

On Demuškin Groups

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Abstract

Let k be an algebraic number field, p an odd prime, and $G_S = \text{Gal}(k_S/k)$, where k_S is the maximal pro- p -extension of k unramified outside the set S of primes dividing p . We consider situations where G_S is a Demuškin group, especially when k is totally real or of CM type.

Several authors gave conditions in order to express the Galois group G_S as a free pro- p -product:

$$G_S = \left(\star_{v \in T} G_v \right) \star F$$

where F is a free pro- p -group, T is a subset of S and $G_v = \text{Gal}(k_v(p)/k_v)$, $k_v(p)$ being the maximal pro- p -extension of the localization of k at v . However these conditions are rather restrictive, implying in particular that the decomposition group G_S^v of an extension of $v \in T$ to k_S is equal to the whole local group G_v . Keeping this in mind, we are all the more interested in the cases where G_S is a Demuškin group without being equal to G_v for any $v \in S$.

When k is totally real, we can give a criterion for G_S to be a Demuškin group involving Iwasawa theory, see Proposition 5. Using numerical data, we obtain many examples when $p = 3$ and k is a real quadratic field.

We classify the Demuškin groups G_S in two types, depending on the existence of a place $v \in S$ such that the natural map

$$H^2(G_S, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2(G_v, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. If such a v exists, we say that G_S is a *Demuškin group of local type*.

When k is totally real or of CM type, we give necessary and sufficient conditions for G_S to be a Demuškin group of local type, involving only the arithmetic of the base field k , see Theorems 1 and 2. Examples are given in the case $p = 3$ and k is an imaginary bi-quadratic field. In these examples we have $G_S = G_S^v$, where v is the unique place of k dividing 3 and $G_S \neq G_v$; this was the situation we were interested in, without allowing the case k is totally real where the Proposition 5 gives an easy characterization.

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1 Introduction

Let us introduce some notations associated with the field k and the prime p . We denote by

- r_1 (resp. r_2) the number of real (resp. complex) archimedean places of k
- k_S the maximal pro- p -extension of k unramified outside S
- k_∞ the cyclotomic $\mathbb{Z}_p$ -extension of k ($k_\infty \subset k_S$), $\Gamma = \text{Gal}(k_\infty/k)$ and $\Lambda = \mathbb{Z}_p[[\Gamma]]$
- μ_p the group of p -th roots of unity, and for a field F , $\mu(F)$ the p -primary part of the group of roots of unity contained in F
- k_v the localization of k at a prime v , U_v (resp. $U_v^{(1)}$) the group of units (resp. principal units) in the ring of integers of k_v and $n_v = (k_v : \mathbb{Q}_\ell)$ the local degree, where v divides the prime ℓ
- $k_v(p)$ the maximal pro- p -extension of k_v
- $G_S = \text{Gal}(k_S/k)$, $G_v = \text{Gal}(k_v(p)/k_v)$
- G_S^v the decomposition group of an extension of v to k_S (G_S^v is defined up to an inner automorphism of G_S)
- $k' = k(\mu_p)$, $\Delta = \text{Gal}(k'/k)$
- $\delta = \begin{cases} 1 & \text{if } \mu_p \subset k \\ 0 & \text{otherwise} \end{cases} \quad \delta_v = \begin{cases} 1 & \text{if } \mu_p \subset k_v \\ 0 & \text{otherwise} \end{cases}$
- E the unit group of the ring of integers of k
- Cl (resp. Cl_S) the p -Sylow of the class group (resp. S -class group) of k
- $V_S = \{a \in k^\times \mid a \in k_v^{\times p} \text{ for } v \in S; a \in U_v k_v^{\times p} \text{ for } v \notin S\} / k^{\times p}$. By comparing class field theory and Kummer theory, we obtain an isomorphism

$$V_S \simeq \text{Hom}_\Delta(\text{Cl}_S(k'), \mu_p) \quad (1)$$

When there is more than one algebraic number field in consideration, we will append the name of the field to the above objects.

Let G be a pro- p -group. For every integer n , let $H^n(G) = H^n(G, \mathbb{Z}/p\mathbb{Z})$, G acting trivially on $\mathbb{Z}/p\mathbb{Z}$. $d(G) := \dim H^1(G)$ is the generator rank of G , and $r(G) := \dim H^2(G)$ is the relation rank of G .

