in \hat{H}^0(X, m\overline{D}) \setminus \{0\}$. Note that $\text{mult}_{z_P}^\preceq(D + (1/m)(\phi)) \in \Theta_{\overline{D}}$ and

$$\ell(\text{mult}_{z_P}^\preceq(D + (1/m)(\phi))) = \text{mult}_\xi(D + (1/m)(\phi)).$$

Thus $\min\{\ell(x) \mid x \in \Theta_{\overline{D}}\} \leq \text{mult}_\xi(D + (1/m)(\phi))$. Therefore, we have (4). $\square$

Since $\widehat{\text{vol}}(\overline{Q}) = \widehat{\text{vol}}(\overline{D})$, by using the integral formula (7.3.1) together with (2) and (3) in the above claim, we can see that $\Theta_{\overline{Q}} + \nu = \Theta_{\overline{D}}$. We choose $x_0 \in \Theta_{\overline{D}}$ such that

$\ell(x_0) = \min\{\ell(x) \mid x \in \Theta_{\overline{D}}\}$. Then there is $y_0 \in \Theta_{\overline{Q}}$ such that $y_0 + \nu = x_0$. As $\ell(y_0) \geq 0$ and $\ell(\nu) = \text{mult}_{\xi}(N)$, by using (4) in the above claim,

$$\mu_{\mathbb{Q},\xi}(\overline{D}) \geq \min\{\ell(x) \mid x \in \Theta_{\overline{D}}\} = \ell(x_0) = \ell(y_0) + \ell(\nu) \geq \ell(\nu) = \text{mult}_{\xi}(N),$$

as required. $\square$

Remark 7.3.6. By virtue of Theorem 7.3.5, we can generalize the necessary and sufficient condition for the existence of Zariski decompositions on arithmetic toric varieties proved in [4, Theorem 8.2] to the case of adelic arithmetic $\mathbb{R}$ -divisors.

7.4. Numerical characterization. We assume that $\dim X = 1$. The following theorem is the main result of this section.

Theorem 7.4.1. *Let $\overline{D}$ be an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor on X . Then $\overline{D}$ is nef if and only if $\overline{D}$ is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D})$ (cf. Theorem 0.4.5 in Introduction).*

Proof. We need to show that if $\overline{D}$ is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D})$, then $\overline{D}$ is nef because the converse follows from Proposition 4.4.2 and Theorem 7.1.1.

First we assume that $\overline{D}$ is big. Since

$$\deg(\overline{D}^2) \leq \widehat{\text{vol}}_{\chi}(\overline{D}) \leq \widehat{\text{vol}}(\overline{D})$$

by Theorem 7.1.1, we have $\deg(\overline{D}^2) = \widehat{\text{vol}}_{\chi}(\overline{D})$ and $\widehat{\text{vol}}_{\chi}(\overline{D}) = \widehat{\text{vol}}(\overline{D})$. Thus, by Theorem 7.1.1 and Theorem 7.3.4, $\overline{D}$ is relatively nef and $\mu_{\mathbb{Q},\xi}(\overline{D}) = 0$ for all $\xi \in X$. On the other hand, by Corollary 6.2.7, there is the greatest element $\overline{Q}$ of $\Upsilon(\overline{D})$. Thus, if we set $\overline{N} := \overline{D} - \overline{Q}$, then $\text{mult}_{\xi}(N) = \mu_{\mathbb{Q},\xi}(\overline{D}) = 0$ for all $\xi \in X$ by Theorem 7.3.5, which means that $N = 0$. Therefore $\widehat{\deg}(\overline{D}|_x) \geq 0$ for all closed point $x \in X$, and hence $\overline{D}$ is nef.

Next we suppose that $\deg(D) > 0$ and $\overline{D}$ is not big. In this case, $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D}) = 0$. Thus, for $\epsilon > 0$, by using Proposition 5.1.2,

$$\epsilon[K : \mathbb{Q}] \deg(D) = \widehat{\deg}((\overline{D} + (0, \epsilon[\infty]))^2) \leq \widehat{\text{vol}}(\overline{D} + (0, \epsilon[\infty])) \leq \epsilon[K : \mathbb{Q}] \deg(D).$$

Therefore, $\widehat{\deg}((\overline{D} + (0, \epsilon[\infty]))^2) = \widehat{\text{vol}}(\overline{D} + (0, \epsilon[\infty])) > 0$, so that, by the previous observation, $\overline{D} + (0, \epsilon[\infty])$ is nef, and hence $\overline{D}$ is also nef.

Finally we consider the case where $\deg(D) = 0$. Then $\widehat{\text{vol}}(\overline{D}) = 0$, so that $\widehat{\deg}(\overline{D}^2) = 0$. By Corollary 7.1.2, $\overline{D} = (\psi)_{\mathbb{R}} + (0, \lambda[\infty])$ for some $\psi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and $\lambda \in \mathbb{R}$. As $\overline{D}$ is pseudo-effective, by (3) in Proposition 4.5.4, $\widehat{\deg}(\overline{A} \cdot \overline{D}) \geq 0$ for any nef adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{A}$ of C^0 -type. Thus we can see that $\lambda \geq 0$, and hence $\overline{D}$ is nef. $\square$

Corollary 7.4.2. *Let $\overline{D}$ and $\overline{P}$ be adelic arithmetic $\mathbb{R}$ -Cartier divisors of C^0 -type on X . Then the following are equivalent:*

- (1) $\overline{P}$ is the greatest element of $\Upsilon(\overline{D})$, that is, $\overline{P} \in \Upsilon(\overline{D})$ and $\overline{L} \leq \overline{P}$ for all $\overline{L} \in \Upsilon(\overline{D})$.
- (2) $\overline{P}$ is an element of $\Upsilon(\overline{D})$ with the following property:

$$\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0 \quad \text{and} \quad \widehat{\deg}(\overline{B}^2) < 0$$

for all integrable adelic arithmetic $\mathbb{R}$ -Cartier divisors $\overline{B}$ of C^0 -type with $0 \leq \overline{B} \leq \overline{D} - \overline{P}$.

Proof. (1) $\implies$ (2) : By Corollary 6.2.7, $\widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}(\overline{P})$. Let $\overline{B}$ be an integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor $\overline{B}$ of C^0 -type with $0 \leq \overline{B} \leq \overline{D} - \overline{P}$. For $0 < \epsilon \leq 1$,

$$\widehat{\deg}((\overline{P} + \epsilon \overline{B})^2) \leq \widehat{\text{vol}}(\overline{P} + \epsilon \overline{B})$$

by Theorem 7.1.1. On the other hand, by using Theorem 5.3.2,

$$\widehat{\deg}((\overline{P} + \epsilon \overline{B})^2) \leq \widehat{\text{vol}}(\overline{P} + \epsilon \overline{B}) \leq \widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2),$$

so that $\widehat{\deg}((\overline{P} + \epsilon \overline{B})^2) \leq \widehat{\deg}(\overline{P}^2)$. Therefore, $2\widehat{\deg}(\overline{P} \cdot \overline{B}) + \epsilon \widehat{\deg}(\overline{B}^2) \leq 0$. In particular, $\widehat{\deg}(\overline{P} \cdot \overline{B}) \leq 0$. Moreover, as $\overline{P}$ is nef and $\overline{B}$ is effective, by (3) in Proposition 4.5.4, we have $\widehat{\deg}(\overline{P} \cdot \overline{B}) \geq 0$, and hence $\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0$.

$\overline{P} + \overline{B}$ is not nef because $\overline{B} \not\geq 0$, so that, by Theorem 7.4.1,

$$\widehat{\deg}((\overline{P} + \overline{B})^2) < \widehat{\text{vol}}(\overline{P} + \overline{B}) = \widehat{\text{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2).$$

Therefore, $\widehat{\deg}(\overline{B}^2) < 0$.

(2) $\implies$ (1) : Let $\overline{L}$ be an element of $\Upsilon(\overline{D})$. If we set $\overline{A} := \max\{\overline{P}, \overline{L}\}$ and $\overline{B} := \overline{A} - \overline{P}$, then $\overline{B}$ is effective, $\overline{A} \leq \overline{D}$ and $\overline{A}$ is nef by Lemma 6.2.5. Moreover,

$$\overline{B} = \overline{A} - \overline{P} \leq \overline{D} - \overline{P}.$$

If we assume $\overline{B} \not\geq 0$, then, by hypothesis, $\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0$ and $\widehat{\deg}(\overline{B}^2) < 0$. On the other hand, as $\overline{A}$ is nef and $\overline{B}$ is effective,

$$0 \leq \widehat{\deg}(\overline{A} \cdot \overline{B}) = \widehat{\deg}(\overline{P} + \overline{B} \cdot \overline{B}) = \widehat{\deg}(\overline{B}^2),$$

which is a contradiction, so that $\overline{B} = 0$, that is, $\overline{P} = \overline{A}$, which means that $\overline{L} \leq \overline{P}$, as required. $\square$

8. DIRICHLET'S UNIT THEOREM FOR ADELIC ARITHMETIC DIVISORS

Let $\mathcal{X}$ be a generically smooth, normal and projective arithmetic variety and $\overline{\mathcal{D}}$ an arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{X}$. In the paper [31], we propose the following question:

Fundamental question. Are the following conditions (1) and (2) equivalent?

- (1) $\overline{\mathcal{D}}$ is pseudo-effective.
- (2) $\overline{\mathcal{D}} + (\overline{\varphi})$ is effective for some $\varphi \in \text{Rat}(\mathcal{X})_{\mathbb{R}}^{\times}$.

We can realize that the above equivalence is a crucial part of the proof of the classical Dirichlet's unit theorem if we trace back it carefully. In this sense, the above question is called *Dirichlet's unit theorem on arithmetic varieties*. Here we would like to consider a similar problem for adelic arithmetic divisors. Throughout this section, let X be a projective, normal and geometrically integral variety over a number field K .

8.1. Fundamental question for adelic divisors. Let $\overline{D}$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . For $\overline{D}$, we can consider the following similar question:

Fundamental question 8.1.1 (Adelic version). Under what conditions are the following properties (1) and (2) equivalent?

- (1) $\overline{D}$ is pseudo-effective.

(2) $\overline{D} + \widehat{(\varphi)}$ is effective for some $\varphi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$.

Clearly (2) implies (1). Indeed, as $-\widehat{(\varphi)}$ is nef, $-\widehat{(\varphi)}$ is pseudo-effective by Proposition 4.4.2, so that $\overline{D} = (\overline{D} + \widehat{(\varphi)}) + (-\widehat{(\varphi)})$ is also pseudo-effective by Corollary 5.3.3. In the case where D is numerically trivial, we have the following affirmative answer of the above question.

Theorem 8.1.2. *If D is numerically trivial and $\overline{D}$ is pseudo-effective, then there is $\varphi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ such that $\overline{D} + \widehat{(\varphi)} \geq 0$ (cf. Theorem 0.4.6 in Introduction).*

8.2. Proof of Theorem 8.1.2. In this subsection, let us consider the proof of Theorem 8.1.2. For the proof of Theorem 8.1.2, we may assume that X is smooth over K , so that throughout this subsection, we assume that X is smooth over K . For each $\sigma \in K(\mathbb{C})$, let $\tilde{\sigma} : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(K)$ denote the induced morphism by $K \xrightarrow{\sigma} \mathbb{C}$. Here we set

$$X_{\sigma}^{\text{an}} := \{x \in X(\mathbb{C}) \mid \pi \circ x = \tilde{\sigma}\},$$

where $\pi : X \rightarrow \text{Spec}(K)$ is the structure morphism.

$$\begin{array}{ccc} \text{Spec}(\mathbb{C}) & \xrightarrow{x} & X \\ & \searrow \tilde{\sigma} & \swarrow \pi \\ & \text{Spec}(K) & \end{array}$$

Then X_{σ}^{an} is a connected component of $X(\mathbb{C})$ and $X(\mathbb{C}) = \bigcup_{\sigma \in K(\mathbb{C})} X_{\sigma}^{\text{an}}$. The following is a key result for the proof of Theorem 8.1.2.

