

p -adic Siegel Eisenstein series of degree n

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1 Introduction

In this paper, we define an Siegel Eisenstein series $G_{k,\chi}^{(n)}$ of degree n and introduce a formula for its Fourier expansion. The definition of $G_{k,\chi}^{(n)}$ is different from the ordinary Siegel Eisenstein series $E_{k,\chi}^{(n)}$. But if χ satisfies a certain condition, $G_{k,\chi}^{(n)}$ coincides with $E_{k,\chi}^{(n)}$. We also introduce the theorem that states the existence of p -adic family of Siegel modular forms that interpolates $G_{k,\chi}^{(n)}$.

2 Statement of the main results

Let F be a totally real field with $[F : \mathbf{Q}] = m$. If K is a number field and v is a finite place of K , then we denote by $\mathcal{O}_K$ and by $\mathcal{O}_v$ the integer ring of K and that of K_v respectively. For an ideal $\mathfrak{n}$ of F , we denote the group of fractional ideals of F relatively prime to $\mathfrak{n}$ by $I_{\mathfrak{n}}$. Let χ be a narrow class character modulo $\mathfrak{n}$, that is, a character $\chi : I_{\mathfrak{n}} \rightarrow \mathbf{C}^\times$ trivial on any principal ideal (a) generated by a totally positive element a such that $a \equiv 1 \pmod{\mathfrak{n}}$. Let $\mathbb{A}_F$ be the adèle ring of F and $\mathbb{A}_F^\times$ the idele group of F . Denote the character of finite order of $\mathbb{A}_F^\times/F^\times$ corresponding to χ by $\tilde{\chi}$.

For an infinite place v of F , let r_v be an element of $\mathbf{Z}/2\mathbf{Z}$ satisfying the following condition.

$$\chi((a)) = \prod_{v|\infty} \operatorname{sgn}(\iota_v(a))^{r_v} \text{ for } a \equiv 1 \pmod{\mathfrak{n}}.$$

Here v runs over the set of m real places of F and ι_v is the real embedding corresponding to v . We define a character sgn_χ of $F^\times$ by

$$\operatorname{sgn}_\chi(a) = \prod_{v|\infty} \operatorname{sgn}^{r_v}(\iota_v(a)).$$

We define a character $\chi_f : (\mathcal{O}_F/\mathfrak{n})^\times \rightarrow \mathbf{C}^\times$ by

$$\chi_f(a) = \operatorname{sgn}_\chi(a)\chi((a)).$$

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Let n be a positive integer. For $0 \leq i \leq n$, we denote by $w_{n,i}$ the matrix given as follows.

$$w_{n,i} = \left(\begin{array}{cc|cc} 0_i & 0 & -1_i & 0 \\ 0 & 1_{n-i} & 0 & 0_{n-i} \\ \hline 1_i & 0 & 0_i & 0 \\ 0 & 0_{n-i} & 0 & 1_{n-i} \end{array} \right).$$

We put $w_n = w_{n,n}$. We define the symplectic group of degree n by

$$\mathrm{Sp}_n(R) = \{g \in \mathrm{GL}_{2g}(R) \mid {}^t g w_n g = w_n\},$$

where R is a commutative ring. For $g \in \mathrm{Sp}_n$, we denote $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ with $a_g, b_g, c_g, d_g \in M_n$. Define the Siegel parabolic subgroup P_n by

$$P_n(R) = \{g \in \mathrm{Sp}_n(R) \mid c_g = 0\}.$$

We define a congruence subgroup $\Gamma_0^{(n)}(\mathfrak{n})$ by

$$\Gamma_0^{(n)}(\mathfrak{n}) = \{g \in \mathrm{Sp}_n(\mathcal{O}_F) \mid c_g \equiv 0 \pmod{\mathfrak{n}}\}.$$

We define the Siegel upper half space of degree n by

$$\mathfrak{H}_n = \{z \in \mathrm{Sym}_n(\mathbf{C}) \mid z = x + iy, \ x, y \in \mathrm{Sym}_n(\mathbf{R}), \ y > 0\}.$$

Let k be a positive integer and assume $\chi_f(-1) = (-1)^k$. Define a Siegel Eisenstein series of degree n , character χ , weight k by

$$E_{k,\chi}^{(n)}(z) = \sum_{g \in P_n(\mathcal{O}_F) \cap \Gamma_0^{(n)}(\mathfrak{n}) \backslash \Gamma_0^{(n)}(\mathfrak{n})} \chi_f^{-1}(\det d_g) \det(c_g z + d_g)^{-k}.$$

Here $z = (z_v)_{v|\infty} \in \prod_{v|\infty} \mathfrak{H}_n$ and $\det(c_g z + d_g)^{-k}$ is defined by

$$\det(c_g z + d_g)^{-k} = \prod_{v|\infty} \det(\iota_v(c_g) z_v + \iota_v(d_g))^{-k}.$$

In the rest of this paper, we assume that $\mathfrak{n}$ is relatively prime to 2 for simplicity.

