

Chaotic string motion in a near pp-wave limit

Shodai Kushiro

*Department of Physics, Kyoto University,
Kitashirakawa Oiwake-cho, Kyoto 606-8502, Japan.*

*Dissertation submitted for Department of Physics, Kyoto University
December, 2024*

Abstract

In this paper, we revisit the classical string motion in a near pp-wave limit of $\text{AdS}_5 \times \text{S}^5$, inspired by the relationship between integrability and chaos in the AdS/CFT correspondence. While the full $\text{AdS}_5 \times \text{S}^5$ background and its exact pp-wave limit are classically integrable, we numerically show that the classical string motion in the near pp-wave limit exhibits chaos. This result is obtained by computing Poincaré sections and Lyapunov spectrum, highlighting the breakdown of integrability in this intermediate regime.

In Chapter 2, we reviewed the connection between the dilatation operator in $\mathcal{N} = 4$ SYM theory and spin-chain Hamiltonians, with particular focus on the two-loop $\text{SU}(2)$ sector corresponding to the XXX spin-chain with next-nearest-neighbor interactions. Chapter 3 introduced the concept of chaos, including level-spacing statistics as a method to characterize quantum chaos. It was shown that the spin-chain system which corresponds to two-loop $\text{SU}(2)$ sector dilatation operator becomes quantum chaotic.

The key finding of this work parallels the behavior of the Henon-Heiles model, derived as a finite-order truncation of a three-particle periodic Toda chain. The Toda chain is integrable, but the truncation breaks its integrability, resulting in chaotic dynamics. Similarly, the near pp-wave limit can be seen as a finite-order truncation of the full $\text{AdS}_5 \times \text{S}^5$ background, where classical chaos emerges as a perturbative artifact.

Contents

1	Introduction	3
1.1	Integrability and Chaos in the AdS/CFT Correspondence	3
1.2	Motivation of our work	4
1.3	Structure of this Thesis	5
2	Spin Chains in $\mathcal{N} = 4$ SYM Theory	7
2.1	The Anomalous Dimensions of Scalar Single-Trace Operators in $\mathcal{N} = 4$ SYM Theory	7
2.2	SU(2) sector dilatation operator: spin-chain Hamiltonian	12
3	Quantum chaos and XXX spin-chain with NNN interactions	15
3.1	Classical chaos	15
3.2	Quantum chaos: Level spacing statistics	17
3.3	Example of level spacing statistics	21
3.4	Chaotic spin chain in AdS/CFT	23
4	Chaotic string motion in a near pp-wave limit	26
4.1	Breaking classical integrability by truncation	27
4.1.1	Three-particle periodic Toda chain	27
4.1.2	Derivation of the Henon-Heiles model	28
4.2	The Hamiltonian of a near pp-wave string	30
4.2.1	A near pp-wave limit of $\text{AdS}_5 \times \text{S}^5$	30
4.2.2	Light-cone Hamiltonian	31
4.3	Chaotic dynamics	33
4.3.1	A stringy ansatz	33

4.3.2	Poincaré sections and Lyapunov spectrum	34
Appendix 4.A	The third conserved charge I_3	36
5	Conculusion and discussion	38
A	Calculation of Lyapunov spectrum	40

Chapter 1

Introduction

1.1 Integrability and Chaos in the AdS/CFT Correspondence

The AdS/CFT correspondence [1–3] is one of the most significant developments in String Theory. A notable example of this duality is the relationship between $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM) theory in the large- N limit and type IIB superstring theory on $AdS_5 \times S^5$. An interesting feature of this duality is that both theories exhibit integrable structures [4]. On the string-theory side, this manifests as the classical integrability of the string sigma model with a target spacetime of $AdS_5 \times S^5$ [5, 6]. On the $\mathcal{N} = 4$ SYM theory side, the integrable structure appears in the dilatation operator [7], which acts on single-trace operators and can be mapped to a spin-chain Hamiltonian. In particular, the one-loop dilatation operator restricted to the scalar $SU(2)$ sector corresponds to the XXX spin-chain [7], a well-known quantum integrable model. These integrability properties have been utilized to test the AdS/CFT correspondence by computing and comparing string energies with the scaling dimensions of single-trace operators (e.g., [8]).

