

CONVOLUTIONS OF CERTAIN ANALYTIC FUNCTIONS

SHINJI SAITA AND SHIGEYOSHI OWA

Abstract. Two subclasses $M^*(\alpha)$ and $N^*(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in the open unit disk U are introduced. The object of the present paper is to derive some convolution properties for functions $f(z)$ belonging to the classes $M^*(\alpha)$ and $N^*(\alpha)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Mathematics Subject Classification (1991): 30C45.

Key Words and Phrases: Analytic function, convolution,
Hadamard product,
Cauchy-Schwarz inequality.

Let $M(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in U)$$

for some α ($\alpha > 1$). And let $N(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in U)$$

for some α ($\alpha > 1$). Recently, Uralegaddi, Ganigi and Sarangi [3] (also, Nishiwaki and Owa [1]) have showed that

(i) if $f(z) \in A$ satisfies

$$(1.2) \quad \sum_{n=2}^{\infty} (n-\alpha) |a_n| \leq \alpha - 1$$

for some α ($1 < \alpha < \frac{3}{2}$), then $f(z) \in M(\alpha)$,

(ii) if $f(z) \in A$ satisfies

$$(1.3) \quad \sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq \alpha - 1$$

for some α ($1 < \alpha < \frac{3}{2}$), then $f(z) \in N(\alpha)$.

By means of the above, we define the subclass $M^*(\alpha)$ of $M(\alpha)$ consisting of $f(z)$ which satisfy (1.2), and the subclass $N^*(\alpha)$ of $N(\alpha)$ consisting of $f(z)$ which satisfy (1.3).

For functions $f_j(x) \in A$ given by

$$(1.4) \quad f_j(x) = x + \sum_{n=2}^{\infty} a_{n,j} x^n,$$

the convolution (or Hadamard product) $(f_1 * f_2)(x)$ of $f_1(x)$ and $f_2(x)$ is defined by

$$(1.5) \quad (f_1 * f_2)(x) = x + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} x^n.$$

In the present paper, we consider the convolutions of functions $f_j(x)$ belonging to $M^*(\alpha)$ and $N^*(\alpha)$.

2. Convolutions of functions in the class $M^*(\alpha)$

Applying the same technique for convolutions due to Owa [2], we first derive

Theorem 1. If $f_j(x) \in M^*(\alpha_j)$ for $j=1, 2, 3, \dots, m$,
then $(f_1 * f_2 * \dots * f_m)(x) \in M^*(\beta)$, where

$$(2.1) \quad \beta = 1 + \frac{\prod_{j=1}^m (\alpha_j - 1)}{\prod_{j=1}^m (2 - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}$$

The result is sharp for functions

$$(2.2) \quad f_j(x) = x + \frac{\alpha_j - 1}{2 - \alpha_j} x^2 \quad (j=1, 2, 3, \dots, m).$$

Proof We use the mathematical induction for the proof.

Let $f_1(x) \in M^*(d_1)$ and $f_2(x) \in M^*(d_2)$. Then

$$\sum_{n=2}^{\infty} (n-d_j) |a_{n,j}| \leq d_j - 1 \quad (j=1,2)$$

gives us that

$$(2.3) \quad \sum_{n=2}^{\infty} \sqrt{\left(\frac{n-d_j}{d_j-1}\right) |a_{n,j}|} \leq 1 \quad (j=1,2)$$

Therefore, an application of Cauchy-Schwarz inequality implies that

$$\begin{aligned} & \left| \sum_{n=2}^{\infty} \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}} |a_{n,1}| |a_{n,2}| \right|^2 \\ & \leq \left(\sum_{n=2}^{\infty} \left(\frac{n-d_1}{d_1-1} \right) |a_{n,1}| \right) \left(\sum_{n=2}^{\infty} \left(\frac{n-d_2}{d_2-1} \right) |a_{n,2}| \right) \leq 1 \end{aligned}$$

Thus, if

$$(2.4) \quad \sum_{n=2}^{\infty} \left(\frac{n-d}{d-1} \right) |a_{n,1}| |a_{n,2}|$$

$$\leq \sum_{n=2}^{\infty} \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}} |a_{n,1}| |a_{n,2}|,$$

or if

$$(2.5) \quad \sqrt{|a_{n,1}| |a_{n,2}|} \leq \left(\frac{s-1}{n-s} \right) \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}}$$

for $n \geq 2$, then $f(x) \in M^*(s)$.