Theorem 8.2.1. *Let $\overline{E} := (0, \sum_{\mathfrak{p} \in M_K} g_{\mathfrak{p}}[\mathfrak{p}] + g_{\infty}[\infty])$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . Let*

$$\lambda_{\mathfrak{p}} := \min \{g_{\mathfrak{p}}(x) \mid x \in X_{\mathfrak{p}}^{\text{an}}\} \quad \text{and} \quad \lambda_{\sigma} := \min \{g_{\infty}(x) \mid x \in X_{\sigma}^{\text{an}}\}$$

for each $\mathfrak{p} \in M_K$ and $\sigma \in K(\mathbb{C})$. Let λ_{∞} be the locally constant function on $X(\mathbb{C})$ given by $\lambda_{\infty}|_{X_{\sigma}^{\text{an}}} = \lambda_{\sigma}$. If $\overline{E}$ is pseudo-effective, then $(0, \sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p}}[\mathfrak{p}] + \lambda_{\infty}[\infty])$ is also pseudo-effective.

Proof. We set

$$\overline{E}' = \left(0, \sum_{\mathfrak{p} \in M_K} g_{\mathfrak{p}}[\mathfrak{p}] + \lambda_{\infty}[\infty]\right) \quad \text{and} \quad \overline{E}'' = \left(0, \sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p}}[\mathfrak{p}] + \lambda_{\infty}[\infty]\right).$$

We assume that $\overline{E}$ is pseudo-effective. First let us see that $\overline{E}'$ is pseudo-effective. Let $\overline{H} = (H, h)$ be a big adelic arithmetic Cartier divisor on X . Let ϵ be a positive number. We set

$$U_{\sigma} = \{x \in X_{\sigma} \mid g_{\infty}(x) < \lambda_{\sigma} + (\epsilon/2)\}$$

and $U = \bigsqcup_{\sigma \in K(\mathbb{C})} U_{\sigma}$. Then, by Lemma 1.1.8, there is a constant $C \geq 1$ depending only on ϵ and h_{∞} such that

$$(8.2.1.1) \quad \sup_{x \in X(\mathbb{C})} \left\{ |s|_{t+bh_{\infty}}^2(x) \right\} \leq C^b \sup_{x \in U} \left\{ |s|_{t+bh_{\infty}}^2(x) \right\}$$

for all $s \in H^0(X(\mathbb{C}), bH)$, $b \in \mathbb{R}_{\geq 0}$ and all locally constant functions t on $X(\mathbb{C})$. Let n be an arbitrary positive integer with $n \geq (2 \log(C))/\epsilon$. Since $\overline{E} + (1/n)\overline{H}$ is big, there

are a positive integer m and $s \in H^0(X, mD + (m/n)H) \setminus \{0\}$ such that $|s|_{mg_p + (m/n)h_p} \leq 1$ for all $p \in M_K \cup \{\infty\}$. In particular,

$$|s|_{(m/n)h_\infty}^2 \leq \exp(mg_\infty).$$

Therefore, $|s|_{(m/n)h_\infty}^2 \leq \exp(m(\lambda_\infty + (\epsilon/2)))$ over U , that is,

$$\sup_{x \in U} \left\{ |s|_{m(\lambda_\infty + (\epsilon/2)) + (m/n)h_\infty}^2 \right\} \leq 1.$$

Thus, by the estimate (8.2.1.1), we have

$$C^{-(m/n)} \sup_{x \in X(\mathbb{C})} \left\{ |s|_{m(\lambda_\infty + (\epsilon/2)) + (m/n)h_\infty}^2 \right\} \leq 1.$$

Since $\log(C)/n \leq \epsilon/2$,

$$\begin{aligned} \sup_{x \in X(\mathbb{C})} \left\{ |s|_{m(\lambda_\infty + \epsilon) + (m/n)h_\infty}^2 \right\} &\leq \sup_{x \in X(\mathbb{C})} \left\{ |s|_{(m/n)\log(C) + m(\lambda_\infty + (\epsilon/2)) + (m/n)h_\infty}^2 \right\} \\ &= C^{-(m/n)} \sup_{x \in X(\mathbb{C})} \left\{ |s|_{m(\lambda_\infty + (\epsilon/2)) + (m/n)h_\infty}^2 \right\} \leq 1 \end{aligned}$$

which yields $\hat{H}^0(X, m(\overline{E}' + (0, \epsilon[\infty]) + (1/n)\overline{H})) \neq \{0\}$. Thus $\overline{E}' + (0, \epsilon[\infty]) + (1/n)\overline{H}$ is big if $n \gg 1$. As a consequence, $\overline{E}' + (0, \epsilon[\infty])$ is pseudo-effective for any positive number ϵ , and hence $\overline{E}'$ is also pseudo-effective.

Next let us see that $\overline{E}''$ is pseudo-effective. We can find a non-empty open set U of $\text{Spec}(O_K)$ such that $g_p = 0$ for all $p \in U \cap M_K$. Let ϵ be a positive number. For $p \in M_K \setminus U$, there is a divisorial absolute value $w_p \in X_p^{\text{an}}$ (cf. Appendix A.1) such that $g_p(w_p) \leq \lambda_p + \epsilon$ because the set of divisorial absolute values in X_p^{an} is a dense subset of X_p^{an} by Theorem 3.3.8. Let $\mathcal{X}$ be a normal model of X such that, for all $p \in M_K \setminus U$, w_p can be realized by a vertical prime divisor on $\mathcal{X}$ over p . Let $\mu : \mathcal{Y} \rightarrow \mathcal{X}$ be an alteration of $\mathcal{X}$ such that $\mathcal{Y}$ is regular and projective over $\text{Spec}(O_K)$ (cf. [15, Theorem 8.2]). Let

$$\mathcal{Y} \xrightarrow{\pi'} \text{Spec}(O_{K'}) \xrightarrow{\nu} \text{Spec}(O_K)$$

be the Stein factorization of $\mathcal{Y} \rightarrow \text{Spec}(O_K)$, where K' is a finite extension field of K .

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\mu} & \mathcal{Y} \\ \pi \downarrow & & \downarrow \pi' \\ \text{Spec}(O_K) & \xleftarrow{\nu} & \text{Spec}(O_{K'}) \end{array}$$

We set $\nu^{-1}(M_K \setminus U) = \{q_1, \dots, q_r\}$ and $Y = \mathcal{Y}_{K'}$. For each i , let v_i be the absolute value of K' at q_i as the extension of $\nu_{\mathcal{Y}(q_i)}$, $\mu_i : Y_{q_i} \rightarrow X_{\mathcal{Y}(q_i)}$ the induced morphism and $g_i := \mu_i^*(g_{\mathcal{Y}(q_i)})$. Let $\pi'^{-1}(q_i) = \Gamma_{i1} \cup \dots \cup \Gamma_{il_i}$ be the irreducible decomposition of $\pi'^{-1}(q_i)$. Let v_{ij} be the absolute value at Γ_{ij} as the extension of v_i . We set

$$\tilde{\lambda}_i = \min\{g_i(v_{ij}) \mid j = 1, \dots, l_i\} \quad \text{and} \quad \tilde{g}_i := g_i - \tilde{\lambda}_i$$

and we choose a_i such that $1 \leq a_i \leq l_i$ and $\tilde{g}_i(v_{ia_i}) = 0$. Note that

$$\lambda_{\mathcal{Y}(q_i)} \leq \tilde{\lambda}_i \leq \lambda_{\mathcal{Y}(q_i)} + \epsilon.$$

Let $\overline{\mathcal{A}} = (\mathcal{A}, \alpha)$ be an ample arithmetic Cartier divisor on $\mathcal{Y}$. Then, by [31, Lemma 3.5.2], we can find an effective and vertical $\mathbb{Q}$ -Cartier divisor $\mathcal{F}$ on $\mathcal{Y}$ with the following properties:

- (a) $\text{Supp}(\mathcal{F}) \subseteq \pi'^{-1}(q_1) \cup \cdots \cup \pi'^{-1}(q_r)$.
- (b) $\mathcal{A} - \mathcal{F}$ is divisorially π' -nef with respect to $\mathcal{A}$, that is,

$$\deg(\mathcal{A} - \mathcal{F} \cdot \mathcal{A}^{d-1} \cdot \Gamma) \geq 0$$

for all vertical prime divisors Γ on $\mathcal{Y}$.

- (c) If Γ is a prime divisor with

$$\Gamma \subseteq \pi'^{-1}(q_1) \cup \cdots \cup \pi'^{-1}(q_r) \quad \text{and} \quad \Gamma \notin \{\Gamma_{1a_1}, \dots, \Gamma_{ra_r}\},$$

then $\deg(\mathcal{A} - \mathcal{F} \cdot \mathcal{A}^{d-1} \cdot \Gamma) = 0$.

Moreover, in the same way as the proof of [31, Claim 3.5.3.3], we can find a constant c such that if we set $\alpha' = \alpha + c$, then

- (d) $\widehat{\deg}((\mathcal{A} - \mathcal{F}, \alpha') \cdot \overline{\mathcal{A}}^{d-1} \cdot (\Gamma, 0)) \geq 0$ for all horizontal prime divisors Γ on $\mathcal{Y}$.