Put

$$\mathcal{P} = \{\mathfrak{p} : \text{a prime of } F \mid \mathfrak{p} \mid \mathfrak{n} \text{ and } \tilde{\chi}_{\mathfrak{p}}^2 \text{ is unramified}\}.$$

Let $g \in \mathrm{Sp}_n(\mathcal{O}_F)$ and assume $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(n)}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ with $0 \leq i_{\mathfrak{p}} \leq n$ if $\mathfrak{p} \in \mathcal{P}$. For $\mathfrak{p} \in \mathcal{P}$, the assumption for g implies $g \bmod \mathfrak{p} \in P_n(\mathcal{O}_F/\mathfrak{p}) w_{i_{\mathfrak{p}}} P_n(\mathcal{O}_F/\mathfrak{p})$. Therefore if $\mathfrak{p} \in \mathcal{P}$, there exist elements $x_{\mathfrak{p}}, y_{\mathfrak{p}} \in \mathrm{GL}(\mathcal{O}_F/\mathfrak{p})$ satisfying

$$g \bmod \mathfrak{p} = \begin{pmatrix} x_{\mathfrak{p}} & * \\ 0 & {}^t x_{\mathfrak{p}}^{-1} \end{pmatrix} w_{i_{\mathfrak{p}}} \begin{pmatrix} y_{\mathfrak{p}} & * \\ 0 & {}^t y_{\mathfrak{p}}^{-1} \end{pmatrix}.$$

We put

$$\chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; g) = \prod_{\substack{\mathfrak{p} | n \\ \mathfrak{p} \notin \mathcal{P}}} (\chi_f)_{\mathfrak{p}} (\det d_g) \prod_{\mathfrak{p} \in \mathcal{P}} (\chi_f)_{\mathfrak{p}} (\det x_{\mathfrak{p}} \det y_{\mathfrak{p}}).$$

Here $(\chi_f)_{\mathfrak{p}}$ is the $\mathfrak{p}$ -component of χ_f .

We define an auxiliary Siegel Eisenstein series $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; z)$ as follows.

$$E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; z) = \sum_g \chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; g)^{-1} \det(c_g z + d_g),$$

where g runs over the set $P_n(F) \cap \mathrm{Sp}_n(\mathcal{O}_F) \backslash \mathrm{Sp}_n(\mathcal{O}_F)$ satisfying the property $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(n)}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ if $\mathfrak{p} \in \mathcal{P}$. By the definition, we have $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; z) = E_{k,\chi}^{(n)}$ if $i_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$.

Let $\mathfrak{p}$ be a prime of F and assume $\mathfrak{p} \in \mathcal{P}$ and $(\mathfrak{p}, 2) = 1$. For $0 \leq i \leq n$ and $s \in \mathbf{C}$, we put $M_{in}(s, \tilde{\chi}_{\mathfrak{p}}) = 0$ if i is odd and put

$$M_{in}(s, \tilde{\chi}_{\mathfrak{p}}) = \tilde{\chi}_{\mathfrak{p}}(-1) N\mathfrak{p}^{-i/2} \prod_{a=0}^{i/2} (1 - N\mathfrak{p}^{-1-2a}) (1 - \tilde{\chi}_{\mathfrak{p}}^2(\mathfrak{p}) N\mathfrak{p}^{-2s+2a+n-i-2}),$$

if i is even. We set

$$m_i(k, \chi) = M_{in}(-k + \frac{n+1}{2}, \tilde{\chi}_{\mathfrak{p}}).$$

By definition, the right hand side does not depend on n .

We define an Eisenstein $G_{k,\chi}^{(n)}$ as a linear combination of $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; z)$.

