

Modification of Rikitake's model for the geomagnetic field

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1 Introduction

In 1958, Tsuneji Rikitake [1] studied the model of two disk dynamos for the occurrence and the reversal of the geomagnetic field, which was described by the 4-dimensional ordinary differential equations :

$$\begin{aligned}\frac{dx}{dt} &= -ax + yz_1 \\ \frac{dy}{dt} &= -ay + xz_2 \\ \frac{dz_1}{dt} &= 1 - xy \\ \frac{dz_2}{dt} &= 1 - xy\end{aligned}\tag{1}$$

where a is a positive constant, the functions $x(t)$ and $y(t)$ of time t are the electric currents of the two disk dynamos, respectively, and the functions $z_1(t)$ and $z_2(t)$ are their angle velocities, respectively. In 1963, John H. Mathews and W.K. Gardner [2] modified this model to an axially stacked type of two disk dynamos, claimed that the reversal of the geomagnetic field may be represented by the change of sign of $x(t) + y(t)$ for t , and found a various results of computer simulations suggesting this change. However no mathematical proof of this change has been obtained. Moreover the author thinks that (1) is not reasonable from the view point of physics, because $z_1(t)$ and $z_2(t)$ of (1) increases without any limit in the case where $x(t) \equiv y(t) \equiv 0$. Therefore we cannot help but build another type of model such that

$$\begin{aligned}\frac{dx}{dt} &= -ax + yz_1 \\ \frac{dy}{dt} &= -ay + xz_2 \\ \frac{dz_1}{dt} &= c(z_1 - z_1^3) - xy \\ \frac{dz_2}{dt} &= c(z_2 - z_2^3) - xy\end{aligned}\tag{2}$$

where c is a positive constant. It is our idea that the constant 1 of the third and fourth equations of (1), which is taken to be a driving force, is replaced by the negative damping term above. In this system (2), we shall show that solutions are bounded in the future, and moreover that the equilibrium point P , where $x = y = 0$, $z_1 = 1$, $z_2 = -1$, is stable and solutions in a neighbourhood of P change the sign of $x(t) + y(t)$. Preliminarily we shall state the variational system of (2) :

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -a & z_1 & y & 0 \\ z_2 & -a & 0 & x \\ -y & -x & c(1 - 3z_1^2) & 0 \\ -y & -x & 0 & c(1 - 3z_2^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad (3)$$

2 Solution behavior of (2)

We shall consider the solution behavior of (2).

Theorem 1

Solutions of (2) are bounded in the future, and in fact for each solution $(x(t), y(t), z_1(t), z_2(t))$ there exists a number T such that

$$x^2(t) + y^2(t) + z_1^2(t) + z_2^2(t) < \frac{(a+c)^2}{2ac} \quad \text{for } t > T$$

Moreover there exists a compact, connected and invariant set K of R^4 such that every solution approaches K as t increases, and especially, in the case where $a > c$, the volume of K is zero in the sense of 4-dimensional Lebesgue measure.

Proof Setting $V(t) = x^2(t) + y^2(t) + z_1^2(t) + z_2^2(t)$, we may see that

$$\frac{d}{dt} V(t) \leq -2aV(t) + \frac{(a+c)^2}{c}$$

which implies our first assertion. Moreover, by the general argument of dynamical system the existence of K above may be claimed. Especially, in the case where $a > c$, the trace of the coefficient matrix of (3) is negative, in fact, which is $-2a + 2c - 2c(z_1^2 + z_2^2)$, and hence by the usual argument of Liouville's theorem the volume of K is equal to zero.