While integrability has been a central focus in studies of the AdS/CFT correspondence, it is equally important to explore chaotic dynamics in this context. Integrable systems are rare in nature, and chaos is a more commonly observed phenomenon, with string theory being no exception. The chaotic behavior of string dynamics can be quantified and visualized using numerical tools such as Lyapunov exponents and Poincaré sections. For instance, classical string motion in the AdS_5 -Schwarzschild background has been

shown to exhibit chaotic dynamics [9]. Chaotic string dynamics have also been studied in various settings, including AdS solitons [10], $\text{AdS}_5 \times \text{T}^{1,1}$ [11], and other $\text{AdS}_5 \times \text{Einstein}_5$ spaces [12]. Further investigations have examined p-brane backgrounds [13–15] and a wide range of other configurations [16–38]. This topic has also been explored in my previous work [39].

In recent years, the study of chaotic phenomena within the framework of AdS/CFT has gained considerable attention. The connection between chaos and the AdS/CFT correspondence is particularly intriguing because it offers insights into quantum information scrambling, thermalization, and the nature of black holes as dynamical systems. Black holes, conjectured to be the fastest scramblers in nature [40–42], have been a focal point of extensive studies on chaos [43–46].

1.2 Motivation of our work

In $\mathcal{N} = 4$ SYM theory, integrability can be broken in certain cases. For instance, when non-planar corrections are included in the dilatation operator, the level-spacing distribution closely resembles the Wigner-Dyson distribution, which indicates quantum chaos [47, 48]. More intriguingly, integrability may also be weakly broken even in the large N limit, as discussed in [49]. When the all-loop integrable spin chain [50–52] is truncated to a finite order, the truncated system may exhibit quantum chaos [53]. This implies that even if the full-order system is integrable, finite-order truncation can lead to non-integrable behavior.

It is instructive to recall the results of [53]. When the full-order dilatation operator is truncated at the two-loop level, the resulting system can be expressed as the Hamiltonian of an XXX spin chain with next-nearest-neighbor (NNN) interactions:

$$H = \epsilon_0 \sum_{i=1}^L \mathbb{1}_{i,i+1} + \lambda_1 \sum_{i=1}^L \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \lambda_2 \sum_{i=1}^L \mathbf{S}_i \cdot \mathbf{S}_{i+2} \quad (1.2.1)$$

with

$$\epsilon_0 = 1 - \frac{3\lambda}{16\pi^2}, \quad \lambda_1 = -4 + \frac{\lambda}{\pi^2}, \quad \lambda_2 = -\frac{\lambda}{4\pi^2}, \quad (1.2.2)$$

where λ represents the 't Hooft coupling, and the spin variable at the i -th site is denoted as $\mathbf{S}_i = (\sigma_i^1, \sigma_i^2, \sigma_i^3)$, with σ_i^a ($a = 1, 2, 3$) being the standard Pauli matrices. The two-loop

spin chain is an example of weakly chaotic spin chains [49], where the energy-level spacing distribution transitions from Poisson at small λ to the Wigner-Dyson distribution at large λ .

Inspired by the findings of [53], we turn our attention to the string-theory side of the AdS/CFT correspondence. One possible approach involves considering a Penrose limit [54, 55] of the $\text{AdS}_5 \times \text{S}^5$ background. When the light-cone coordinates are chosen using the global AdS time and an angular variable in S^5 , the resulting background becomes the maximally supersymmetric pp-wave background [56, 57]. Type IIB superstring theory on the pp-wave background is exactly solvable, as the world-sheet theory is described by a collection of massive free bosons and fermions [58]. The AdS/CFT dictionary at the pp-wave level was established by Berenstein, Maldacena, and Nastase [8]. Sub-leading corrections can also be included in the Penrose limit, leading to a near pp-wave background with non-trivial interactions on the world-sheet [59].

An intriguing question is whether the world-sheet theory in the near pp-wave limit is classically integrable. Previous studies [18] based on a point-particle ansatz found no evidence of classical chaos in this regime. However, their setup was significantly restricted. In this paper, we revisit this question using a stringy ansatz and demonstrate classical chaos through the computation of Poincaré sections and the Lyapunov spectrum. Consequently, while the pp-wave string and the $\text{AdS}_5 \times \text{S}^5$ string are integrable, the world-sheet theory in the intermediate Penrose limit is found to be non-integrable.