A motivation for our problem is given by the following local result

1. If $\mu_p \not\subset k_v$, then G_v is a free pro- p -group.
2. If $\mu_p \subset k_v$, then G_v is a Demuškin group.

Definition 1 G is defined to be a Demuškin group if $d(G)$ is finite, $r(G) = 1$, and the cup-product

$$H^1(G) \times H^1(G) \rightarrow H^2(G) \text{ is a perfect pairing.}$$

Let us now consider the global situation. The Galois group G_S has the following well-known property:

$\text{cd}(G_S)$ (the cohomological dimension of G_S) is ≤ 2 , and $d(G_S)$ is finite.

and this property holds also for a Demuškin group. After the case G_S is a free pro- p -group (or equivalently $cd(G_S) = 1$), the case G_S is a Demuškin group might be considered as the easiest situation to handle in restricted ramification Theory.

In order to evaluate the relation rank of G_S , the Poitou-Tate duality gives the following exact sequence

$$0 \rightarrow \mu_p(k) \rightarrow \prod_{v \in S} \mu_p(k_v) \rightarrow H^2(G_S)^* \rightarrow V_S \rightarrow 0 \quad (2)$$

where A^* denotes the Pontrjagin dual of a locally compact abelian group. In the sequence (2), the local group $\mu_p(k_v)$ is in duality with $H^2(G_v)$ and the map $\mu_p(k_v) \rightarrow H^2(G_S)^*$ corresponds to the composition $G_v \rightarrow G_S^v \rightarrow G_S$.

We will use the following relation between the generator rank and the relation rank of G_S giving the value of its Euler-Poincaré characteristic:

$$\chi(G_S) := 1 - d(G_S) + r(G_S) = -r_2 \quad (3)$$

Let us suppose that G_S is a Demuškin group. Then the relation rank of G_S is equal to one, so there are only two possibilities:

1. $V_S = 0$ and $\delta + 1 = \sum_{v \in S} \delta_v$ —we will say that G_S is a *Demuškin group of local type*. Then the natural map $H^2(G_S) \rightarrow H^2(G_v)$ is an isomorphism, for every $v \in S$ such that $\delta_v = 1$.
2. $\dim V_S = 1$ and $\delta = \sum_{v \in S} \delta_v$ —we will say that G_S is a *Demuškin group of global type*. Then, $H^2(G_S) \rightarrow H^2(G_v)$ is the zero map for every $v \in S$.

In any case there are at most two primes $v \in S$ such that $\delta_v = 1$. Furthermore such primes can not split in k_S ; otherwise there would exist a finite extension K in k such that $\sum_{v \in S(K)} \geq p > 2$, which is absurd because $G_S(K)$, having finite index in G_S , is also a Demuškin group. So we have proved the following

Proposition 1 *Suppose that G_S is a Demuškin group and there exists $v \in S$ such that $\delta_v = 1$. Then $G_S = G_S^v$.*

2 Known Results

We quote here some results relevant to our problem.

Suppose first that k contains μ_p and G_S is a Demuškin group. Then, by Proposition 1, the places of S do not split in k_S and we have $|S| = 1$ (resp. $|S| = 2$) if G_S is a Demuškin group of global (resp. local) type. Conversely, if k contains μ_p , using a result of Kuz'min, see [4, Fundamental Theorem, A & B(a)], we have

1. if $|S(k_S)| = 1$, then G_S is a free pro- p -group.

2. if $|S(k_S)| = 2$, then $G_S = G_{v_1} = G_{v_2}$, where $S(k) = \{v_1, v_2\}$, hence G_S is a Demuškin group.

Hence, we deduce the following (essentially due to Kuz'min)

Proposition 2 *Suppose that k contains μ_p and $S = S_p$. Then G_S is a Demuškin group if and only if $S = \{v_1, v_2\}$ and v_i does not split in k_S for $i = 1, 2$. Moreover, we have $G_S = G_{v_1} = G_{v_2}$, and G_S is a Demuškin group of local type.*

Ku'zmin gave the following example: Let $p = 3$ and $k = \mathbb{Q}(\sqrt{-3}, \sqrt{15})$. Then G_S is a Demuškin group.