Next we shall consider the equilibrium points of (2), say $P(x, y, z_1, z_2)$, and its stability in terms of the eigenvalues of the coefficient matrix of (3) substituted with P , say $\lambda_k (1 \leq k \leq 4)$. The equation of P is the following

$$-ax + yz_1 = -ay + xz_2 = 0 \quad (4)$$

$$c(z_1 - z_1^3) = c(z_2 - z_2^3) = xy \quad (5)$$

Since the set $x = y = 0$ is invariant, we shall treat first of all this case, where (5) is reduced to $z_1 - z_1^3 = z_2 - z_2^3 = 0$, which implies that z_1 and z_2 are any one of 0, +1 and -1, respectively. Therefore we may have the nine equilibrium points as the table below, where $i = \sqrt{-1}$.

z_1	z_2	λ_1	λ_2	λ_3	λ_4
0	0	$-a$	$-a$	$2c$	$2c$
0	1	$-a$	$-a$	$2c$	$-2c$
0	-1	$-a$	$-a$	$2c$	$-2c$
1	0	$-a$	$-a$	$-2c$	$2c$
-1	0	$-a$	$-a$	$-2c$	$2c$
1	1	$1 - a$	$-1 - a$	$-2c$	$-2c$
-1	-1	$1 - a$	$-1 - a$	$-2c$	$-2c$
1	-1	$-a + i$	$-a - i$	$-2c$	$-2c$
-1	1	$-a + i$	$-a - i$	$-2c$	$-2c$

The equilibrium point $P(0, 0, z_1, z_2)$, where either $z_1 = 1, z_2 = -1$ or $z_1 = -1, z_2 = 1$, is asymptotically stable, and $x(t) + y(t)$ oscillates around zero, namely, the sign of $x(t) + y(t)$ changes to t . This result may be illustrated by computer simulations in the later part.

Next we shall treat the case where $xy \neq 0$, which implies that $z_1 z_2 = a^2$. Since (5) implies that $z_1 - z_1^3 = z_2 - z_2^3$, it follows that either $z_1 = z_2$ or $z_1^2 + z_1 z_2 + z_2^2 = 1$. We may see that the former case yields more four equilibrium points $P(x, y, z_1, z_2)$ as in the following table, where $A = \sqrt{ac(1 - a^2)}$, by assuming that $0 < a < 1$.

x	y	z_1	z_2
A	A	a	a
$-A$	$-A$	a	a
A	$-A$	$-a$	$-a$
$-A$	A	$-a$	$-a$

The characteristic equation of λ is common to these four equilibriums, which is reduced to $\lambda = c(1 - 2a^2)$ and

$$\lambda^3 + (2a - c + 2a^2c)\lambda^2 + 2a^3c\lambda + 4a^2c(1 - a^2) = 0. \quad (6)$$

Since $1 > a^2$, (6) has one negative root, and it is seen from Routh-Hurwitz type of criterion that the real parts of the remaining two roots are all negative and all positive if $a^2 > \frac{1}{2}$ and if $a^2 < \frac{1}{2}$, respectively.

Remark 1 [1] has a wrong point to the part corresponding the above. In fact, the entries of the determinant between (25) and (26) in [1, p.96] is not correct, and hence his assertion such that $\mu > 0$ cannot be derived.

In the later part we shall show the results of computer simulations to (2).

Numerical results of (2) by Mathematica ($z_1(0) \neq z_2(0)$)