Definition 2.1. If $\mathcal{P} \neq \emptyset$, we define

$$G_{k,\chi}^{(n)}(z) = \sum_{\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}} \left(\prod_{\mathfrak{p} \in \mathcal{P}} m_{i_{\mathfrak{p}}}(k, \chi) \right) E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}; z).$$

Here $\{i_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}}$ runs over all the non-empty subsets of $\prod_{\mathfrak{p} \in \mathcal{P}} \{0, \dots, n\}$. If $\mathcal{P} = \emptyset$, we define

$$G_{k,\chi}^{(n)} = E_{k,\chi}^{(n)}.$$

Remark 2.1. We can define $G_{k,\chi}^{(n)}$ more naturally by using the intertwining operator. But to shorten the statement, we define $G_{k,\chi}^{(n)}$ in this way. (See subsection 3.1)

The first main theorem of this paper is the result for Fourier coefficients for $G_{k,\chi}^{(n)}$. We prepare some notation.

Let $B \in \mathrm{Sym}_n^*(\mathcal{O}_F)$ be a half integral matrix of size n . Put $r = \mathrm{rank} B$. There exists a matrix $A \in \mathrm{GL}_n(F)$ such that

$${}^tABA = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix},$$

with $B' \in \text{Sym}_r(F)$. Then $\det B' \in F^\times / F^{\times 2}$ does not depend on the choice of A . If r is even we denote by χ_B the narrow class character of F associated with the extension $F(\sqrt{(-1)^{r/2} \det B'})/F$ by the global class field theory.

For a prime $\mathfrak{p}$ of F such that $\mathfrak{p} \nmid \mathfrak{n}$, there exists a matrix $U \in \text{GL}_n(\mathcal{O}_{\mathfrak{p}})$ that satisfies

$${}^t U B U = \begin{pmatrix} B'_p & 0 \\ 0 & 0 \end{pmatrix},$$

with $B'_p \in \text{Sym}_r^*(\mathcal{O}_{\mathfrak{p}})$. The matrix B'_p is unique up to unimodular equivalence. Therefore $\Phi_p^{(r)}(B'_p; T)$ does not depend on the choice of U , where $\Phi_p^{(r)}(B'_p; T)$ is the polynomial obtained by the Siegel series. (In the notation of [4] 13.6. Theorem, we have $\Phi_p^{(r)}(B'_p; T) = f_{B'_p}(T)$.) Thus we put $\Phi_p^{(r)}(B, T) = \Phi_p^{(r)}(B'_p, T)$.

Theorem 2.1. *Let $0 \leq B \in \text{Sym}_n^*(\mathcal{O}_F)$ be a half integral positive semi-definite matrix of size n and $k > n + 1$ an integer. Let χ be a primitive narrow class character of F of conductor $\mathfrak{n}$. Put $r = \text{rank } B$. Then the following assertions hold.*

If r is even, then $a(B, G_{k, \chi}^{(n)})$ is given by

$$2^{rm/2} \prod_{\mathfrak{p} \nmid \mathfrak{n}} \Phi_p^{(r)}(B; \chi(\mathfrak{p}) \mathfrak{N} \mathfrak{p}^{k-r-1}) \\ \times L(1-k, \chi)^{-1} L^{(n)}(1+r/2-k, \chi_h \chi) \prod_{i=1}^{r/2} L^{(n)}(1+2i-2k, \chi^2)^{-1}.$$

If r is odd, then $a(B, G_{k, \chi}^{(n)})$ is given by

$$2^{(r+1)m/2} \prod_{\mathfrak{p} \nmid \mathfrak{n}} \Phi_p^{(r)}(B; \chi(\mathfrak{p}) \mathfrak{N} \mathfrak{p}^{k-r-1}) \\ \times L(1-k, \chi)^{-1} \prod_{i=1}^{(r-1)/2} L^{(n)}(1+2i-k, \chi^2)^{-1}.$$

For a Hecke L -function $L(s, \chi)$ and an ideal $\mathfrak{n}$, we denote $L^{(n)}(s, \chi) = \prod_{\mathfrak{p} \nmid \mathfrak{n}} (1 - \chi(\mathfrak{p}) \mathfrak{N} \mathfrak{p}^{-s})^{-1}$, where the index $\mathfrak{p}$ runs over the set of primes of F relatively prime to the conductor of χ and the ideal $\mathfrak{n}$.

From this theorem and the definition of $G_{k, \chi}^{(n)}$, we have a formula for the Fourier coefficients of $E_{k, \chi}^{(n)}$ if $\mathcal{P} = \emptyset$.