In [6], Wingberg gives necessary and sufficient conditions for G_S to be equal, up to a free pro- p -group F , to the free pro- p -product of the local groups G_v for $v \in (S \setminus S_0)$, where $S_0 \subset S$ is a maximal subset with the property (+):

$$G_S = \left(\bigstar_{v \in (S \setminus S_0)} G_v \right) \star F \quad \text{with (+) : } \sum_{w \in S_0} \delta_w = \delta$$

In the particular case of Demuškin groups, here are the conditions (see loc. cit. Theorem (i) and Corollary b):

Proposition 3 *Let $v \in S$ such that $\delta_v = 1$. Then $G_S = G_v$ (hence G_S is a Demuškin group of local type) if and only if*

$$S_0 = S \setminus \{v\} \text{ has the property (+), } r_2 = n_v, \text{ and } V_{S_0}^S = 0,$$

where $V_{S_0}^S = \{a \in k^\times \mid a \in k_v^{\times p} \text{ for } v \in S_0; a \in U_v k_v^{\times p} \text{ for } v \notin S\} / k^{\times p}$.

Remark. Except the case $\delta = 1$ studied by Kuz'min, there is no example of field k satisfying Proposition 3.

3 Totally real fields

In order to tackle the totally real case, we will use the following result, see [1, Prop. 1]

Proposition 4 *Let G be a pro- p -group having a closed normal subgroup H such that the quotient $\Gamma = G/H$ is a one-generator free pro- p -group. Then the following conditions are equivalent*

1. G is a two-generator Demuškin group.
2. H is a one-generator free pro- p -group.

We suppose now that the field k is totally real. Then by the equality (3), we have $d(G_S) = 1 + r(G_S)$. Hence, by the previous Proposition, G_S is a Demuškin group if and only if the group $H = \text{Gal}(k_S/k_\infty)$ is isomorphic to $\mathbb{Z}_p$, and from a property of pro- p -groups, this is true if and only if the maximal abelian pro- p -quotient H^{ab} is isomorphic to $\mathbb{Z}_p$. We take the following notations of Iwasawa's theory:

- $H = \text{Gal}(k_S/k_\infty)$, $\mathcal{X} = H^{ab}$ the maximal Λ -module unramified outside S
- k_n the n -th layer of the cyclotomic extension k_∞/k
- $X = \varprojlim \text{Cl}(k_n)$, $X' = \varprojlim \text{Cl}_S(k_n)$, the projective limits being taken via the norm maps. By class field theory, X (resp. X') is isomorphic to the Galois group of the maximal abelian pro- p -extension of k_∞ which is unramified (resp. unramified and where the places of $S(k_\infty)$ split totally).
- $\Delta \xrightarrow{\omega} \mathbb{Z}_p^\times$ the Teichmüller describing the action of Δ on $\zeta \in \mu_p$: $\sigma(\zeta) = \zeta^{\omega(\sigma)}$. For a $\mathbb{Z}_p[\Delta]$ -module M , we denote by M_ω the ω -component of M :

$$M_\omega = \{m \in M \mid \delta.m = \omega(a)m \text{ for every } \delta \in \Delta\}$$

- $\lambda(M)$ and $\mu(M)$ the Iwasawa invariants of a Noetherian Λ -module M .

We use the following standard results of Iwasawa's theory, see for instance [3]

$\mathcal{X}(k)$ and $X(k')_\omega$ are Noetherian torsion Λ -modules having the same λ and μ invariants, and $\mathcal{X}$ has no finite submodule other than 0.