Claim 8.2.1.2. *Let $\nu : \mathcal{Y}' \rightarrow \mathcal{Y}$ be a birational morphism of normal and projective arithmetic varieties. Then, for any pseudo-effective $\mathbb{R}$ -Cartier divisor $\mathcal{E}$ of C^0 -type on $\mathcal{Y}'$,*

$$\widehat{\deg}(\nu^*((\mathcal{A} - \mathcal{F}, \alpha')) \cdot \nu^*(\overline{\mathcal{A}})^{d-1} \cdot \overline{\mathcal{E}}) \geq 0.$$

Proof. By virtue of the above (b) and (d) together with the projection formula,

$$\widehat{\deg}(\nu^*((\mathcal{A} - \mathcal{F}, \alpha')) \cdot \nu^*(\overline{\mathcal{A}})^{d-1} \cdot (\Gamma', 0)) \geq 0$$

for all prime divisors Γ' on $\mathcal{Y}'$. Thus, as $c_1(\nu^*(\overline{\mathcal{A}}))$ is semi-positive, the assertion of the claim holds if $\overline{\mathcal{E}}$ is big. Therefore, if $\overline{\mathcal{B}}$ is a big arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on $\mathcal{Y}'$ and $n > 0$, then

$$\widehat{\deg}(\nu^*((\mathcal{A} - \mathcal{F}, \alpha')) \cdot \nu^*(\overline{\mathcal{A}})^{d-1} \cdot (\overline{\mathcal{E}} + (1/n)\overline{\mathcal{B}})) \geq 0.$$

and hence the claim follows. $\square$

Let us choose $f_i \in O_{K'}$ such that $f_i O_{K'} = \mathfrak{q}_i^{e_i}$ for some $e_i \in \mathbb{Z}_{>0}$. Then we can find $d_i \in \mathbb{R}$ with $d_i \pi'^*((f_i))^a = \tilde{\lambda}_i[\mathfrak{q}_i]$. Therefore, if we set

$$\vartheta := f_1^{d_1} \cdots f_r^{d_r} \quad \text{and} \quad \lambda'_\infty := \mu^*(\lambda_\infty) + \sum_{i=1}^r d_i \log |f_i|^2,$$

then

$$\mu^*(\overline{E}') = \left(0, \sum_{i=1}^r \tilde{g}_i[\mathfrak{q}_i] + \lambda'_\infty[\infty] \right) + \pi'^*(\widehat{(\vartheta)}).$$

By Theorem 4.1.3, there is a sequence $\{\mathcal{Y}_n\}_{n=1}^\infty$ of normal models of Y with the following properties:

- (i) For each $n \geq 1$, there is a birational morphism $\nu_n : \mathcal{Y}_n \rightarrow \mathcal{Y}$.
- (ii) For each $n \geq 1$, there are an $\mathbb{R}$ -Cartier divisor $\mathcal{E}_n$ on $\mathcal{Y}_n$ and

$$\varphi_{n, \mathfrak{q}_1} \in C^0(Y_{\mathfrak{q}_1}^{\text{an}}), \dots, \varphi_{n, \mathfrak{q}_r} \in C^0(Y_{\mathfrak{q}_r}^{\text{an}})$$

such that $0 \leq \varphi_{n, \mathfrak{q}_i} \leq 1/n$ for $i = 1, \dots, r$ and

$$\mu^*(\overline{E}') + \left(0, \sum_{i=1}^r \varphi_{n, \mathfrak{q}_i}[\mathfrak{q}_i] \right) = (\mathcal{E}_n, \mu^*(\lambda_\infty))^a.$$

Claim 8.2.1.3. *If we set $\Delta_n := \widehat{\deg}(\nu_n^*((\mathcal{A} - \mathcal{F}, \alpha')) \cdot \nu_n^*(\overline{\mathcal{A}})^{d-1} \cdot (\mathcal{E}_n, \mu^*(\lambda_\infty))^a)$, then $\Delta_n \geq 0$ and*

$$\Delta_n = \sum_{i=1}^r \log \#(O_{K'}/\mathfrak{q}_i) \widehat{\deg}_{\mathfrak{q}_i} \left((\mathcal{A} - \mathcal{F})_{(\mathfrak{q}_i)} \cdot \mathcal{A}_{(\mathfrak{q}_i)}^{d-1}; \varphi_{n, \mathfrak{q}_i} \right) + \frac{1}{2} \left(\sum_{\sigma' \in K'(\mathbb{C})} (\lambda'_\infty)_{\sigma'} \right) \int_{Y(\mathbb{C})} c_1(\overline{\mathcal{A}})^d.$$

Proof. The first assertion follows from Claim 8.2.1.2. As

$$(\mathcal{E}_n, \mu^*(\lambda_\infty))^a = \left(0, \sum_{i=1}^r (\tilde{g}_i + \varphi_{n, \mathfrak{q}_i})[\mathfrak{q}_i] + \lambda'_\infty[\infty] \right) + \pi'^*(\widehat{(\vartheta)}),$$

we have

$$\Delta_n = \sum_{i=1}^r \log \#(O_{K'}/\mathfrak{q}_i) \widehat{\deg}_{\mathfrak{q}_i} \left((\mathcal{A} - \mathcal{F})_{(\mathfrak{q}_i)} \cdot \mathcal{A}_{(\mathfrak{q}_i)}^{d-1}; \tilde{g}_i + \varphi_{n, \mathfrak{q}_i} \right) + \frac{1}{2} \left(\sum_{\sigma' \in K'(\mathbb{C})} (\lambda'_\infty)_{\sigma'} \right) \int_{Y(\mathbb{C})} c_1(\overline{\mathcal{A}})^d.$$

On the other hand, by using property (c) and $\tilde{g}_i(v_{i, \mathfrak{q}_i}) = 0$ ($i = 1, \dots, r$),

$$\widehat{\deg}_{\mathfrak{q}_i} \left((\mathcal{A} - \mathcal{F})_{(\mathfrak{q}_i)} \cdot \mathcal{A}_{(\mathfrak{q}_i)}^{d-1}; \tilde{g}_i \right) = 0.$$

Thus the second assertion follows. $\square$

Since $\mathcal{A} - \mathcal{F}$ is divisorially π' -nef with respect to $\mathcal{A}$ and $0 \leq \varphi_{n, \mathfrak{q}_i} \leq 1/n$, by using (2.4.1), we have

$$0 \leq \widehat{\deg}_{\mathfrak{q}_i} \left((\mathcal{A} - \mathcal{F})_{(\mathfrak{q}_i)} \cdot \mathcal{A}_{(\mathfrak{q}_i)}^{d-1}; \varphi_{n, \mathfrak{q}_i} \right) \leq (1/n) \widehat{\deg}_{\mathfrak{q}_i} \left((\mathcal{A} - \mathcal{F})_{(\mathfrak{q}_i)} \cdot \mathcal{A}_{(\mathfrak{q}_i)}^{d-1}; 1 \right),$$

and hence $\sum_{\sigma' \in K'(\mathbb{C})} (\lambda'_\infty)_{\sigma'} \geq 0$ by Claim 8.2.1.3, so that $(0, \lambda'_\infty[\infty])$ is pseudo-effective by [31, Proposition 3.5]. Therefore, $(0, \lambda'_\infty[\infty]) + \pi'^*(\widehat{(\vartheta)})$ is also pseudo-effective by Proposition 4.4.2 and Corollary 5.3.3. Note that

$$\begin{aligned} (0, \lambda'_\infty[\infty]) + \pi'^*(\widehat{(\vartheta)}) &= \left(0, \sum_{i=1}^r \tilde{\lambda}_i[\mathfrak{q}_i] + \mu^*(\lambda_\infty)[\infty] \right) \\ &\leq \left(0, \sum_{i=1}^r (\lambda_{\mathfrak{p}(\mathfrak{q}_i)} + \epsilon)[\mathfrak{q}_i] + \mu^*(\lambda_\infty)[\infty] \right) \end{aligned}$$

and

$$\begin{aligned} \mu_* \left(0, \sum_{i=1}^r (\lambda_{\mathfrak{p}(\mathfrak{q}_i)} + \epsilon)[\mathfrak{q}_i] + \mu^*(\lambda_\infty)[\infty] \right) \\ = \deg(\mu) \left(0, \sum_{\mathfrak{p} \in M_K \setminus U} (\lambda_{\mathfrak{p}} + \epsilon)[\mathfrak{p}] + \lambda_\infty[\infty] \right). \end{aligned}$$

Thus, $\overline{E}'' + \epsilon(0, \sum_{\mathfrak{p} \in M_K \setminus U} [\mathfrak{p}])$ is pseudo-effective for all $\epsilon > 0$, and hence $\overline{E}''$ is also pseudo-effective. $\square$

We also need the following theorem, which is a consequence of the Hodge index theorem.

Theorem 8.2.2. *Let $\overline{D} = (D, g)$ be an adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type on X . If D is numerically trivial and $\overline{D}$ is pseudo-effective, then $D = (\vartheta)$ for some $\vartheta \in \text{Rat}(X)_{\mathbb{R}}^{\times}$.*

Proof. By using Lemma 1.1.6, there are a normal model $\mathcal{X}$ of X over $\text{Spec}(O_K)$, a non-empty Zariski open set U of $\text{Spec}(O_K)$ and an $\mathbb{R}$ -Cartier divisor $\mathcal{D}'$ on $\mathcal{X}$ such that $g_p = g_{(\mathcal{X}_p, \mathcal{D}'_p)}$ for all $p \in U$. As $g_q - g_{(\mathcal{X}_q, \mathcal{D}'_q)}$ is a continuous function on X_q for each $q \in M_K \setminus U$, there is a constant C_q such that $g_q \leq g_{(\mathcal{X}_q, \mathcal{D}'_q)} + C_q$ on X_q . Therefore, we can find an $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ such that

$$g_p = g_{(\mathcal{X}_p, \mathcal{D}_p)} \quad (\forall p \in U) \quad \text{and} \quad g_q \leq g_{(\mathcal{X}_q, \mathcal{D}_q)} \quad (\forall q \in M_K \setminus U),$$

that is, $\overline{D} \leq (\mathcal{D}, g_{\infty})^a$. Let $\mu : \mathcal{Y} \rightarrow \mathcal{X}$ be an alteration of $\mathcal{X}$ such that $\mathcal{Y}$ is regular and projective over $\text{Spec}(O_K)$ (cf. [15, Theorem 8.2]). Let

$$\mathcal{Y} \xrightarrow{\pi'} \text{Spec}(O_{K'}) \xrightarrow{\nu} \text{Spec}(O_K)$$

be the Stein factorization of $\mathcal{Y} \rightarrow \text{Spec}(O_K)$, where K' is a finite extension field of K . Let $\overline{\mathcal{A}}$ be an ample arithmetic divisor on $\mathcal{Y}$. By [31, Lemma 2.2.2], there is a vertical and effective $\mathbb{R}$ -Cartier divisor $\mathcal{E}$ on $\mathcal{Y}$ such that $\mu^*(\mathcal{D}) + \mathcal{E}$ is divisorially π -numerically trivial. Moreover, we can find an F_{∞} -invariant $\mu^*(\mathcal{D})$ -Green function g_0 of C^{∞} -type on $\mathcal{Y}(\mathbb{C})$ such that $c_1(\mu^*(\mathcal{D}), g_0) = 0$ and $\mu^*(g) \leq g_0$. By the Hodge index theorem (cf. [31, Theorem 2.2.5]),

$$\widehat{\deg}((\mu^*(\mathcal{D}) + \mathcal{E}, g_0)^2 \cdot \overline{\mathcal{A}}^{d-1}) \leq 0.$$

If $\widehat{\deg}((\mu^*(\mathcal{D}) + \mathcal{E}, g_0)^2 \cdot \overline{\mathcal{A}}^{d-1}) < 0$, then, by [31, Lemma 2.3.2], $(\mu^*(\mathcal{D}) + \mathcal{E}, g_0)$ is not pseudo-effective, so that $\mu^*(\overline{D})$ is not pseudo-effective because $\mu^*(\overline{D}) \leq (\mu^*(\mathcal{D}) + \mathcal{E}, g_0)^a$, which is a contradiction. Therefore,

$$\widehat{\deg}((\mu^*(\mathcal{D}) + \mathcal{E}, g_0)^2 \cdot \overline{\mathcal{A}}^{d-1}) = 0,$$

and hence, by using the equality condition of the Hodge index theorem (cf. [31, Theorem 2.2.5]), $\mu^*(D) = (\varphi)$ for some $\varphi \in \text{Rat}(\mathcal{Y})_{\mathbb{R}}^{\times}$. Let $N : \text{Rat}(\mathcal{Y})^{\times} \rightarrow \text{Rat}(X)^{\times}$ be the norm map of the finite fields extension $\text{Rat}(\mathcal{Y}) \supseteq \text{Rat}(X)$. We set $\varphi = \varphi_1^{a_1} \cdots \varphi_r^{a_r}$ for some $\varphi_1, \dots, \varphi_r \in \text{Rat}(\mathcal{Y})^{\times}$ and $a_1, \dots, a_r \in \mathbb{R}$. Then we have

$$\deg(\mu)D = \mu_*(\mu^*(D)) = \mu_*((\varphi)) = \sum_{i=1}^r a_i \mu_*((\varphi_i)) = \sum_{i=1}^r a_i (N(\varphi_i)),$$

as required. $\square$

Proof of Theorem 8.1.2. Let us start the proof of Theorem 8.1.2. By Theorem 8.2.2, there is $\vartheta_1 \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ such that $D = (\vartheta_1)$. If we set $\overline{E} = \overline{D} - (\widehat{\vartheta_1})$, then $\overline{E}$ is pseudo-effective by Proposition 4.4.2 and Corollary 5.3.3, and we can set $\overline{E}$ as follows:

$$\overline{E} = \left(0, \sum_{p \in M_K} g_p[p] + g_{\infty}[\infty] \right).$$