Remark 2.2. When $F = \mathbf{Q}$, Katsurada [1] proved the explicit formula for $\Phi_p^{(r)}(B, T)$, thus in this case we have the explicit formula for Fourier coefficients of $G_{k, \chi}^{(n)}$.

For a prime $\mathfrak{p}$ of F , we put

$$P_+(\mathfrak{p}^\alpha) = \{a \mathcal{O}_F \mid a: \text{positive definite and } a \equiv \text{mod } \mathfrak{p}^\alpha\}.$$

We define the narrow ray class group of F of conductor $\mathfrak{p}^\alpha$ by $\text{Cl}_F(\mathfrak{p}^\alpha) = I_{\mathfrak{p}}/P_+(\mathfrak{p}^\alpha)$. We consider the projective limit

$$\text{Cl}_F(\mathfrak{p}^\infty) = \varprojlim \text{Cl}_F(\mathfrak{p}^\alpha).$$

We put $G = \text{Cl}_F(\mathfrak{p}^\infty)$.

Let Ω be the completion of $\overline{F}_{\mathfrak{p}}$ and A the integer ring of Ω . We fix the embedding of $\overline{\mathbf{Q}}$ to $\mathbf{C}$ and Ω . We denote $\text{Meas}(G, A)$ by the bounded $\mathfrak{p}$ -adic measure on G with values in A . Let p be the residual characteristic of $F_{\mathfrak{p}}$. Since $I_{\mathfrak{p}}$ can be considered as a dense subgroup of G and the norm map $N : I_{\mathfrak{p}} \rightarrow \mathbf{Z}_p^\times$ is continuous, we can extend N to G . We denote the extended character by the same letter. Let ω be the Teichmüller character of $\mathbf{Z}_p^\times$ and put $\omega_F = \omega \circ N$.

Theorem 2.2. *Let $\mathfrak{p}$ be a prime of F such that $(\mathfrak{p}, 2) = 1$, p a residual characteristic of $F_{\mathfrak{p}}$ and χ a narrow ray class character of conductor $\mathfrak{p}^\nu$. Denote $\mathcal{O}_F[\chi]$ by the ring generated by $\text{Im}(\chi)$ over $\mathcal{O}_F$. Then there exists a formal Fourier expansion $\mathbf{G}^{(n)}(\chi; T)$*

$$\mathbf{G}^{(n)}(\chi; T) = \sum_{0 \leq B \in \text{Sym}_n^{(*)}(\mathcal{O}_F)} \mathbf{a}(B; T) \mathbf{e}(Bz),$$

where $\mathbf{a}(B; T)$ is an element of the quotient ring of the formal power series ring $\text{Frac } \mathcal{O}_F[\chi][T]$, and satisfies the following condition. If $k > n + 1$ and $\chi \cdot \omega_F^{-k}$ is not the trivial character modulo $\mathfrak{p}$

$$\mathbf{G}^{(n)}(\chi; u^k - 1) = G_{k, \chi \cdot \omega_F^{-k}},$$

where u is a fixed generator of $1 + \mathbf{Z}_p$. Moreover, there exists a nonzero formal power series $\mathbf{b}(T)$ and a $\mathfrak{p}$ -adic measure $\mu_B \in \text{Meas}(G, A)$ for each B that satisfy

$$\mathbf{b}(u^s - 1) \mathbf{a}(B; u^s - 1) = \int_G \chi(x) N(x)^{-1} \langle N(x) \rangle^s d\mu_B, \quad \text{for } s \in \mathbf{Z}_p.$$

Here for $a \in \mathbf{Z}_p^\times$, we put $\langle a \rangle = a\omega^{-1}(a)$.

Remark 2.3. In the interpolation property, we assume the character is not trivial character mod $\mathfrak{p}$. H. Kawamura [2] proved the existence p -adic family of Siegel Eisenstein series that interpolates Eisenstein series with trivial character modulo p .

3 Sketch of the proof of the main theorem

Since we can derive theorem 2.2 by theorem 2.1 and the existence of $\mathfrak{p}$ -adic Heck L -functions for totally real fields, we only prove theorem 2.1.

3.1 Definition of an Eisenstein series $\tilde{G}_{k,\chi}^{(n)}$

In this subsection, we give a more natural definition of $G_{k,\chi}^{(n)}$.