This implies that $\mu(\mathcal{X}) = 0$ if and only if $\mathcal{X}$ is a free $\mathbb{Z}_p$ -module, and in that case the $\mathbb{Z}_p$ -rank of $\mathcal{X}$ is equal to $\lambda(\mathcal{X})$. Hence, we obtain:

Proposition 5 *Let k be a totally real field. Then G_S is a Demuškin group if and only if $\mu(X(k')_\omega) = 0$ and $\lambda(X(k')_\omega) = 1$.*

Example. Let $k = \mathbb{Q}(\mu_p)^+$ be the maximal real subfield of $\mathbb{Q}(\mu_p)$. Then $\mu(X(k'))$ is equal to 0 by the Theorem of Ferrero-Washington, and for $p < 125000$, $\lambda(X(k')_\omega)$ is equal to the irregularity index $i(p)$ of p , see [5, Remark following Corollary 10.3]. So for $p < 125000$, G_S is a Demuškin group if and only if $i(p) = 1$. In this case, the unique place $v \in S$ is totally ramified in k_S , because $\text{Cl}(k) = 0$ (*Vandiver's Conjecture*) is true for $p < 125000$. In particular we have $G_S = G_S^v$, although G_S is a Demuškin group of global type.

We will now consider the case where G_S is a Demuškin group of local type for a totally real field k . The discussion at the end of the introduction shows that we must suppose that there exists a unique $v \in S$ such that $\delta_v = 1$ and v does not split in any S -ramified extension of k . We have the following criterion using only the arithmetic of k , see [1, Th. 1]:

Theorem 1 *Let k be a totally real field, having a unique $v \in S$ such that $\delta_v = 1$, and suppose that v does not split in k_∞/k . Then the following conditions are equivalent:*

1. G_S is a Demuškin group (of local type).
2. Let $w(F)$ be the cardinal of $\mu(F)$ for a field F , and let h , R_p , D be respectively the class number, the p -adic regulator, and the discriminant of k . Then

$$\frac{w(k')hR_p}{w(k_v)\sqrt{D}} \prod_{w \in S} (1 - \text{Norm}_{k_w/\mathbb{Q}_p} w^{-1}) \text{ is a } p\text{-adic unit.}$$

$$3. X'(k')_\omega = 0$$

4. The map $E/E^p \rightarrow \prod_{w \in S} U_w/U_w^p$ is injective, and the p -Hilbert class field of k (the maximal abelian unramified pro- p -extension of k) is contained in k_∞ .

Example. $k = \mathbb{Q}(\sqrt{d})$ is a real quadratic field, and $p = 3$

We suppose that $d > 1$ is square free integer. Let $k^* = \mathbb{Q}(\sqrt{-m})$ the “mirror field”, where $m = d/3$ if 3 divides d , and $m = 3d$ otherwise. In standard way, we have $X(k')_\omega = X(k^*)$, and also similar results for $X'(k^*)$, $\text{Cl}(k^*)$, $\text{Cl}_S(k^*)$...

By the Theorem of Ferrero-Washington, we have $\mu(X(k^*)) = 0$. Hence by Proposition 5, we obtain:

$$G_S(k) \text{ is a Demuškin group} \iff \lambda(X(k^*)) = 1$$

We use the tables of [2] giving the λ -invariant of $X(k^*)$ for $0 < m < 100000$. The case $\lambda = 1$ happens quite often (23489 times), giving us a good provision of Demuškin groups. We give examples in two cases, just to illustrate different situations which were discussed: $d \equiv -3 \pmod{9}$, and $d \equiv 2 \pmod{3}$.

$d \equiv -3 \pmod{9}$ In this case, 3 is ramified in k , and $\mu_3 \subset k_v$, where $\{v\} = S$. Here are the first ten values of $m \equiv -1 \pmod{3}$ for which $\lambda(X(k^*)) = 1$: $m = 2, 5, 11, 17, 23, 26, 29, 38, 53, 59$. Hence

$$G_S(\mathbb{Q}\sqrt{d}) \text{ is a Demuškin group of local type for } d = 6, 15, 33, 51, 69, 78, 87, 114, 159, 177 \dots$$

It is also possible to use the last condition of Theorem 1: Here, k_∞/k is totally ramified at v , hence the 3-Hilbert class field of k is disjoint to k_∞ . So we obtain:

Proposition 6 *Let $k = \mathbb{Q}\sqrt{d}$, where d is a positive square free integer such that $d \equiv -3 \pmod{9}$, and let $p = 3$. Then $G_S(k)$ is a Demuškin group (of local type) if and only if $\text{Cl}(k) = 0$ and the fundamental unit ϵ of k is not a cube in U_v .*