Let $\lambda_{\mathfrak{p}} := \min \{g_{\mathfrak{p}}(x) \mid x \in X_{\mathfrak{p}}^{\text{an}}\}$ and $\lambda_{\sigma} := \min \{g_{\infty}(x) \mid x \in X_{\sigma}^{\text{an}}\}$ for each $\mathfrak{p} \in M_K$ and $\sigma \in K(\mathbb{C})$. Let λ_{∞} be the locally constant function on $X(\mathbb{C})$ given by $\lambda_{\infty}|_{X_{\sigma}^{\text{an}}} = \lambda_{\sigma}$. Then,

$$\overline{E} \geq \left(0, \sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p}}[\mathfrak{p}] + \lambda_{\infty}[\infty]\right)$$

and, by Theorem 8.2.1, $(0, \sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p}}[\mathfrak{p}] + \lambda_{\infty}[\infty])$ is pseudo-effective. Moreover, there is $\vartheta_2 \in K_{\mathbb{R}}^{\times}$ such that

$$\left(0, \sum_{\mathfrak{p} \in M_K} \lambda_{\mathfrak{p}}[\mathfrak{p}] + \lambda_{\infty}[\infty]\right) - (\widehat{\vartheta_2}) = (0, \lambda'_{\infty}[\infty])$$

for some F_{∞} -invariant locally constant function λ'_{∞} on $X(\mathbb{C})$. Note that $(0, \lambda'_{\infty}[\infty])$ is pseudo-effective by Proposition 4.4.2 and Corollary 5.3.3. In particular,

$$\mu = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in K(\mathbb{C})} (\lambda'_{\infty})_{\sigma} \geq 0.$$

If we set $\xi_{\sigma} := (\lambda'_{\infty})_{\sigma} - \mu$, then $\sum_{\sigma \in K(\mathbb{C})} \xi_{\sigma} = 0$ and $\xi_{\sigma} = \xi_{\bar{\sigma}}$ ($\forall \sigma \in K(\mathbb{C})$). Thus, by using the classical Dirichlet's unit theorem, there is $u \in (O_K)_{\mathbb{R}}^{\times}$ such that $(0, (\lambda'_{\infty} - \mu)[\infty]) = (\widehat{u})$, that is, $(0, \lambda'_{\infty}[\infty]) \geq (\widehat{u})$. Therefore,

$$\overline{D} - (\widehat{\vartheta_1}) - (\widehat{\vartheta_2}) = \overline{E} - (\widehat{\vartheta_2}) \geq (0, \lambda'_{\infty}[\infty]) \geq (\widehat{u}),$$

so that, if we set $\varphi = \vartheta_1^{-1} \vartheta_2^{-1} u^{-1} \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, then $\overline{D} + (\widehat{\varphi}) \geq 0$. $\square$

Remark 8.2.3. In [9], Chen and Moriawaki give a nef arithmetic $\mathbb{R}$ -Cartier divisor $\overline{D} = (D, g)$ of C^0 -type on $\mathbb{P}_K^1$ such that D is ample and $\overline{D} + (\widehat{\phi})$ is not effective for all $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$.

Question 8.2.4 (Existence of weak Zariski decomposition). If $\overline{D}$ is pseudo-effective, then does it follow that there are a birational morphism $f: Y \rightarrow X$ of normal projective varieties over K and a nef arithmetic $\mathbb{R}$ -Cartier divisor $\overline{P}$ of C^0 -type on Y such that $\overline{D} \geq \overline{P}$?

Appendix A. CHARACTERIZATION OF RELATIVELY NEF CARTIER DIVISORS

In this appendix, we consider a characterization of relatively nef Cartier divisors in terms of asymptotic multiplicities. Let k be a field and v a complete discrete absolute value of k . Let ϖ be a uniformizing parameter of k° . Note that the absolute value v is not necessarily non-trivial.

A.1. Asymptotic multiplicity. Let $\mathcal{X}$ be a $(d+1)$ -dimensional, proper and normal variety over k° (cf. Conventions and terminology 0.5.4), that is, the Krull dimension of $\mathcal{X}$ is $d+1$, $\mathcal{X}$ is proper over $\text{Spec}(k^{\circ})$ and $\mathcal{X}$ is integral and normal. We denote the rational function field of $\mathcal{X}$ by $\text{Rat}(\mathcal{X})$. Let $\text{WDiv}(\mathcal{X})$ and $\text{Div}(\mathcal{X})$ denote the group of Weil divisors on $\mathcal{X}$ and the group of Cartier divisors on $\mathcal{X}$, respectively. In addition, for a point $x \in \mathcal{X}$, let $\text{Div}(\mathcal{X}; x)$ be the subgroup of $\text{WDiv}(\mathcal{X})$ consisting of Weil divisors $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{D} = (\phi)$ around x for some $\phi \in \text{Rat}(\mathcal{X})^{\times}$, that is, $\mathcal{D}$ is a Cartier divisor around x . Note that

$$\text{Div}(\mathcal{X}) \subseteq \text{Div}(\mathcal{X}; x) \subseteq \text{WDiv}(\mathcal{X}).$$

For example, if x is a regular point of $\mathcal{X}$, then $\text{Div}(\mathcal{X}; x) = \text{WDiv}(\mathcal{X})$. We set

$$\begin{cases} \text{WDiv}(\mathcal{X})_{\mathbb{R}} := \text{WDiv}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \text{Div}(\mathcal{X})_{\mathbb{R}} := \text{Div}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \text{Div}(\mathcal{X}; x)_{\mathbb{R}} := \text{Div}(\mathcal{X}; x) \otimes_{\mathbb{Z}} \mathbb{R}. \end{cases}$$

Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$ and let $\text{Rat}(\mathcal{X})_{\mathbb{K}}^{\times} := (\text{Rat}(\mathcal{X})^{\times}, \times) \otimes_{\mathbb{Z}} \mathbb{K}$. Note that the homomorphism

$$(\cdot) : \text{Rat}(\mathcal{X})^{\times} \rightarrow \text{Div}(\mathcal{X})$$

given by $f \mapsto (f)$ has the natural extension

$$(\cdot) : \text{Rat}(\mathcal{X})_{\mathbb{K}}^{\times} \rightarrow \text{Div}(\mathcal{X})_{\mathbb{R}},$$

that is, for $\phi = f_1^{\otimes a_1} \cdots f_r^{\otimes a_r} \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^{\times}$ ($f_1, \dots, f_r \in \text{Rat}(\mathcal{X})^{\times}$, $a_1, \dots, a_r \in \mathbb{K}$),

$$(\phi) = a_1(f_1) + \cdots + a_r(f_r).$$

Let $\mathcal{D}$ be an $\mathbb{R}$ -Weil divisor on $\mathcal{X}$, that is, $\mathcal{D} \in \text{WDiv}(\mathcal{X})_{\mathbb{R}}$. We define $\Gamma^{\times}(\mathcal{X}, \mathcal{D})$ and $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D})$ to be

$$\begin{cases} \Gamma^{\times}(\mathcal{X}, \mathcal{D}) := \{\phi \in \text{Rat}(\mathcal{X})^{\times} \mid \mathcal{D} + (\phi) \geq 0\}, \\ \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) := \{\phi \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^{\times} \mid \mathcal{D} + (\phi) \geq 0\}. \end{cases}$$

Let $w : \text{Rat}(\mathcal{X}) \rightarrow \mathbb{R} \cup \{\infty\}$ be an additive discrete non-trivial valuation over k . Namely, w satisfies the following conditions:

- (1) $w(f \cdot g) = w(f) + w(g)$ for all $f, g \in \text{Rat}(\mathcal{X})$.
- (2) $w(f + g) \geq \min\{w(f), w(g)\}$ for all $f, g \in \text{Rat}(\mathcal{X})$.
- (3) $f = 0$ if and only if $w(f) = \infty$.
- (4) $w(a) = -\log v(a)$ for all $a \in k^{\times}$.

Let $\mathcal{O}_w$ be the valuation ring of w and m_w its maximal ideal, that is,

$$\mathcal{O}_w = \{f \in \text{Rat}(\mathcal{X}) \mid w(f) \geq 0\} \quad \text{and} \quad m_w = \{f \in \text{Rat}(\mathcal{X}) \mid w(f) > 0\}.$$

Note that $k^{\circ} \subseteq \mathcal{O}_w$ and $k^{\circ\circ} \subseteq m_w$, so that $\mathcal{O}_w/m_w$ is a $k^{\circ}/k^{\circ\circ}$ -algebra. We say that w is a *divisorial valuation of $\text{Rat}(\mathcal{X})$ over k* if $\text{trdeg}_{k^{\circ}/k^{\circ\circ}} \mathcal{O}_w/m_w = d$. For a divisorial valuation w of $\text{Rat}(\mathcal{X})$ over k , there are a normal variety $\mathcal{V}$ over k° , a vertical prime divisor Γ on $\mathcal{V}$ and a birational morphism $\mu : \mathcal{V} \rightarrow \mathcal{X}$ over $\text{Spec}(k^{\circ})$ such that $w = a \text{ord}_{\Gamma}$ for some $a \in \mathbb{R}_{>0}$. Indeed, it can be shown as follows: We may assume that v is non-trivial. Otherwise the assertion follows from [38, Chapter VI, § 14, Theorem 31]. We choose $x_1, \dots, x_d \in \mathcal{O}_w$ such that $x_1, \dots, x_d$ form a transcendental basis of $\mathcal{O}_w/m_w$ over $k^{\circ}/k^{\circ\circ}$. Then $\text{Rat}(\mathcal{X})$ is a finite extension of $k(x_1, \dots, x_d)$ and the transcendental degree of $k^{\circ}[x_1, \dots, x_d]/k^{\circ}[x_1, \dots, x_d] \cap m_w$ over $k^{\circ}/k^{\circ\circ}$ is d . Let R be the normalization of $k^{\circ}[x_1, \dots, x_d]$ in $\text{Rat}(\mathcal{X})$. Note that R is finite over $k^{\circ}[x_1, \dots, x_d]$ because k° is excellent. In addition, $R \subseteq \mathcal{O}_w$, $R \cap m_w$ is a prime ideal of R and $\text{trdeg}_{k^{\circ}/k^{\circ\circ}}(R/R \cap m_w) = d$, which prove the assertion.