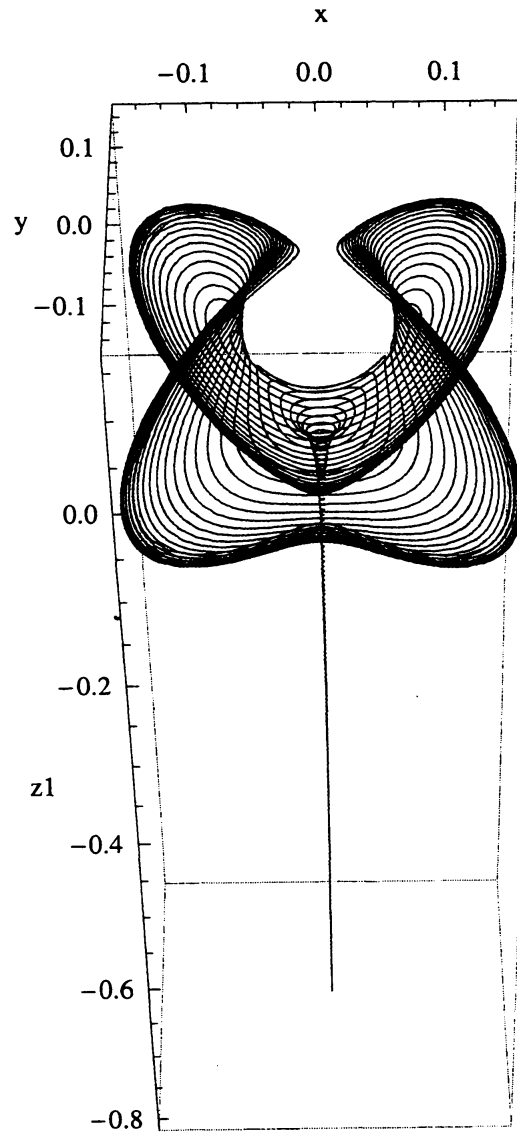
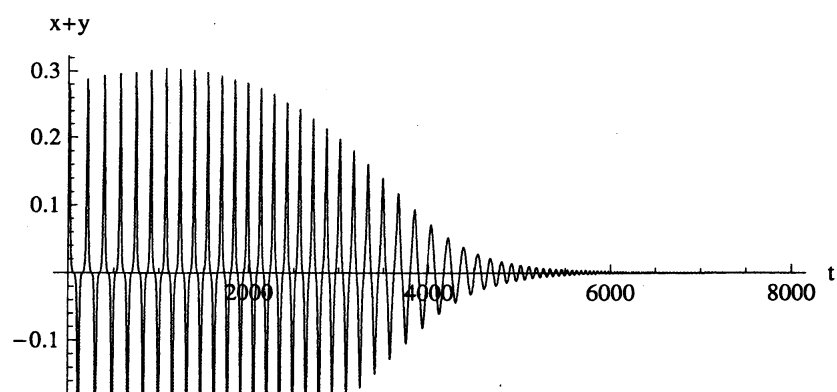
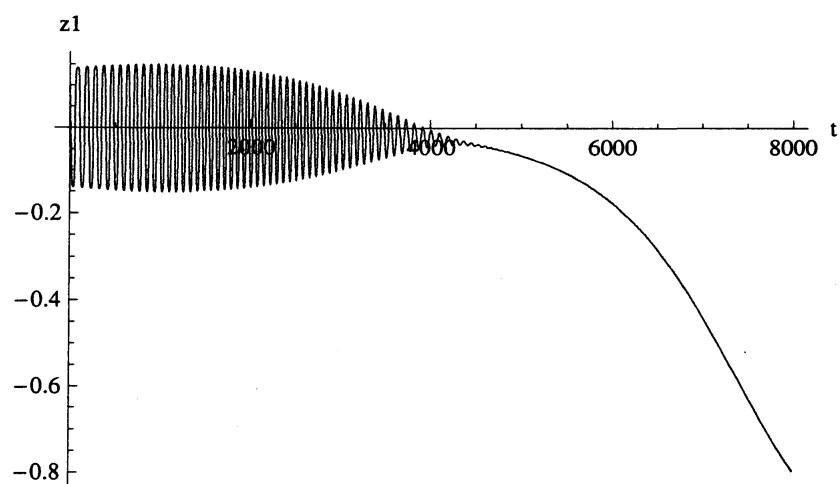