For a place v of F , we denote the space for the normalized induction by

$$\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\tilde{\chi}_{\mathfrak{p}} | \cdot |_{\mathfrak{p}}^s).$$

We define the intertwining operator

$$M_{w_n}^{(s)} : \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\tilde{\chi}_v | \cdot |_v^s) \rightarrow \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\tilde{\chi}_v^{-1} | \cdot |_v^{-s})$$

by

$$M_{w_n}^{(s)}(f)(g) = \int_{\mathrm{Sym}_n(F_v)} f \left(w_n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx,$$

for $g \in \mathrm{Sp}_n(F_v)$. Here we take a Haar measure of $\mathrm{Sym}_n(F_v)$ so that we have $\int_{\mathrm{Sym}_n(\mathcal{O}_v)} dx = 1$. The integral is convergent if $\mathrm{Re} s$ is sufficiently large and has meromorphic continuation to the whole complex plane.

We define a compact subgroup $C_{0,v}$ of $\mathrm{Sp}_n(F_v)$ as follows.

- (i) If v is real or $\tilde{\chi}_v$ is unramified then we define $C_{0,v} = C_v$.
- (ii) If $v = \mathfrak{p}$ is a finite place and $\tilde{\chi}_{\mathfrak{p}}$ is ramified then we define

$$C_{0,v} = \{g \in \mathrm{Sp}_n(\mathcal{O}_v) \mid c_g \equiv 0 \pmod{\mathfrak{p}^\nu}\}.$$

Here $\mathfrak{p}^\nu$ is the conductor of $\tilde{\chi}_v$.

We define a character κ_v of $C_{0,v}$ as follows.

- (i) If v is real then we define

$$\kappa_v \left(\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \right) = \det(u + iv)^{-k}.$$

- (ii) If v is finite and $\tilde{\chi}_v$ is unramified then we define $\kappa_v = 1$.
- (iii) If v is finite and $\tilde{\chi}_v$ is ramified then we define

$$\kappa_v(\gamma) = \tilde{\chi}_v(\det d_\gamma).$$

We denote by $\phi_v(s, \cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\tilde{\chi}_v | \cdot |_v^s)$ satisfying the following conditions.

$$\begin{aligned} \mathrm{supp} \phi_v(s, \cdot) &= P_n(F_v) C_{0,v}, \\ \phi_v(s, g\gamma) &= \kappa_v(\gamma) \phi_v(s, g) \quad \text{for all } \gamma \in C_{0,v}, \\ \phi_v(s, 1) &= 1. \end{aligned}$$

We also denote by $\phi'_v(-s, \cdot)$ the element of $\text{Ind}_{P_n}^{\text{Sp}_n}(\tilde{\chi}_v^{-1} | \cdot |_v^{-s})$ satisfying the following conditions.

$$\begin{aligned} \text{supp} \phi'_v(-s, \cdot) &= P_n(F_v) w_n C_{0,v}, \\ \phi'_v(-s, g\gamma) &= \kappa_v(\gamma) \phi'_v(-s, g) \quad \text{for all } \gamma \in C_{0,v}. \\ \phi'_v(-s, w_n) &= 1. \end{aligned} \quad (3.1)$$

For a place v of F , we define $\varphi_v(s, \cdot) \in \text{Ind}_{P_n}^{\text{Sp}_n}(\tilde{\chi}_v | \cdot |_v^s)$ as follows.

(i) If v is real or $\tilde{\chi}_v$ is unramified then we define

$$\varphi_v(s, g) = \phi_v(s, g).$$

(ii) If v is finite and $\tilde{\chi}_v$ is ramified then we define

$$\varphi_v(s, g) = M_{w_n}^{(-s)}(\phi'_v(-s, \cdot))(g). \quad (3.2)$$

For $g = (g_v)_v \in \text{Sp}_n(\mathbb{A}_F)$, we put

$$\varphi(s, g) = \prod_v \varphi_v(s, g_v),$$

where v runs over the set of the places of F .

We define an Eisenstein series on $\text{Sp}_n(\mathbb{A}_F)$ by

$$\tilde{\mathcal{G}}_{s,\chi}^{(n)}(g) = \sum_{\gamma \in P_n(F) \backslash \text{Sp}_n(F)} \varphi(s, \gamma g).$$

We define $\tilde{G}_{k,\chi}^{(n)}$ by the function on $\prod_{v|\infty} \mathfrak{H}_n$ corresponding to $\tilde{\mathcal{G}}_{k-(n+1)/2,\chi}^{(n)}$.