$d \equiv 2 \pmod{3}$ In this case, 3 remains prime in k , and for every integer n , the unique ideal of k_n dividing 3 is principal, hence $\text{Cl}(k_n) = \text{Cl}_S(k_n)$, which implies $X(k) = X'(k)$. Here are the first values of d such that G_S is a Demuškin group:

$$G_S(\mathbb{Q}\sqrt{d}) \text{ is a Demuškin group of global type for } d = 29, 74, 113, 122, 131, 137, 173, 182, 206, 251, 254, 257 \dots$$

4 CM fields

In this section, we generalize a process used by Kuz'min to construct Demuškin groups, see [4, Proposition 5.4], where he considered an extension k/k^+ of CM type, the field k containing μ_p . In order to find new examples, we make the following weaker hypothesis:

k/k^+ is an extension of CM type, with Galois group $J = \text{Gal}(k/k^+)$.

For a $\mathbb{Z}_p[J]$ -module M , we denote by M^+ and M^- respectively the invariant and anti-invariant part of M under the action of J .

Taking the inflation followed by the restriction, for every integer i , we have isomorphisms

$$H^i(G_S(k^+)) \xrightarrow{\text{inf}} H^i(\text{Gal}(k_S/k^+)) \xrightarrow{\text{res}} H^i(G_S(k))^+$$

for the following reasons: the order of J is prime to p , so the restriction maps are isomorphisms. The isomorphisms hold trivially for $i = 0$, and also for $i \geq 3$ because the cohomology groups vanish. The inflation $H^1(G_S(k^+)) \rightarrow H^1(\text{Gal}(k_S/k^+))$ is an isomorphism, because $G_S(k^+)$ is the maximal pro- p -quotient of $\text{Gal}(k_S/k_S^+)$. At last, from the description of $H^2(G_S)$ by the sequence (2), $H^2(G_S(k^+))$ and $H^2(G_S(k))^+$ are isomorphic. In the case of Demuškin groups, we can make a more precise statement, see [1, Prop. 10]:

Proposition 7 *If $G_S(k)$ is a Demuškin group, then $G_S(k^+)$ is also a Demuškin group, and*

$$H^2(G_S(k)) = H^2(G_S(k))^+.$$

If we restrict to Demuškin groups of local type, the previous proposition has the following converse, see [1, Th. 2]:

Theorem 2 *Let k/k^+ be an extension of CM type, such that $G_S(k^+)$ is a Demuškin group, and suppose that there exists $v \in S(k)$ such that $\mu_p \subset k_v$. Let us also denote by v the place of k^+ dividing v . Then $G_S(k)$ is a Demuškin group if and only if the following conditions hold:*

- (i) $\mu_p \subset k_v^+$
- (ii) $S(k^+) = \{v\}$ and $|S(k)| = 1 + \delta$
- (iii) $X'(k)^- = 0$

In this case, $G_S(k)$ and $G_S(k^+)$ are both Demuškin groups of local type.

We now derive some consequences of Theorem 2. We suppose first that $\mu_p \subset k$. Then $X'(k)^- = X'(k)_\omega$. Hence, by using Theorem 1 and Proposition 7, we obtain the following result, see [1, Cor. 1]:

Corollary 1 (compare with [4, Proposition 5.4]) *Let k/k^+ be an extension of CM type, such that k contains μ_p . Then the following conditions are equivalent:*

1. $G_S(k)$ is a Demuškin group.
2. $G_S(k^+)$ is a Demuškin group of local type, and $|S(k^+)| = 1$.
3. $S(k^+)$ has only one element v , $\mu_p \subset k^+$, v does not split in k_∞^+/k^+ and $X'(k)^- = 0$.

In this case we have $G_S = G_S^w = G_w$, where w is one of the two places of k dividing v .