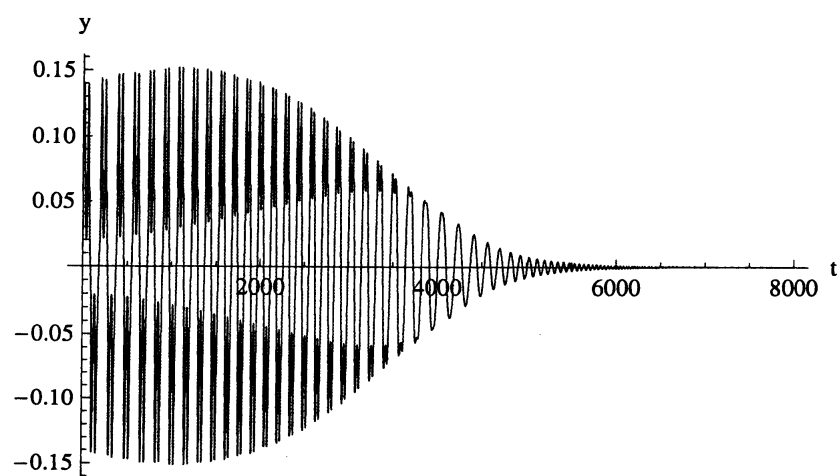
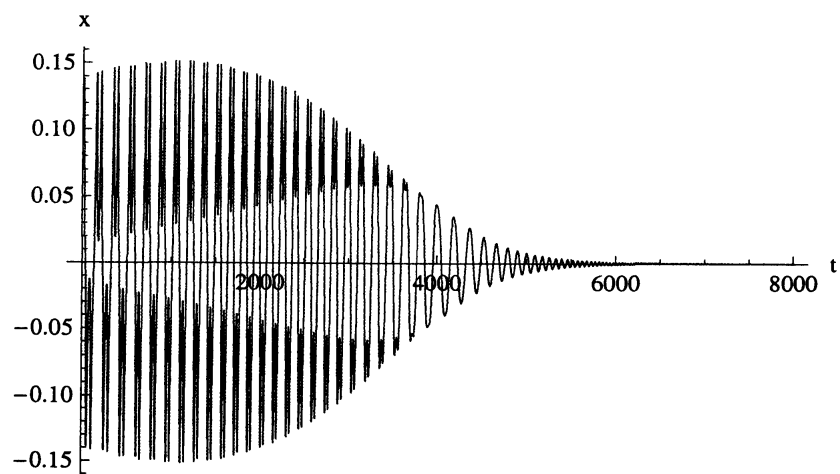