We denote the set of all divisorial valuations of $\text{Rat}(\mathcal{X})$ over k by $\text{DVal}_k(\mathcal{X})$. As $\mathcal{X}$ is proper and separated over $\text{Spec}(k^{\circ})$, there is a unique morphism $t : \text{Spec}(\mathcal{O}_w) \rightarrow \mathcal{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{Spec}(\text{Rat}(\mathcal{X})) & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow & \nearrow t & \downarrow \\ \text{Spec}(\mathcal{O}_w) & \xrightarrow{\quad} & \text{Spec}(k^{\circ}) \end{array}$$

Let x be the image of the closed point m_w by t . The point x is called the *center of w on $\mathcal{X}$* . Note that $x \in \mathcal{X}_\circ$ (the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$). For $\mathcal{D} \in \text{Div}(\mathcal{X}; x)$, $\text{mult}_w(\mathcal{D})$ is defined by $w(f)$, where f is a local equation of $\mathcal{D}$ at x . In this way, we have a map

$$\text{mult}_w : \text{Div}(\mathcal{X}; x) \rightarrow \mathbb{Z}.$$

It is easy to see that mult_w is a homomorphism, so that we have the natural extension

$$\text{mult}_w : \text{Div}(\mathcal{X}; x)_{\mathbb{R}} \rightarrow \mathbb{R},$$

that is,

$$\text{mult}_w(a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r) = a_1 \text{mult}_w(\mathcal{D}_1) + \cdots + a_r \text{mult}_w(\mathcal{D}_r),$$

where $\mathcal{D}_1, \dots, \mathcal{D}_r \in \text{Div}(\mathcal{X}; x)$ and $a_1, \dots, a_r \in \mathbb{R}$.

For $\mathcal{D} \in \text{Div}(\mathcal{X}; x)_{\mathbb{R}}$, we define $\mu_{\mathbb{K}, w}(\mathcal{D})$ to be

$$\mu_{\mathbb{K}, w}(\mathcal{D}) := \begin{cases} \inf \{ \text{mult}_w(\mathcal{D} + (\phi)) \mid \phi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) \} & \text{if } \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

which is called the $\mathbb{K}$ -asymptotic multiplicity of $\mathcal{D}$ at w . Here we give one additional definition. An $\mathbb{R}$ -Cartier divisor $\mathcal{D}$ (i.e. $\mathcal{D} \in \text{Div}(\mathcal{X})_{\mathbb{R}}$) is said to be *big* if $\mathcal{D}$ is big on the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. First let us observe elementary properties of the asymptotic multiplicity. The arithmetic version can be found in [29, Proposition 2.1 and Theorem 2.5] and Proposition 7.2.1.

Proposition A.1.1. *Let w be a divisorial valuation of $\text{Rat}(\mathcal{X})$ over k and x the center of w on $\mathcal{X}$. For $\mathcal{D}, \mathcal{E} \in \text{Div}(\mathcal{X}; x)_{\mathbb{R}}$, we have the following:*

- (1) $\mu_{\mathbb{K}, w}(\mathcal{D} + \mathcal{E}) \leq \mu_{\mathbb{K}, w}(\mathcal{D}) + \mu_{\mathbb{K}, w}(\mathcal{E})$.
- (2) If $\mathcal{D} \leq \mathcal{E}$, then $\mu_{\mathbb{K}, w}(\mathcal{E}) \leq \mu_{\mathbb{K}, w}(\mathcal{D}) + \text{mult}_w(\mathcal{E} - \mathcal{D})$.
- (3) $\mu_{\mathbb{K}, w}(\mathcal{D} + (\phi)) = \mu_{\mathbb{K}, w}(\mathcal{D})$ for $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^{\times}$.
- (4) $\mu_{\mathbb{K}, w}(a\mathcal{D}) = a\mu_{\mathbb{K}, w}(\mathcal{D})$ for $a \in \mathbb{K}_{>0}$.
- (5) $0 \leq \mu_{\mathbb{R}, w}(\mathcal{D}) \leq \mu_{\mathbb{Q}, w}(\mathcal{D})$.
- (6) Let $v : \mathcal{Y} \rightarrow \mathcal{X}$ be a birational morphism of proper and normal varieties over k° .
 - (6.1) If $\mathcal{D}$ is an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$, then $\mu_{\mathbb{K}, w}(v^*(\mathcal{D})) = \mu_{\mathbb{K}, w}(\mathcal{D})$.
 - (6.2) Let x and y be the centers of w on $\mathcal{X}$ and $\mathcal{Y}$, respectively (note that $v(y) = x$). We assume that v is an isomorphism over x . Then, for $\mathcal{D}' \in \text{Div}(\mathcal{Y}; y)$,

$$\mu_{\mathbb{K}, w}(v_*(\mathcal{D}')) \leq \mu_{\mathbb{K}, w}(\mathcal{D}').$$

- (7) If $\mathcal{D}$ is an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$ and $\mathcal{D}$ is relatively nef with respect to $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ (cf. Conventions and terminology 0.5.6) and big, then $\mu_{\mathbb{K}, w}(\mathcal{D}) = 0$.
- (8) If $\mathcal{D}$ is an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$ and $\mathcal{D}$ is big, then $\mu_{\mathbb{Q}, w}(\mathcal{D}) = \mu_{\mathbb{R}, w}(\mathcal{D})$.

Proof. (1) If $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D} + \mathcal{E}) = \emptyset$, then either $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) = \emptyset$ or $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{E}) = \emptyset$, so that we may assume that $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D} + \mathcal{E}) \neq \emptyset$. Thus we may also assume that $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) \neq \emptyset$ and $\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{E}) \neq \emptyset$. Therefore, the assertion follows because $\phi\psi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D} + \mathcal{E})$ for all $\phi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D})$ and $\psi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{E})$.

(2) is derived from (1).

(3) The assertion follows from the following:

$$\psi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) \iff \psi\phi^{-1} \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D} + (\phi)).$$

(4) Note that $\psi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D})$ if and only if $\psi^a \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, a\mathcal{D})$, and that

$$\text{mult}_w(a\mathcal{D} + (\psi^a)) = a \text{mult}_w(\mathcal{D} + (\psi)),$$

which implies (4).

(5) is obvious.

(6.1) For $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^{\times}$, $\mathcal{D} + (\phi)_{\mathcal{X}} \geq 0$ if and only if $\nu^*(\mathcal{D}) + (\phi)_{\mathcal{Y}} \geq 0$. Thus

$$\Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \mathcal{D}) = \Gamma_{\mathbb{K}}^{\times}(\mathcal{Y}, \nu^*(\mathcal{D})).$$

Moreover,

$$\text{mult}_w(\mathcal{D} + (\phi)_{\mathcal{X}}) = \text{mult}_w(\nu^*(\mathcal{D}) + (\phi)_{\mathcal{Y}}).$$

Therefore, we have (6.1).

(6.2) Let $\phi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{Y}, \mathcal{D}')$, that is, $\mathcal{D}' + (\phi)_{\mathcal{Y}} \geq 0$. Then

$$0 \leq \nu_*(\mathcal{D}' + (\phi)_{\mathcal{Y}}) = \nu_*(\mathcal{D}') + (\phi)_{\mathcal{X}}.$$

The above observation means that $\Gamma_{\mathbb{K}}^{\times}(\mathcal{Y}, \mathcal{D}') \subseteq \Gamma_{\mathbb{K}}^{\times}(\mathcal{X}, \nu_*(\mathcal{D}'))$. Moreover, by our assumption,

$$\text{mult}_w(\mathcal{D}' + (\phi)_{\mathcal{Y}}) = \text{mult}_w(\nu_*(\mathcal{D}') + (\phi)_{\mathcal{X}})$$

for $\phi \in \Gamma_{\mathbb{K}}^{\times}(\mathcal{Y}, \mathcal{D}')$. Thus the assertion follows.

(7) Let us begin with the following claim:

Claim A.1.1.1. *If $\mathcal{X}$ is projective over $\text{Spec}(k^{\circ})$, then, for any Cartier divisor $\mathcal{E}$ on $\mathcal{X}$, there are effective Cartier divisors $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{X}$ such that $\mathcal{E} = \mathcal{A} - \mathcal{B}$.*

Proof. Let $\mathcal{H}$ be an ample Cartier divisor on $\mathcal{X}$. Let $\mathcal{E} = e_1\Gamma_1 + \cdots + e_r\Gamma_r$ be the decomposition as a Weil divisor. As $\mathcal{H}$ is ample, for a sufficiently large l , there is $\phi \in H^0(\mathcal{X}, l\mathcal{H}) \setminus \{0\}$ such that $\mathcal{A} := l\mathcal{H} + (\phi)$ is effective and $\text{mult}_{\Gamma_i}(\mathcal{A}) \geq e_i$ for $i = 1, \dots, r$. Thus $\mathcal{B} := \mathcal{A} - \mathcal{E}$ is effective and $\mathcal{E} = \mathcal{A} - \mathcal{B}$. $\square$

Let us go back to the proof of (7). By (5), it is sufficient to show that $\mu_{\mathbb{Q},w}(\mathcal{D}) = 0$. By using Chow's lemma together with (6.1), we may assume that $\mathcal{X}$ is projective over $\text{Spec}(k^{\circ})$. First we assume that $\mathcal{D}$ is an ample $\mathbb{Q}$ -Cartier divisor. Then there is $\phi \in \Gamma_{\mathbb{Q}}^{\times}(\mathcal{X}, \mathcal{D})$ such that $\text{mult}_w(\mathcal{D} + (\phi)) = 0$, and hence $\mu_{\mathbb{Q},w}(\mathcal{D}) = 0$.

Next we assume that $\mathcal{D}$ is ample. By using Claim A.1.1.1, we can set

$$\mathcal{D} = a_1\mathcal{D}_1 + \cdots + a_r\mathcal{D}_r,$$

where $\mathcal{D}_1, \dots, \mathcal{D}_r$ are effective Cartier divisors and $a_1, \dots, a_r \in \mathbb{R}$. For any $n > 0$, there are $\delta_1, \dots, \delta_r \in \mathbb{R}$ such that $0 < \delta_i < 1/n$ and $a_i - \delta_i \in \mathbb{Q}$ for all i and that $(a_1 - \delta_1)\mathcal{D}_1 + \cdots + (a_r - \delta_r)\mathcal{D}_r$ is ample. Then, by (2) and the previous case,

$$\begin{aligned} \mu_{\mathbb{Q},w}(\mathcal{D}) &\leq \mu_{\mathbb{Q},w}((a_1 - \delta_1)\mathcal{D}_1 + \cdots + (a_r - \delta_r)\mathcal{D}_r) + \text{mult}_{\mathbb{Q},w}(\delta_1\mathcal{D}_1 + \cdots + \delta_r\mathcal{D}_r) \\ &\leq \delta_1 \text{mult}_w(\mathcal{D}_1) + \cdots + \delta_r \text{mult}_w(\mathcal{D}_r) \\ &\leq (1/n)(\text{mult}_w(\mathcal{D}_1) + \cdots + \text{mult}_w(\mathcal{D}_r)), \end{aligned}$$

which proves the assertion in this case.

Let us consider a general case.