Example. Let $p = 3$, let $k^+ = \mathbb{Q}\sqrt{d}$ be a real quadratic field, and $k = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$. In the previous section, we gave the first values of d such that $G_S(k^+)$ is a Demuškin group of local type. We obtain:

$G_{S_3}(\mathbb{Q}(\sqrt{-3}, \sqrt{d}))$ is a Demuškin group for $d = 6, 15, 33, 51, 69, 78, 87, 114, 159, 177 \dots$

We suppose now that $\mu_p \not\subset k$. If $G_S(k^+)$ is a Demuškin group, and if the condition (ii) of the Theorem holds, then we have $|S(k_\infty)| = 1$. And under this hypothesis, the conditions $X'(k_\infty)^- = 0$ and $\text{Cl}_S(k)^- = 0$ are equivalent, because $\text{Cl}_S(k)^- = X'(k_\infty)^-_{\Gamma}$, see [5, Lemma 13.15] for an analogous result. Hence we obtain the following

Corollary 2 *Let k/k^+ be an extension of CM type such that k does not contain μ_p , and suppose that there exists $v \in S(k)$ such that k_v contains μ_p . Then the following conditions are equivalent:*

1. $G_S(k)$ is a Demuškin group.
2. $S(k) = \{v\}$, $\mu_p \subset k_v^+$, $G_S(k^+)$ is a Demuškin group and $\text{Cl}_S(k)^- = 0$.

In this case $G_S(k) = G_S^v(k)$, but $G_S(k) \neq G_v(k)$.

The last assertion of the Corollary comes from the fact that the Euler-Poincaré characteristic $-r_2(k)$ of $G_S(k)$ and the Euler-Poincaré characteristic $-n_v(k)$ of $G_v(k)$ are different.

Example. Let $p = 3$, and let us find the bi-quadratic fields k satisfying Corollary 2. The real maximal subfield $k^+ = \mathbb{Q}\sqrt{d}$, where d is a square free integer, must be of type given by Proposition 6. k must have a unique place v dividing 3, and the inertia group of v in $k/\mathbb{Q}$ is non trivial (because $k^+/\mathbb{Q}$ is ramified) and cyclic, by class field theory. Hence the inertia field k^v is an imaginary (otherwise k would be real) quadratic field: $k^v = \mathbb{Q}\sqrt{-d'}$, where d is a square free integer congruent to 1 (mod 3), because 3 remains prime in k^v . Let $\tilde{k} = \mathbb{Q}\sqrt{-dd'}$ be the other quadratic subfield of k . We have canonically

$$\text{Cl}_S(k)^- \simeq \text{Cl}_S(k^v) \oplus \text{Cl}_S(\tilde{k})$$

Furthermore $\text{Cl}_S(k^v) = \text{Cl}(k^v)$ because the prime 3 is principal in k^v , and $\text{Cl}_S(\tilde{k}) = \text{Cl}(\tilde{k})$ because the prime ideal dividing 3 in $\tilde{k}$ has order 1 or 2 in $\text{Cl}(\tilde{k})$, $\tilde{k}/\mathbb{Q}$ being totally ramified at 3. Hence we obtain the following

Proposition 8 Let $p = 3$ and $k = \mathbb{Q}(\sqrt{d}, \sqrt{-d'})$, where d and d' are square free integers such that $d \equiv -3 \pmod{9}$ and $d' \equiv 1 \pmod{3}$. Then G_S is a Demuškin group if and only if the following conditions hold:

1. $G_S(\mathbb{Q}\sqrt{d})$ is a Demuškin group.
2. $Cl(\mathbb{Q}\sqrt{-d'}) = Cl(\mathbb{Q}\sqrt{-dd'}) = 0$.

Let m be the square free integer such that $\tilde{k} = \mathbb{Q}\sqrt{-m}$. The following table gives the value of m when $m < 500$. The first line and the first column give respectively the first values d' and d satisfying the congruences of Proposition 8, such that 3 does not divide the class number of $\mathbb{Q}\sqrt{-d'}$ and $G_S(\mathbb{Q}\sqrt{d})$ is a Demuškin group (See the list given in the previous section). $\nexists$ means that 3 divides the class number of $\tilde{k}$. The table shows that $G_S(k)$ is a Demuškin group, with $k = \mathbb{Q}(\sqrt{d}, \sqrt{-d'})$, except the cases $(d', d) = (37, 6)$ or $(7, 33)$.

$d \downarrow \xrightarrow{d'}$	1	7	10	13	19	22	34	37	43	46
6	6	42	15	78	114	33	51	222	258	69
15	15	105	6	195	285	330				
33	33	231	330	429		6				
51	51	357					6			
69	69	483								6
78	78		195	6		429				

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