Figure 1: Case 1 $a = b$
 $(a, b, c) = (0.002, 0.002, 0.001)$,
 $(x(0), y(0), z_1(0), z_2(0)) = (0.1, 0.1, 0.1, 0.101)$



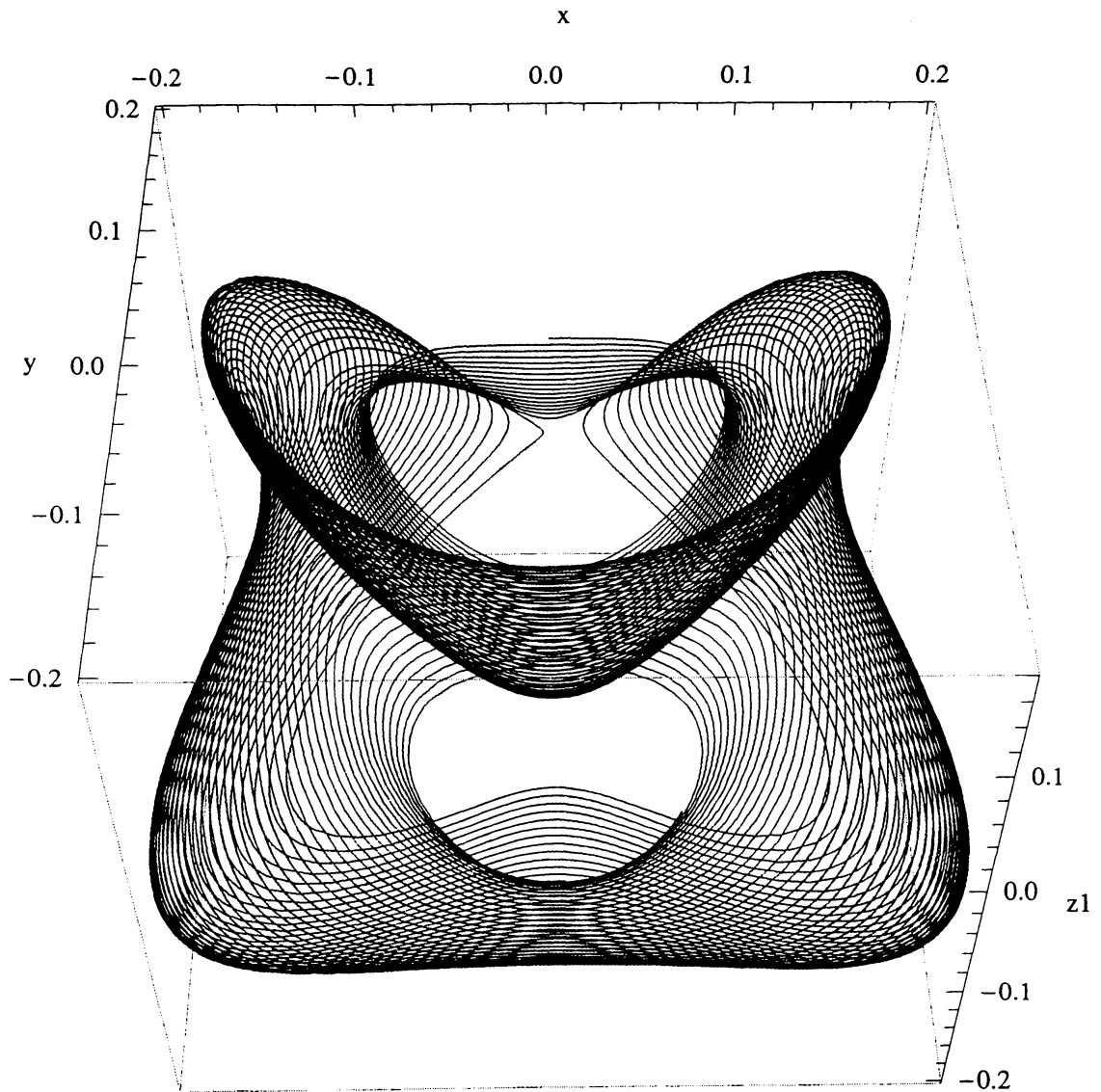