Furthermore, since (2.3) gives that

$$\sqrt{|a_{n,j}|} \leq \sqrt{\frac{d_j-1}{n-d_j}} \quad (j=1,2)$$

if

$$(2.6) \quad \sqrt{\frac{(d_1-1)(d_2-1)}{(n-d_1)(n-d_2)}} \leq \left(\frac{s-1}{n-s} \right) \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}},$$

or if

$$(2.7) \quad \frac{n-s}{s-1} \leq \frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}$$

for $n \geq 2$, then we have $f(x) \in M^*(s)$.

It follows from (2.7) that

$$(2.8) \quad s \geq 1 + \frac{(n-1)(d_1-1)(d_2-1)}{(n-d_1)(n-d_2) + (d_1-1)(d_2-1)} \equiv h(n)$$

for $n \geq 2$. Note that $h(n)$ is decreasing for $n \geq 2$.

Consequently, we have

$$(2.9) \quad \delta \geq h(2)$$

$$= 1 + \frac{(d_1-1)(d_2-1)}{(2-d_1)(2-d_2) + (d_1-1)(d_2-1)},$$

which shows that $f(x) \in M^*(\delta)$, where

$$(2.10) \quad \delta = 1 + \frac{(d_1-1)(d_2-1)}{(2-d_1)(2-d_2) + (d_1-1)(d_2-1)}$$

Next, we suppose that

$$(2.11) \quad (f_1 * f_2 * \cdots * f_m)(x) \in M^*(\beta'),$$

where

$$(2.12) \quad \beta' = 1 + \frac{\prod_{j=1}^m (d_j-1)}{\prod_{j=1}^m (2-d_j) + \prod_{j=1}^m (d_j-1)}$$

Then, by means of the previous proof, we have

$$(f_1 * f_2 * \cdots * f_m * f_{m+1})(x) \in M^*(\beta),$$

where

$$(2.13) \quad \beta = 1 + \frac{(\beta'-1)(d_{m+1}-1)}{(2-\beta')(2-d_{m+1}) + (\beta'-1)(d_{m+1}-1)}$$

Since

$$(\beta' - 1)(\alpha_{m+1} - 1) = \frac{\prod_{j=1}^{m+1} (\alpha_j - 1)}{\prod_{j=1}^m (2 - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}$$

and

$$(2 - \beta')(2 - \alpha_{m+1}) = \frac{\prod_{j=1}^{m+1} (2 - \alpha_j)}{\prod_{j=1}^m (2 - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)},$$

(2.13) gives

$$(2.14) \quad \beta = 1 + \frac{\prod_{j=1}^{m+1} (\alpha_j - 1)}{\prod_{j=1}^{m+1} (2 - \alpha_j) + \prod_{j=1}^{m+1} (\alpha_j - 1)}$$

Further, consider the functions $f_j(x)$ defined by (2.2). For such functions $f_j(x)$, we have

$$(f_1 * f_2 * \dots * f_m)(x) = x + \left(\prod_{j=1}^m \frac{\alpha_j - 1}{2 - \alpha_j} \right) x^2 = x + A_2 x^2$$

with $A_2 = \prod_{j=1}^m \frac{\alpha_j - 1}{2 - \alpha_j}$.

It follows that

$$(2.15) \quad \sum_{n=2}^{\infty} \frac{n - \beta}{\beta - 1} |A_n| = 1.$$

This completes the proof of Theorem 1.

Letting $\alpha_j = \alpha$ for $j = 1, 2, 3, \dots, m$ in Theorem 1, we have

Corollary 1. If $f_j(x) \in M^*(\alpha)$ for $j=1, 2, 3, \dots, m$,
 then $(f_1 * f_2 * \dots * f_m)(x) \in M^*(\beta)$, where

$$(2.16) \quad \beta = 1 + \frac{(\alpha-1)^m}{(2-\alpha)^m + (\alpha-1)^m}.$$

The result is sharp for functions

$$(2.17) \quad f_j(x) = x + \frac{\alpha-1}{2-\alpha} x^2 \quad (j=1, 2, 3, \dots, m)$$