Claim A.1.1.2. *There are an ample $\mathbb{Q}$ -Cartier divisor $\mathcal{A}$ on $\mathcal{X}$ and $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{Q}}^{\times}$ such that $\mathcal{E} := \mathcal{D} - \mathcal{A} + (\phi)$ is effective.*

Proof. If ν is trivial, then $k^{\circ} = k$, so that the assertion is obvious. We assume that ν is non-trivial. Let $\mathcal{A}'$ be an ample Cartier divisor on $\mathcal{X}$. Let X be the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(k^{\circ})$, $D := \mathcal{D} \cap X$ and $A' := \mathcal{A}' \cap X$. Then, as D is big, there are $n \in \mathbb{Z}_{>0}$ and $\phi_1 \in \text{Rat}(\mathcal{X})^{\times}$ such that

$$nD - A' + (\phi_1) \geq 0$$

on X . Therefore, we can find $m \in \mathbb{Z}_{>0}$ such that $n\mathcal{D} - \mathcal{A}' + (\phi_1) + m(\varpi) \geq 0$, and hence, $\mathcal{D} - (1/n)\mathcal{A}' + (\phi_1^{1/n} \varpi^{m/n}) \geq 0$, as required. $\square$

As $\mathcal{A} + (1-\epsilon)\mathcal{E} = \epsilon\mathcal{A} + (1-\epsilon)(\mathcal{D} + (\phi))$ is ample for $0 < \epsilon < 1$, by using (2), (3) and the previous assertion in the case where $\mathcal{D}$ is ample,

$$\begin{aligned} \mu_{\mathbb{Q},w}(\mathcal{D}) &= \mu_{\mathbb{Q},w}(\mathcal{A} + \mathcal{E} + (\phi^{-1})) = \mu_{\mathbb{Q},w}(\mathcal{A} + \mathcal{E}) = \mu_{\mathbb{Q},w}(\mathcal{A} + (1-\epsilon)\mathcal{E} + \epsilon\mathcal{E}) \\ &\leq \mu_{\mathbb{Q},w}(\mathcal{A} + (1-\epsilon)\mathcal{E}) + \epsilon \operatorname{mult}_w(\mathcal{E}) \leq \epsilon \operatorname{mult}_w(\mathcal{E}), \end{aligned}$$

and hence $\mu_{\mathbb{Q},w}(\mathcal{D}) = 0$.

(8) In the same way as (7), we may assume that $\mathcal{X}$ is projective over $\operatorname{Spec}(k^\circ)$. Let $\phi = \phi_1^{a_1} \cdots \phi_r^{a_r} \in \Gamma_{\mathbb{R}}^\times(\mathcal{X}, \mathcal{D})$, where $\phi_1, \dots, \phi_r \in \operatorname{Rat}(\mathcal{X})^\times$ and $a_1, \dots, a_r \in \mathbb{R}$. By Claim A.1.1.1, for each i , there are effective Cartier divisors $\mathcal{A}_i$ and $\mathcal{B}_i$ on $\mathcal{X}$ such that $(\phi_i) = \mathcal{A}_i - \mathcal{B}_i$. Here we consider the map $f : \mathbb{Q}^r \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$f(t_1, \dots, t_r) = \mu_{\mathbb{Q},w}(\mathcal{D} + t_1 \mathcal{B}_1 + \cdots + t_r \mathcal{B}_r).$$

Claim A.1.1.3. $\lim_{\substack{(t_1, \dots, t_r) \rightarrow (0, \dots, 0) \\ (t_1, \dots, t_r) \in \mathbb{Q}^r}} f(t_1, \dots, t_r) = f(0, \dots, 0) = \mu_{\mathbb{Q},w}(\mathcal{D}).$

Proof. First note that there is a positive rational number c such that $\mathcal{D} + t_1 \mathcal{B}_1 + \cdots + t_r \mathcal{B}_r$ is big for $(t_1, \dots, t_r) \in (-c, \infty)^r \cap \mathbb{Q}^r$. Moreover, by using (1) and (4), we can see that f is a convex function over $\mathbb{Q}$, that is, $f(\lambda t + (1-\lambda)t') \leq \lambda f(t) + (1-\lambda)f(t')$ for $t, t' \in \mathbb{Q}^r$ and $\lambda \in [0, 1] \cap \mathbb{Q}$. Therefore, by virtue of [28, Proposition 1.3.1], there is a continuous function $\tilde{f} : (-c, \infty)^r \rightarrow \mathbb{R}$ such that $\tilde{f} = f$ on $(-c, \infty)^r \cap \mathbb{Q}^r$, which shows the assertion of the claim. $\square$

For each $i = 1, \dots, r$ and $n \in \mathbb{Z}_{>0}$, we choose $t_{i,n} \in \mathbb{R}$ and $t'_{i,n} \in \mathbb{Q}$ such that $a_i + t_{i,n} \in \mathbb{Q}$ and $0 \leq t_{i,n} \leq t'_{i,n} \leq 1/n$. Then

$$\begin{aligned} \mathcal{D} + \sum_i t'_{i,n} \mathcal{B}_i + \sum_i (a_i + t_{i,n})(\phi_i) &\geq \mathcal{D} + \sum_i t_{i,n} \mathcal{B}_i + \sum_i (a_i + t_{i,n})(\phi_i) \\ &= \mathcal{D} + \sum_i a_i(\phi_i) + \sum_i t_{i,n} \mathcal{A}_i \geq 0, \end{aligned}$$

and hence

$$\mu_{\mathbb{Q},w}(\mathcal{D} + \sum_i t'_{i,n} \mathcal{B}_i) \leq \operatorname{mult}_w(\mathcal{D} + \sum_i t'_{i,n} \mathcal{B}_i + \sum_i (a_i + t_{i,n})(\phi_i)).$$

Thus, taking the limits as $n \rightarrow \infty$ together with Claim A.1.1.3, we have

$$\mu_{\mathbb{Q},w}(\mathcal{D}) \leq \operatorname{mult}_w(\mathcal{D} + \sum_i a_i(\phi_i)),$$

which gives rise to $\mu_{\mathbb{Q},w}(\mathcal{D}) \leq \mu_{\mathbb{R},w}(\mathcal{D})$, so that (8) follows from (5). $\square$

A.2. Sectional decomposition. Let $\mathcal{X}$ be a regular and proper variety over k° . Let $\mathcal{D}$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$. We assume that $\Gamma^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$. We set

$$\begin{cases} \operatorname{Bs}(\mathcal{D}) := \bigcap_{\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})} \operatorname{Supp}_{\mathbb{R}}(\mathcal{D} + (\phi)), \\ \mathcal{F}(\mathcal{D}) := \sum_{\Gamma} \inf \{ \operatorname{mult}_{\Gamma}(\mathcal{D} + (\phi)) \mid \phi \in \Gamma^\times(\mathcal{X}, \mathcal{D}) \} \cdot \Gamma, \\ \mathcal{P}(\mathcal{D}) := \mathcal{D} - \mathcal{F}(\mathcal{D}), \end{cases}$$

where Γ runs over all prime divisors on $\mathcal{X}$. Note that the above “inf” can be replaced by “min” because the set $\{ \operatorname{mult}_{\Gamma}(\mathcal{D} + (\phi)) \mid \phi \in \Gamma^\times(\mathcal{X}, \mathcal{D}) \}$ is discrete in $\mathbb{R}$. The decomposition $\mathcal{D} = \mathcal{P}(\mathcal{D}) + \mathcal{F}(\mathcal{D})$ is called the *sectional decomposition* of $\mathcal{D}$.

Lemma A.2.1. (1) *The natural inclusion map*

$$H^0(\mathcal{X}, \mathcal{P}(\mathcal{D})) \rightarrow H^0(\mathcal{X}, \mathcal{D})$$

is bijective.

(2) $\text{codim Bs}(\mathcal{P}(\mathcal{D})) \geq 2$.

Proof. By our construction, $\mathcal{D} + (\phi) \geq \mathcal{F}(\mathcal{D})$ for all $\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})$. Thus (1) follows. Moreover, if $\text{codim Bs}(\mathcal{P}(\mathcal{D})) = 1$, then there is a prime divisor Γ such that

$$\text{mult}_\Gamma(\mathcal{P}(\mathcal{D}) + (\phi)) > 0$$

for all $\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})$, that is, $\text{mult}_\Gamma(\mathcal{D} + (\phi)) > \text{mult}_\Gamma(\mathcal{F}(\mathcal{D}))$ for all $\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})$, which is a contradiction. $\square$

From now on, we assume that $\Gamma_\mathbb{Q}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$. We set

$$N(\mathcal{D}) := \{m \in \mathbb{Z}_{\geq 1} \mid \Gamma^\times(\mathcal{X}, m\mathcal{D}) \neq \emptyset\}.$$

Note that $N(\mathcal{D}) \neq \emptyset$. For $m \in N(\mathcal{D})$, we set $\mathcal{F}_m := \mathcal{F}(m\mathcal{D})$ and $\mathcal{P}_m := \mathcal{P}(m\mathcal{D})$.

Lemma A.2.2. (1) $\mathcal{F}_m + \mathcal{F}_{m'} \geq \mathcal{F}_{m+m'}$ for $m, m' \in N(\mathcal{D})$. In particular,

$$\inf_{m \in N(\mathcal{D})} \left\{ \frac{\text{mult}_w(\mathcal{F}_m)}{m} \right\} = \lim_{\substack{m \rightarrow \infty, \\ m \in N(\mathcal{D})}} \frac{\text{mult}_w(\mathcal{F}_m)}{m}$$

for all $w \in \text{DVal}_k(\mathcal{X})$ (cf. [33, Chapter 3, 98]).

$$(2) \mu_{\mathbb{Q}, \Gamma}(\mathcal{D}) = \inf_{m \in N(\mathcal{D})} \left\{ \frac{\text{mult}_\Gamma(\mathcal{F}_m)}{m} \right\} \text{ for all prime divisors } \Gamma \text{ on } \mathcal{X}.$$

Proof. (1) is obvious because $\phi\phi' \in \Gamma^\times(\mathcal{X}, (m+m')\mathcal{D})$ for all $\phi \in \Gamma^\times(\mathcal{X}, m\mathcal{D})$ and $\phi' \in \Gamma^\times(\mathcal{X}, m'\mathcal{D})$. For (2), note that

$$\Gamma_\mathbb{Q}^\times(\mathcal{X}, \mathcal{D}) = \bigcup_{m \in N(\mathcal{D})} (\Gamma^\times(\mathcal{X}, m\mathcal{D}))^{1/m}.$$

$\square$

A.3. Characterization in terms of μ_w . The following theorem is a characterization of relatively nef Cartier divisors in terms of the asymptotic multiplicity.

Theorem A.3.1. *Let $\mathcal{X}$ be a $(d+1)$ -dimensional, proper and normal variety over k° and let $\mathcal{D}$ be an $\mathbb{R}$ -Cartier divisor on $\mathcal{X}$. If $\Gamma_\mathbb{Q}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$ and $\mu_{\mathbb{Q}, w}(\mathcal{D}) = 0$ for all $w \in \text{DVal}_k(\mathcal{X})$, then $\mathcal{D}$ is relatively nef. In particular, if $\mathcal{D}$ is big, then the following are equivalent:*

- (1) $\mathcal{D}$ is relatively nef with respect to $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$.
- (2) $\mu_{\mathbb{Q}, w}(\mathcal{D}) = 0$ for all $w \in \text{DVal}_k(\mathcal{X})$.
- (3) $\mu_{\mathbb{R}, w}(\mathcal{D}) = 0$ for all $w \in \text{DVal}_k(\mathcal{X})$.