We can prove the proposition below by explicit calculation of the value of the intertwining operator. We omit the proof.

Proposition 3.1. *Assume that n is relatively prime to 2. Then we have*

$$G_{k,\chi}^{(n)} = \tilde{G}_{k,\chi}^{(n)}.$$

3.2 Functional equation of Whittaker functions

The key ingredient for the proof of the main theorem is the following theorem by T. Ikeda (Kyoto University).

Theorem 3.1 (T. Ikeda). *Let k be a local field. Let ψ be a nontrivial additive character of k and χ be a quasi character of $k^\times$. Suppose $B \in \text{Sym}_n(k)$ and $\det B \neq 0$. For $f \in \text{Ind}_{P_n}^{\text{Sp}_n}(\tilde{\chi}_p | \cdot |_p^s)$, we put*

$$W_B(f)(g) = \int_{\text{Sym}_n(k)} f \left(w_n \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} \right) \psi(-\text{Tr} Bx) dx.$$

Then

$$W_B \circ M_{w_n} = \chi(\det B)^{-1} |\det B|^{-s} c(s, B) W_B.$$

The notation is as follows.

Let n be even. D_B is defined by $D_B = (-1)^{n/2} \det B$. χ_B is the character of $k^\times$ corresponding to $k(\sqrt{D_B})/k$. $c(s, B)$ is given as follows.

$$\begin{aligned} c(s, B) &= |2|^{-ns} \frac{\alpha(D_B)}{\alpha(1)} \chi(2)^{-n} \varepsilon'(s + \frac{1}{2}, \chi \chi_B, \psi) \\ &\quad \times \varepsilon'(s - \frac{n-1}{2}, \chi, \psi)^{-1} \prod_{r=1}^{n/2} \varepsilon'(2s - n + 2r, \chi^2, \psi)^{-1}. \end{aligned}$$

Here $\varepsilon(s, \omega, \psi)$ is the epsilon factor, $\varepsilon'(s, \omega, \psi) = \varepsilon(s, \omega, \psi)^{\frac{L(1-s, \omega^{-1})}{L(s, \omega)}}$ and $\alpha(*)$ is the Weil index.

$$\frac{\alpha(1)}{\alpha(D_B)} = \varepsilon(\frac{1}{2}, \chi_B, \psi).$$

Let n be odd. Then

$$\begin{aligned} c(s, B) &= |2|^{-(n-1)s} \chi(2)^{-(n-1)} \zeta_B \\ &\quad \times \varepsilon'(s - \frac{n-1}{2}, \chi, \psi)^{-1} \prod_{r=1}^{(n-1)/2} \varepsilon'(2s - n + 2r, \chi^2, \psi)^{-1}. \end{aligned}$$

Here

$$\zeta_B = ((-1)^{(n-1)/2}, \det B)(-1, -1)^{(n^2-1)/8} h(B),$$

and $(*, *)$ is the Hilbert symbol.

3.3 Sketch of the proof

sketch of the proof of theorem 2.1. For simplicity, we assume the class number of F is one. Denote Φ by the Siegel operator. Then by the definition of $G_{k, \chi}^{(n)}$, we have $\Phi G_{k, \chi}^{(n)} = G_{k, \chi}^{(n-1)}(z)$. Thus it is enough to compute $a(B, G_{k, \chi}^{(n)})$ when $\det B \neq 0$. We can prove that $a(B, G_{k, \chi}^{(n)})$ has Euler product expression. By [3] (4.34K), (4.35K), [4] 13.6. Theorem, we know the Euler factor at infinite places and unramified places. Thus it is enough to compute the Euler factors at ramified places. Let $\mathfrak{p} \mid n$. Then the Euler factor at $\mathfrak{p}$ is given by

$$\begin{aligned} &\int_{\text{Sym}_n(F_{\mathfrak{p}})} \varphi_v \left(k - (n+1)/2, w_n \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} \right) \mathbf{e}(-\text{Tr} Bx) dx \\ &= W_B \circ M_{w_n}(\phi'(-k + (n+1)/2, \cdot))(1_n). \end{aligned}$$

By theorem 3.1, it is enough to compute $W_B(\phi'(-k + (n+1)/2, \cdot))(1_n)$, but it is easy to verify that this equals to 1. $\square$

References

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