3. Convolutions of functions in the class $N^*(\alpha)$

For convolutions of $f(x) \in N^*(\alpha)$, we have

Theorem 2. If $f_j(x) \in N^*(\alpha_j)$ for $j=1, 2, 3, \dots, m$,
 then $(f_1 * f_2 * \dots * f_m)(x) \in N^*(\beta)$, where

$$(3.1) \quad \beta = 1 + \frac{\prod_{j=1}^m (\alpha_j - 1)}{2^{m-1} \prod_{j=1}^m (2 - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}$$

The result is sharp for functions

$$(3.2) \quad f_j(x) = x + \frac{\alpha_j - 1}{2(2 - \alpha_j)} x^2 \quad (j=1, 2, 3, \dots, m)$$

Proof As in the proof of Theorem 1, for $f_1(x) \in N^*(d_1)$ and $f_2(x) \in N^*(d_2)$, the following inequality.

$$(3.3) \quad \sum_{n=2}^{\infty} \frac{n(n-s)}{s-1} |a_{n,1}| |a_{n,2}| \leq 1$$

implies $(f_1 * f_2)(x) \in N^*(s)$. Spending the same manner of the proof in Theorem 1, we obtain that

$$(3.4) \quad s \geq 1 + \frac{(n-1)(d_1-1)(d_2-1)}{n(n-d_1)(n-d_2) + (d_1-1)(d_2-1)}$$

for $n \geq 2$. The right hand side of (3.4) takes its maximum value for $n=2$, because it is a decreasing function of $n \geq 2$. Therefore, we have $(f_1 * f_2)(x) \in N^*(s)$ with

$$(3.5) \quad s = 1 + \frac{(d_1-1)(d_2-1)}{2(2-d_1)(2-d_2) + (d_1-1)(d_2-1)}$$

Moreover, suppose that

$$(f_1 * f_2 * \dots * f_m)(x) \in N^*(\beta)' ,$$

where

$$(3.6) \quad \beta' = 1 + \frac{\prod_{j=1}^m (d_j - 1)}{2^{m-1} \prod_{j=1}^m (2 - d_j) + \prod_{j=1}^m (d_j - 1)}$$

Then we get

$$(f_1 * f_2 * \dots * f_m * f_{m+1})(x) \in N^*(\beta),$$

where

$$(3.7) \quad \beta = 1 + \frac{(\beta' - 1)(\alpha_{m+1} - 1)}{2(2 - \beta')(2 - \alpha_{m+1}) + (\beta' - 1)(\alpha_{m+1} - 1)}$$

$$= 1 + \frac{\prod_{j=1}^{m+1} (\alpha_j - 1)}{2^{m-1} \prod_{j=1}^{m+1} (2 - \alpha_j) + \prod_{j=1}^{m+1} (\alpha_j - 1)}$$

Further, taking $f_j(x)$ given by (3.2), we know that the result of Theorem 2 is sharp.

Finally, letting $\alpha_j = d$ for $j = 1, 2, 3, \dots, m$ in Theorem 2,

Corollary 2. If $f_j(x) \in N^*(d)$ for $j = 1, 2, 3, \dots, m$,

then $(f_1 * f_2 * \dots * f_m)(x) \in N^*(\beta)$, where

$$(3.8) \quad \beta = 1 + \frac{(d-1)^m}{2^{m-1}(2-d)^m + (d-1)^m}.$$

The result is sharp for

$$(3.9) \quad f_j(x) = x + \frac{d-1}{2(2-d)} x^2 \quad (j=1, 2, 3, \dots, m).$$

Remark Uraleghaddi and Desai [4] have showed some convolution properties of univalent functions with positive coefficients. Our result in the present paper are generalization theorems of the results by Uraleghaddi and Desai [4].

References

- [1] J. Nishiwaki and S. Owa, Coefficient inequalities for certain analytic functions, preprint.
- [2] S. Owa, The quasi-Hadamard products of certain analytic functions, in Current Topics in Analytic Function Theory (edited by H.M. Srivastava and S. Owa; World Scientific; Singapore, New Jersey, London and Hong Kong, 1992), 234-251.
- [3] B.A. Uraleghaddi, M.D. Ganigi and S.M. Sarangi, Univalent functions with positive coefficients, Tam Kang J. Math. 25 (1994), 225-230.
- [4] B.A. Uraleghaddi and A.R. Desai, Convolutions of univalent functions with positive coefficients, Tam Kang J. Math. 29 (1998), 279-285.

Department of Mathematics
 Kinki University
 Higashi-Osaka, Osaka 577-8502
 Japan