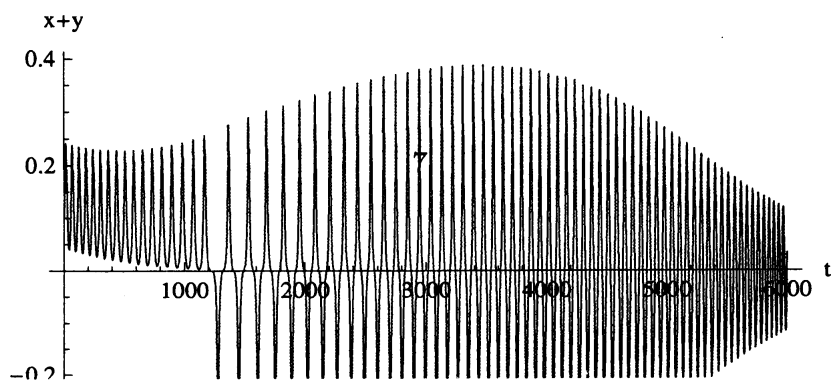
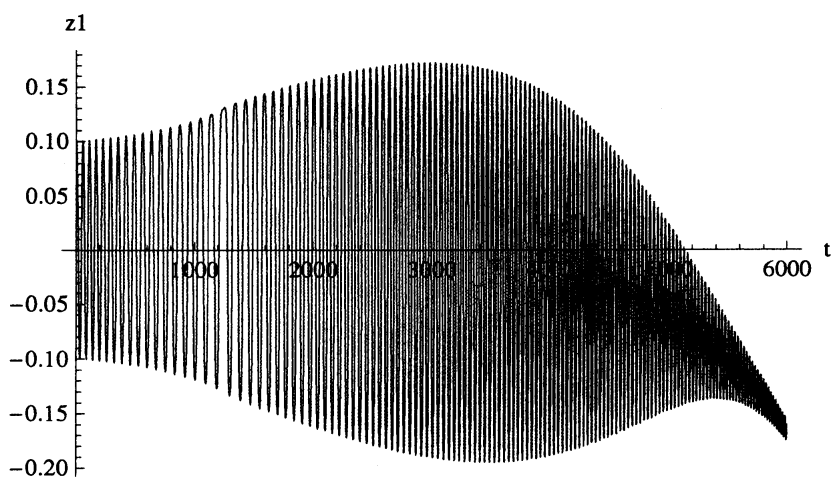
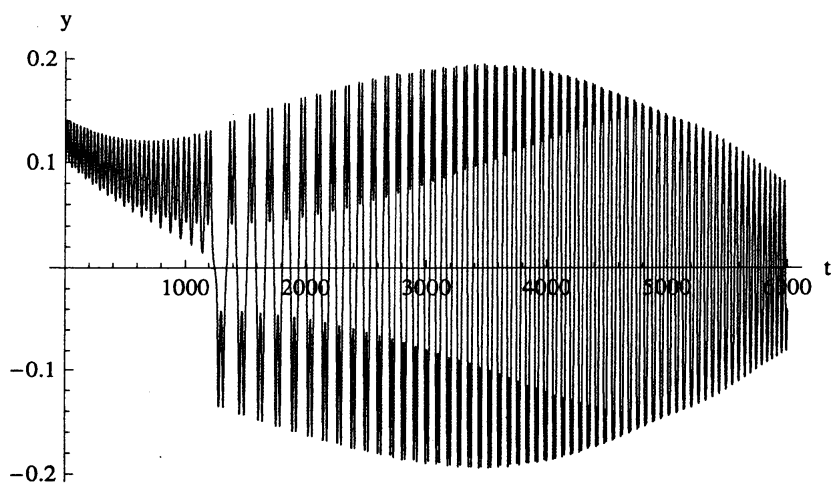
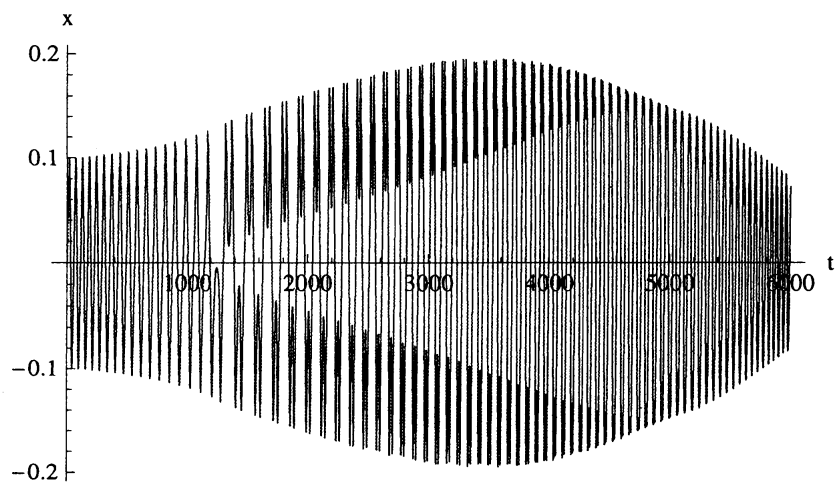