Proof. Let us begin with the following claim:

Claim A.3.1.1. *Let $\mathcal{Y}$ be a normal and proper variety over k° and let $\nu: \mathcal{Y} \rightarrow \mathcal{X}$ be a dominant morphism over $\text{Spec}(k^\circ)$ such that $\text{Rat}(\mathcal{Y})$ is algebraic over $\text{Rat}(\mathcal{X})$. If*

$$\Gamma_\mathbb{Q}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset \quad \text{and} \quad \mu_{\mathbb{Q}, w}(\mathcal{D}) = 0$$

for all $w \in \text{DVal}_k(\mathcal{X})$, then $\mu_{\mathbb{Q}, w'}(\nu^(\mathcal{D})) = 0$ for all $w' \in \text{DVal}_k(\mathcal{Y})$.*

Proof. Let w' be a divisorial valuation of $\text{Rat}(\mathcal{Y})$ over k and let w be the restriction of w' to $\text{Rat}(\mathcal{X})$. As $\text{Rat}(\mathcal{Y})$ is algebraic over $\text{Rat}(\mathcal{X})$, we can see that $\mathcal{O}_{w'}/m_{w'}$ is algebraic over $\mathcal{O}_w/m_w$, so that w is a divisorial valuation of $\text{Rat}(\mathcal{X})$ over k . Then, for an $\mathbb{R}$ -Cartier divisor $\mathcal{L}$ on $\mathcal{X}$, $\text{mult}_{w'}(\nu^*(\mathcal{L})) = \text{mult}_w(\mathcal{L})$. Thus,

$$\begin{aligned} \mu_{\mathbb{Q}, w'}(\nu^*(\mathcal{D})) &= \inf \left\{ \text{mult}_{w'}(\nu^*(\mathcal{D}) + (\psi)) \mid \psi \in \Gamma_{\mathbb{Q}}^{\times}(\mathcal{Y}, \nu^*(\mathcal{D})) \right\} \\ &\leq \inf \left\{ \text{mult}_{w'}(\nu^*(\mathcal{D} + (\phi))) \mid \phi \in \Gamma_{\mathbb{Q}}^{\times}(\mathcal{X}, \mathcal{D}) \right\} \\ &= \inf \left\{ \text{mult}_w(\mathcal{D} + (\phi)) \mid \phi \in \Gamma_{\mathbb{Q}}^{\times}(\mathcal{X}, \mathcal{D}) \right\} = \mu_{\mathbb{Q}, w}(\mathcal{D}), \end{aligned}$$

which prove the claim. $\square$

Claim A.3.1.2. *We may assume that $\mathcal{X}$ is regular and projective over $\text{Spec}(k^{\circ})$.*

Proof. We assume that the theorem holds if $\mathcal{X}$ is regular and projective. By de Jong's theorem [15], there is a regular and projective variety $\mathcal{Y}$ over k° together with a dominant morphism $\mu : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec}(k^{\circ})$ such that $\text{Rat}(\mathcal{Y})$ is algebraic over $\text{Rat}(\mathcal{X})$. By the previous claim and our assumption, we can see that $\nu^*(\mathcal{D})$ is relatively nef, so that $\mathcal{D}$ is also relatively nef. $\square$

Let C be an irreducible and reduced curve on $\mathcal{X}_{\circ}$. Let us see $(\mathcal{D} \cdot C) \geq 0$. Clearly we may assume that $\mathcal{D}$ is effective and $C \subseteq \text{Supp}_{\mathbb{R}}(\mathcal{D})$. There is a succession of blowing-ups $\rho : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ at closed points such that the strict transform $\tilde{C}$ of C is regular (cf. [20, Theorem 1.101]). If $(\rho^*(\mathcal{D}) \cdot \tilde{C}) \geq 0$, then $(\mathcal{D} \cdot C) = (\rho^*(\mathcal{D}) \cdot \tilde{C}) \geq 0$, so that we may assume that C is regular.

Let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ be the blowing-up along C and let $\mathcal{E}$ be the exceptional set of π . Let $\mathcal{D}'$ be the strict transform of $\mathcal{D}$. Then $\pi^*(\mathcal{D}) = \mathcal{D}' + e\mathcal{E}$ for some $e \in \mathbb{Z}_{>0}$. Let H be a very ample divisor on $\mathcal{E}$. Choosing general members $H_1, \dots, H_{d-1}$ of $|H|$, we set $C' = H_1 \cap \dots \cap H_{d-1}$ and $\pi_*(C') = aC$ for some $a \in \mathbb{Z}_{>0}$. As $H_1, \dots, H_{d-1}$ are general, $C' \not\subseteq \text{Supp}_{\mathbb{R}}(\mathcal{D}') \cap \mathcal{E}$. If $(\mathcal{E} \cdot C') = (\mathcal{O}_{\mathcal{X}}(\mathcal{E})|_{\mathcal{E}} \cdot H^{d-1}) \geq 0$, then

$$a(\mathcal{D} \cdot C) = (\pi^*(\mathcal{D}) \cdot C') = (\mathcal{D}' \cdot C') + e(\mathcal{E} \cdot C') \geq 0.$$

Thus we may assume $(\mathcal{O}_{\mathcal{X}}(\mathcal{E})|_{\mathcal{E}} \cdot H^{d-1}) < 0$.

Let $m\pi^*(\mathcal{D}) = \mathcal{P}_m + \mathcal{F}_m$ be the sectional decomposition of $m\pi^*(\mathcal{D})$. By virtue of (1) in Lemma A.2.2, there are finitely many prime divisors $\Gamma_1, \dots, \Gamma_r$ of $\mathcal{Y}$ such that

$$\mathcal{F}_m = a_{m,1}\Gamma_1 + \dots + a_{m,r}\Gamma_r$$

for some $a_{m,1}, \dots, a_{m,r} \in \mathbb{R}_{\geq 0}$. First we assume that $\Gamma_i \neq \mathcal{E}$ for all $i = 1, \dots, r$. Choosing general members $H_1, \dots, H_{d-2} \in |H|$ if necessary, we have

$$C' \not\subseteq (\text{Bs}(\mathcal{P}_m) \cup \Gamma_1 \cup \dots \cup \Gamma_r) \cap \mathcal{E},$$

and hence

$$am(\mathcal{D} \cdot C) = (\pi^*(m\mathcal{D}) \cdot C') = (\mathcal{P}_m \cdot C') + \sum_{i=1}^r a_{m,i}(\Gamma_i \cdot C') \geq 0.$$

Therefore, we may assume that $\Gamma_1 = \mathcal{E}$. By (2) in Lemma A.2.2 and our assumption, $\lim_{m \rightarrow \infty} a_{m,1}/m = 0$. For any $\epsilon > 0$, we choose m such that $0 \leq a_{m,1}/m \leq \epsilon$. As before, choosing general members $H_1, \dots, H_{d-2} \in |H|$ if necessary,

$$C' \not\subseteq (\text{Bs}(\mathcal{P}_m) \cup \Gamma_2 \cup \dots \cup \Gamma_r) \cap \mathcal{E}$$

holds, so that $(\mathcal{P}_m \cdot C') \geq 0$ and $(\Gamma_i \cdot C') \geq 0$ for $i = 2, \dots, r$. Thus

$$a(\mathcal{D} \cdot C) = (\pi^*(\mathcal{D}) \cdot C') \geq (a_{m,1}/m)(\mathcal{E} \cdot C') \geq \epsilon(\mathcal{O}_{\mathcal{X}}(\mathcal{E})|_{\mathcal{E}} \cdot H^{d-2}).$$

Therefore $(\mathcal{D} \cdot C) \geq 0$ because ϵ is an arbitrary small number.

Finally, the last assertion of the theorem follows from the first assertion, (7) and (8) in Proposition A.1.1. $\square$

As a corollary, we have the following characterization of relatively nef $\mathbb{R}$ -Cartier divisors. It is proved in [6, Theorem 5.11 and Lemma 5.12] in the case where the characteristic of k°/k^∞ is zero. In general, it seems to be proved by Thuillier. Note that our proof is based on de Jong's alteration.

Corollary A.3.2. *Let X be a proper and normal variety over k and let L be an $\mathbb{R}$ -Cartier divisor on X . Let $\mathcal{X}$ be a normal model of X over $\text{Spec}(k^\circ)$ and let $\mathcal{L}$ be an $\mathbb{R}$ -Cartier on $\mathcal{X}$ with $\mathcal{L} \cap X = L$. We assume that there is a sequence $\{(\mathcal{X}_n, \mathcal{L}_n)\}_{n=1}^\infty$ with the following properties:*

- (1) $\mathcal{X}_n$ is a normal model of X over $\text{Spec}(k^\circ)$.
- (2) $\mathcal{L}_n$ is a relatively nef $\mathbb{R}$ -Cartier divisor on $\mathcal{X}_n$ such that $\mathcal{L}_n \cap X = L$.
- (3) $\lim_{n \rightarrow \infty} \text{mult}_w(\mathcal{L}_n) = \text{mult}_w(\mathcal{L})$ for all $w \in \text{DVal}_k(\mathcal{X})$.

Then $\mathcal{L}$ is relatively nef.

Proof. If ν is trivial, then the assertion is obvious, so that we assume that ν is non-trivial. Clearly we may assume that there is a birational morphism $\nu_n : \mathcal{X}_n \rightarrow \mathcal{X}$ over k° . By using Chow's lemma, we have a birational morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}$ over k° such that $\mathcal{X}'$ is projective over $\text{Spec}(k^\circ)$. Let X' be the generic fiber of $\mathcal{X}' \rightarrow \text{Spec}(k^\circ)$. Let $\mathcal{X}'_n$ be the normalization of the main component of $\mathcal{X}_n \times_{\mathcal{X}} \mathcal{X}'$, and let $\mu_n : \mathcal{X}'_n \rightarrow \mathcal{X}_n$ be the induced morphism. We set

$$\mathcal{L}' := \mu^*(\mathcal{L}), \quad L' := \mathcal{L}' \cap X' \quad \text{and} \quad \mathcal{L}'_n := \mu_n^*(\mathcal{L}_n).$$

Then $\mathcal{X}'_n$ is a model of X' over $\text{Spec}(k^\circ)$ and $\mathcal{L}'_n$ is a relatively nef $\mathbb{R}$ -Cartier divisor on $\mathcal{X}'_n$ such that $\mathcal{L}'_n \cap X' = L'$. Moreover,

$$\text{mult}_w(\mathcal{L}') = \text{mult}_w(\mathcal{L}) \quad \text{and} \quad \text{mult}_w(\mathcal{L}'_n) = \text{mult}_w(\mathcal{L}_n)$$

for all $w \in \text{DVal}_k(\mathcal{X})$. Therefore, we may assume that $\mathcal{X}$ is projective.

Let $\mathcal{A}$ be a relatively nef and big Cartier divisor on $\mathcal{X}$. As L is nef on X , $\mathcal{L} + \epsilon \mathcal{A}$ is nef and big on X for $\epsilon > 0$, so that, by virtue of Theorem A.3.1, it is sufficient to see that

$$\mu_{\mathbb{R},w}(\mathcal{L} + \epsilon \mathcal{A}) = 0$$

for all $w \in \text{DVal}_k(\mathcal{X})$ and $\epsilon > 0$. Replacing $\mathcal{X}$ by a suitable birational model, we may assume that there is a vertical prime divisor Γ on $\mathcal{X}$ such that $w = a \text{ord}_\Gamma$ for some positive number a . Let $\mathcal{X}_\circ = a_1 \Gamma_1 + \dots + a_r \Gamma_r$ be the irreducible decomposition of the central fiber $\mathcal{X}_\circ$ of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ as a Weil divisor. Renumbering $\Gamma_1, \dots, \Gamma_r$, we may set $\Gamma = \Gamma_1$. Let w_{Γ_i} be the additive valuation over k arising from Γ_i . Note that $w = w_{\Gamma_1}$. For a positive number δ , there is N such that

$$\left| \text{mult}_{w_{\Gamma_i}}(\mathcal{L}) - \text{mult}_{w_{\Gamma_i}}(\mathcal{L}_n) \right| \leq a_i \delta$$

for all $n \geq N$ and $i = 1, \dots, r$. Then $(\nu_n)_*(\mathcal{L}_n) - \delta \mathcal{X}_\circ \leq \mathcal{L}$ for $n \geq N$. Therefore, as $\mathcal{L}_n - \delta(\mathcal{X}_n)_\circ + \epsilon \nu_n^*(\mathcal{A})$ is relatively nef and big, by (2), (6.2) and (7) in Proposition A.1.1,

$$\begin{aligned}
0 &\leq \mu_{\mathbb{R},w}(\mathcal{L} + \epsilon \mathcal{A}) \\
&\leq \mu_{\mathbb{R},w}((\nu_n)_*(\mathcal{L}_n) - \delta \mathcal{X}_\circ + \epsilon \mathcal{A}) \\
&\quad + \text{mult}_w(\mathcal{L} - (\nu_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \quad (\because (2)) \\
&= \mu_{\mathbb{R},w}((\nu_n)_*(\mathcal{L}_n - \delta(\mathcal{X}_n)_\circ + \epsilon \nu_n^*(\mathcal{A}))) \\
&\quad + \text{mult}_w(\mathcal{L} - (\nu_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \\
&\leq \mu_{\mathbb{R},w}(\mathcal{L}_n - \delta(\mathcal{X}_n)_\circ + \epsilon \nu_n^*(\mathcal{A})) \\
&\quad + \text{mult}_w(\mathcal{L} - (\nu_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \quad (\because (6.2)) \\
&= \text{mult}_w(\mathcal{L} - (\nu_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \quad (\because (7)) \\
&\leq |\text{mult}_w(\mathcal{L}) - \text{mult}_w(\mathcal{L}_n)| + \delta \text{mult}_w(\mathcal{X}_\circ) \leq 2a_1 \delta
\end{aligned}$$

for $n \geq N$. Thus $\mu_{\mathbb{R},w}(\mathcal{L} + \epsilon \mathcal{A}) = 0$ because δ is an arbitrary positive number. $\square$

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INDEX

- $\text{Bs}(\mathcal{D})$, 97
- $\hat{\chi}(M, \|\cdot\|)$, 7
- $\hat{\chi}(X, \overline{D})$, 44
- $C^0(M)$, 6
- $C^0_\eta(X^{\text{an}})$, 21
- $C^0_{F_\infty}(X(\mathbb{C}))$, 4, 39
- $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$, 5, 47
- $\widehat{\deg}(\overline{D}|_x)$, 42
- $\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1})$, 3
- $\widehat{\deg}_{\infty}(\overline{D}|_x)$, 42
- $\widehat{\deg}_{\text{p}}(\overline{D}|_x)$, 42
- $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$, 27
- $\widehat{\deg}_v(\overline{D}|_x)$, 26
- $\widehat{\deg}_v(\overline{L}_1 \cdots \overline{L}_{d+1})$, 32
- $\widehat{\deg}_v(\overline{L}_1 \cdots \overline{L}_d; \phi)$, 29
- $D_1 \leq D_2$, 25
- D_W , 15
- $\mathcal{D}^a$, 41
- $\mathcal{D}_{(\text{p})}$, 41
- $\overline{D}^\tau$, 40
- $\overline{D}_1 \leq \overline{D}_2$, 40
- $\overline{\mathcal{D}}^a$, 41
- $\widehat{\text{Div}}^a_{\text{int}}(X)_{\mathbb{R}}$, 46
- $\widehat{\text{Div}}_{C^0}(X)_{\mathbb{R}}$, 4, 40
- $\text{Div}(X)$, 7, 15
- $\text{Div}(X)_{\mathbb{K}}$, 15
- $\text{Div}(X)_{\mathbb{R}}$, 7
- $\text{Div}(\mathcal{X})$, 93
- $\text{Div}(\mathcal{X})_{\mathbb{R}}$, 94
- $\text{Div}(\mathcal{X}; x)$, 93
- $\text{Div}(\mathcal{X}; x)_{\mathbb{R}}$, 94
- $\text{Div}^a_{C^0}(X)_{\mathbb{R}}$, 25
- $\text{DVal}_k(\mathcal{X})$, 94
- f^{an} , 17
- F_∞ , 4, 39
- $\mathcal{F}(\mathcal{D})$, 97
- $G(\overline{D}; V_\bullet)$, 84
- $g(\mathcal{X}, \mathcal{D})$, 4, 22
- $\Gamma^\times(\mathcal{X}, \mathcal{D})$, 94
- $\Gamma^\times_{\mathbb{K}}(\mathcal{X}, \mathcal{D})$, 94
- $\widehat{\Gamma}^\times_{\mathbb{K}}(X, \overline{D})$, 79
- $\hat{h}^0(M, \|\cdot\|)$, 7
- $\hat{H}^0(X, \overline{D}')$, 43
- $\hat{H}^0(X, \overline{D})$, 5, 43
- $\hat{h}^0(X, \overline{D})$, 43
- $H^0(X, D)$, 5, 21
- $H^0(X, D)$, 43
- $H^0(\mathcal{X}, \mathcal{D})$, 2
- K_{p} , 4, 39
- k° , 6
- $k^{\circ\circ}$, 6
- k_ν , 12
- $\log|\mathcal{S}|$, 34
- $\max\{D_1, \dots, D_r\}$, 8
- $\min\{D_1, \dots, D_r\}$, 8
- $\mathbb{M}(X; \Psi)$, 36
- $\mathbb{M}(X^{\text{an}}; \Psi)$, 34
- $\mu_{\mathbb{K}, w}(\mathcal{D})$, 95
- $\mu_{\mathbb{K}, \xi}(\overline{D})$, 79
- $\text{mult}^{\sim}_{z_p}$, 84
- mult_w , 95
- $\mu_{\overline{L}_1 \cdots \overline{L}_d}$, 39
- $N(\mathcal{D})$, 98
- $\nu_\xi(\overline{D})$, 81
- $\|\phi\|_{g_\infty}$, 2
- $\|\phi\|_{g_p}$, 5, 43
- $|\phi|_{g_p}$, 43
- $\|\phi\|_g$, 21
- ord_Γ , 7
- ord_γ , 7
- $\text{ord}^{\sim}_{z_p}(f)$, 83
- ord_A , 7
- $(\varphi)_{\mathbb{K}}$, 78
- $\mathcal{P}(\mathcal{D})$, 97
- (φ) , 40
- $(\varphi)_{\mathbb{K}}$, 78
- $r_{\mathcal{X}}$, 17
- $\text{Rat}(X)$, 15
- $\text{Rat}(X)_{\mathbb{K}}^\times$, 15, 78
- $\text{Rat}(\mathcal{X})$, 93
- $\text{Rat}(\mathcal{X})_{\mathbb{K}}^\times$, 94
- $\Sigma(\overline{D}; Q)$, 65, 71
- $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, 67
- $\text{Spec}^{\text{an}}_k(A)$, 17
- $\text{Supp}_{\mathbb{K}}(D)$, 15
- $\text{Supp}_W(D)$, 15
- $\Theta(\overline{D}; V_\bullet)$, 84
- $\Upsilon(\overline{D})$, 6, 76
- $\Upsilon(\overline{D}; R)$, 73
- $\Upsilon(\mathcal{D})$, 3
- $\Upsilon_{\text{rel}}(\overline{D})$, 72
- $\text{vol}(\overline{D})$, 5, 44
- $\text{vol}(\overline{D}; V_\bullet)$, 84
- $\widehat{\text{vol}}(\mathcal{D})$, 2
- $\widehat{\text{vol}}_{\mathcal{X}}(\overline{D})$, 44
- $\widehat{\text{vol}}_{\mathcal{X}}(\overline{D}; V_\bullet)$, 84
- ν_{p} , 4, 39
- $\text{WDiv}(\mathcal{X})$, 93
- $\text{WDiv}(\mathcal{X})_{\mathbb{R}}$, 94
- $X(\mathbb{C})$, 4, 39
- X^{an} , 3, 17
- X_{p} , 4
- $\mathcal{X}_{(\text{p})}$, 41

$\mathcal{X}_\circ$, 12

adelic arithmetic principal divisor, 40
adelic arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type, 4, 39

adelic $\mathbb{R}$ -Cartier divisor of C^0 -type, 25
arithmetic $\mathbb{R}$ -Cartier divisor of C^0 -type, 1, 20
arithmetic variety, 7
associated absolute value, 16

associated adelic arithmetic $\mathbb{R}$ -Cartier divisor, 41

associated adelic $\mathbb{R}$ -Cartier divisor, 26
associated global adelic $\mathbb{R}$ -Cartier divisor, 41
associated $\mathbb{K}$ -Weil divisor, 15
associated prime of the multiplicative semi-norm, 16

big adelic arithmetic $\mathbb{R}$ -Cartier divisor, 44
big $\mathbb{R}$ -Cartier divisor, 95
birational system of models, 34

center of valuation, 95
 $\hat{\chi}$ -volume of adelic arithmetic $\mathbb{R}$ -Cartier divisor, 44
continuous plurisubharmonic function, 18
convex lattice hull, 8

D -Green function of $(C^0 \cap \text{PSH})$ -type, 18
 D -Green function of C^0 -type, 18
defining model, 40
Dirichlet's unit theorem, 87
divisorial valuation, 94

extended arithmetic asymptotic multiplicity, 81

fractional ideal sheaf, 33

global adelic $\mathbb{R}$ -Cartier divisor of C^0 -type, 40
global degree, 42
global model function, 36
Green function induced by model, 4
Green function of C^0 -type, 3, 20

integrable adelic arithmetic $\mathbb{R}$ -Cartier divisor, 5, 46

integrable adelic $\mathbb{R}$ -Cartier divisor, 25
integrable arithmetic $\mathbb{R}$ -Cartier divisor, 2

$\mathbb{K}$ -asymptotic multiplicity, 79, 95
 $\mathbb{K}$ -Cartier divisor, 15
 $\mathbb{K}$ -principal divisor, 15
 $\mathbb{K}$ -rational function, 15
 $\mathbb{K}$ -support, 15

local degree, 26

model, 7

model function, 34
model of (X, D) , 3, 22
model of D , 22
multiplicative semi-norm, 16

nef adelic arithmetic $\mathbb{R}$ -Cartier divisor, 44
normal model, 7

of $(C^0 \cap \text{PSH})$ -type, 4, 24

pseudo-effective arithmetic $\mathbb{R}$ -Cartier divisor, 44

$\mathbb{R}$ -Cartier divisor, 7
radon measure induced by local intersection, 39

reduction map, 17
regular model, 7
relatively nef, 7, 20
relatively nef adelic arithmetic $\mathbb{R}$ -Cartier divisor, 44

relatively nef adelic $\mathbb{R}$ -Cartier divisor, 25
relatively nef global adelic $\mathbb{R}$ -Cartier divisor, 46

residue field, 16

sectional decomposition, 97
support as a Weil-divisor, 15

truncation of the adelic arithmetic $\mathbb{R}$ -Cartier divisor, 40

variety over a noetherian integral scheme, 7
vertical curve, 7
vertical fractional ideal sheaf, 33
volume of adelic arithmetic $\mathbb{R}$ -Cartier divisor, 44

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