Figure 2: Case 2 $a \neq b$
 $(a, b, c) = (0.001, 0.00102, 0.001)$,
 $(x(0), y(0), z_1(0), z_2(0)) = (0.0, 0.1, 0.1, 0.101)$



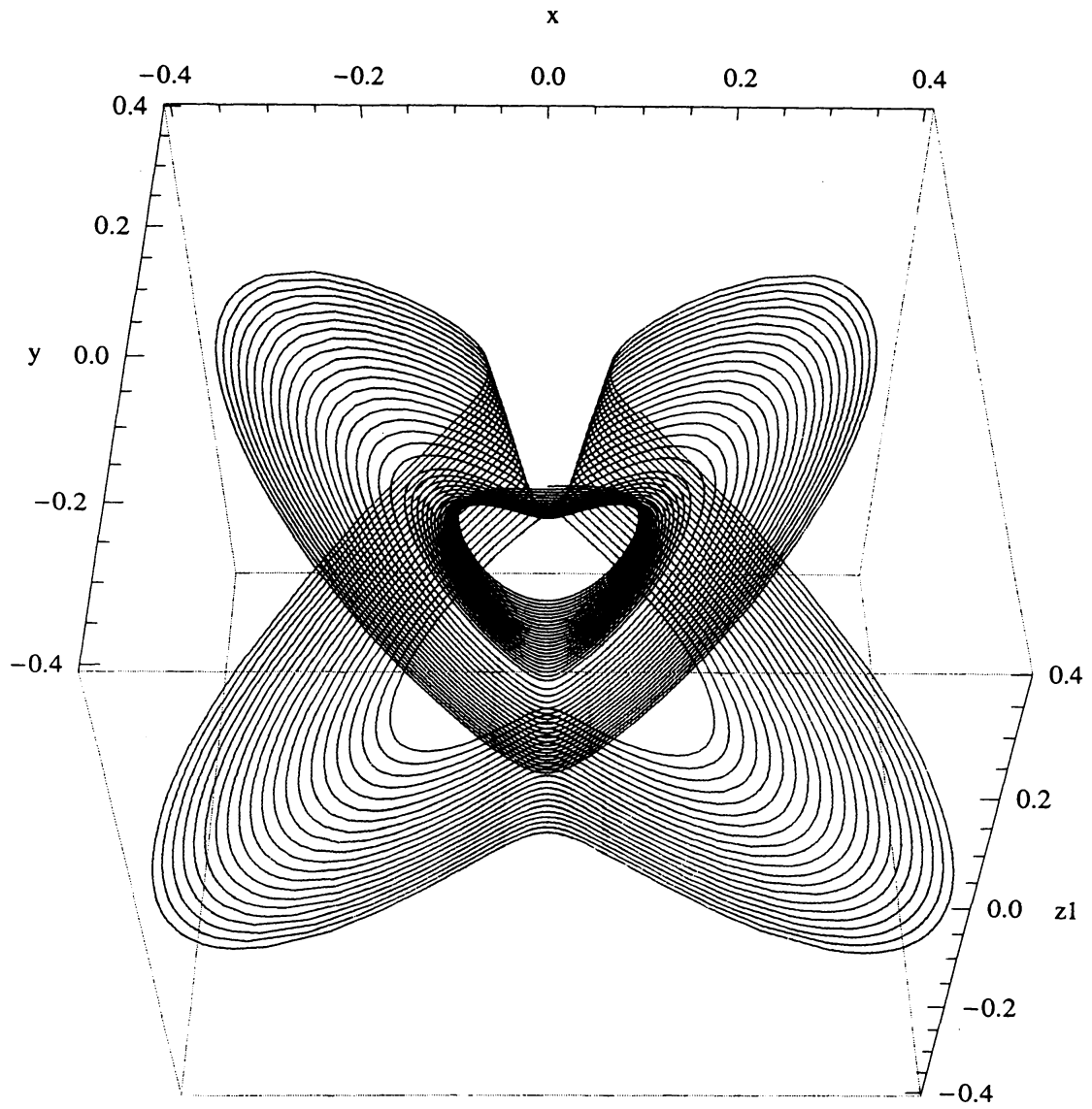
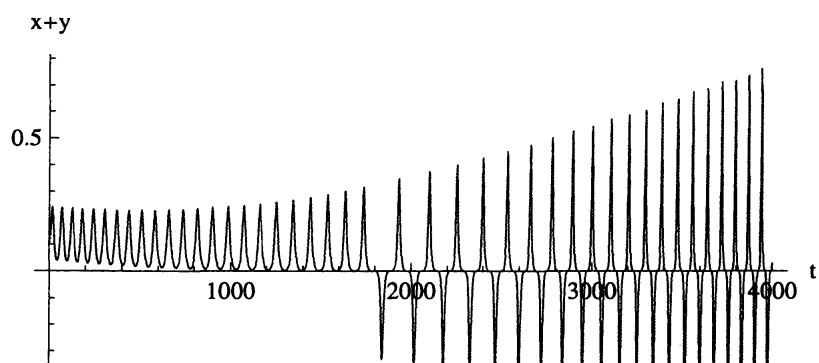
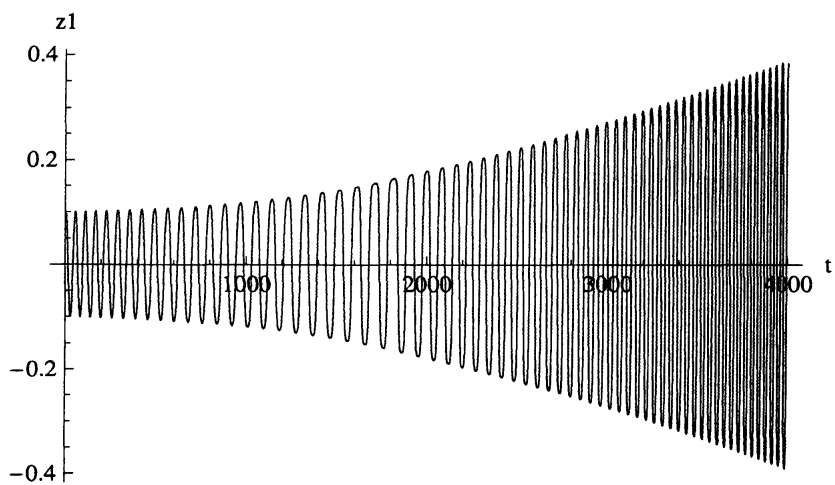
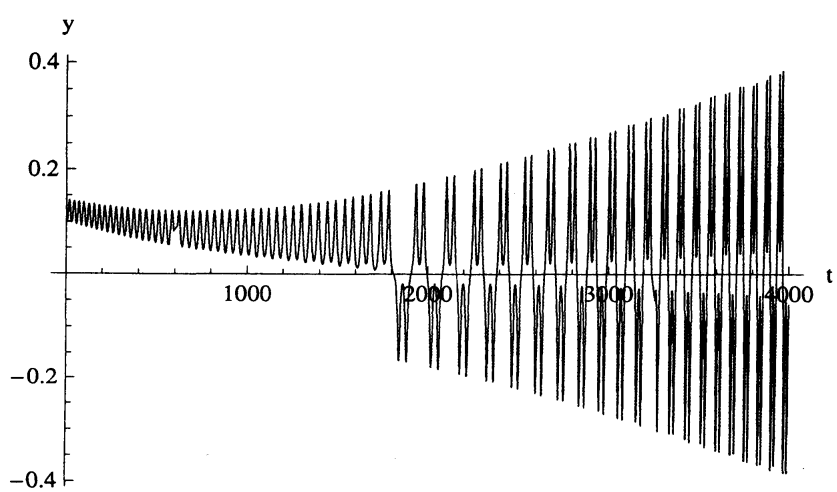
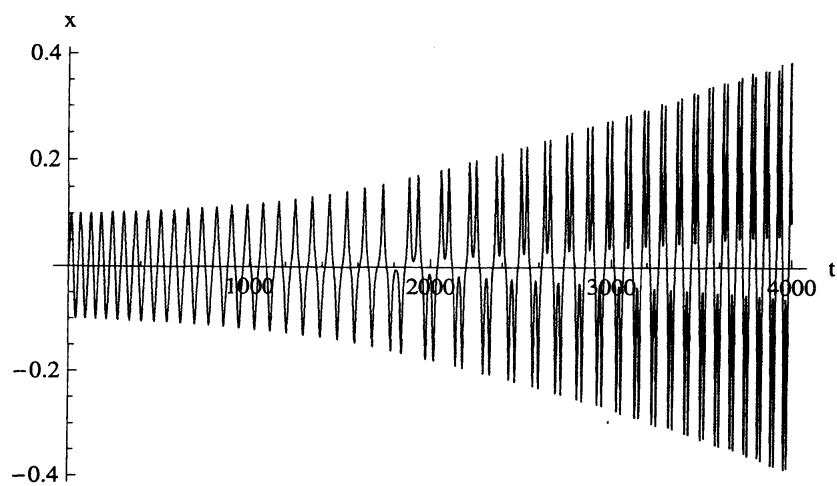


Figure 3: Case 3 $a \neq b$ and $z_1(0) \sim z_2(0)$
 $(a, b, c) = (0.001, 0.00102, 0.001)$,
 $(x(0), y(0), z_1(0), z_2(0)) = (0.0, 0.1, 0.1, 0.1001)$



References

- [1] Rikitake, T., Oscillations of a system of disk dynamos, Proc. Cambr. Phil. Soc., 54(1958), 89-105
- [2] Mathews, T., and Gardner, W., Field reversals of paleo magnetic type in coupled disk dynamos, U.S. Naval Res. Lab. Rep. 5886(1963).