1.3 Structure of this Thesis

The structure of this thesis is as follows. This thesis is divided into four chapters, each focusing on a specific aspect of our research.

In Chapter 2, we provide a review of the anomalous dimension of single-trace operators in $\mathcal{N} = 4$ SYM theory. The one-loop planar dilatation operator corresponds to the XXX spin chain Hamiltonian, whereas the two-loop operator corresponds to the XXX spin chain Hamiltonian with next-to-nearest-neighbor (NNN) interactions.

In Chapter 3, we review the concepts of classical chaos and quantum chaos, with a particular focus on the chaotic properties of the XXX spin chain with NNN interactions.

In Chapter 4, building on our work [60], we numerically demonstrate chaotic string motion in the near pp-wave limit of $\text{AdS}_5 \times \text{S}^5$.

Chapter 2

Spin Chains in $\mathcal{N} = 4$ SYM Theory

In this chapter, we review the anomalous dimensions of single-trace operators in $\mathcal{N} = 4$ SYM theory. One way to test the AdS/CFT correspondence is by calculating the anomalous dimensions on the gauge theory side and comparing them with the energy spectrum on the gravity side. At that time, the calculation of the anomalous dimensions of scalar single-trace operators in $\mathcal{N} = 4$ SYM theory can be mapped to a one-dimensional spin chain [7, 61].

2.1 The Anomalous Dimensions of Scalar Single-Trace Operators in $\mathcal{N} = 4$ SYM Theory

For simplicity, we focus on scalar single-trace operators of the form:

$$\mathcal{O}_{i_1 i_2 \dots i_L}(x) = \frac{(2\pi)^L}{\sqrt{C_{i_1 i_2 \dots i_L}} N^{L/2}} \text{Tr} \left(\phi^{i_1}(x) \phi^{i_2}(x) \dots \phi^{i_L}(x) \right), \quad (2.1.1)$$

where ϕ^i , ($i = 1, \dots, 6$) are $\text{SU}(N)$ adjoint scalars, and $C_{i_1 i_2 \dots i_L}$ is a symmetry factor. To calculate the anomalous dimensions, we analyze the two-point functions of the operators $\mathcal{O}(x)$:

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(y) \rangle \approx \frac{1}{|x - y|^{2\Delta}}, \quad (2.1.2)$$

where the scaling dimension Δ is expressed as $\Delta = \Delta_0 + \gamma$, with Δ_0 being the bare dimension and γ the anomalous dimension. For small values of the coupling constant

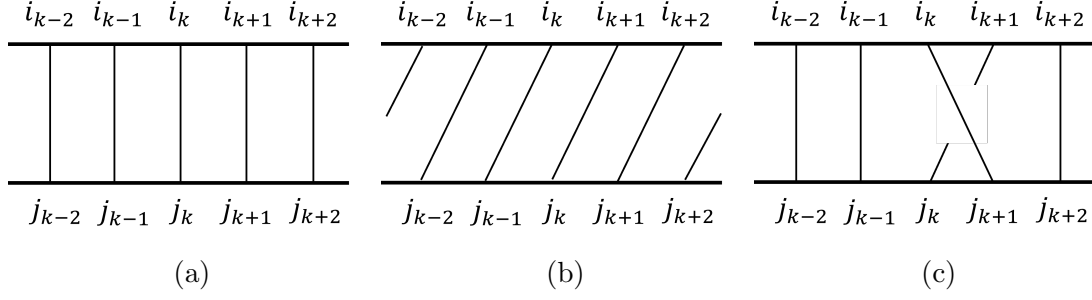


Figure 2.1: How to contract scalar fields at tree-level. i_k means a scalar fields $(\phi^{i_k})^A_B(x)$. (a) and (b) are planer diagrams while (c) is non-planer diagram.

g_{YM} , the two-point function (2.1.2) can be approximated as:

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(y) \rangle \approx \frac{1}{|x-y|^{2\Delta_0}} (1 - \gamma \ln \mu^2 |x-y|^2), \quad (2.1.3)$$

where μ is the UV cutoff scale.

Let us now consider the large N limit. First, we compute the two-point functions at tree level. The propagator for the scalar fields is given by:

$$\langle (\phi^i)^A_B(x) (\phi^j)^C_D(y) \rangle = \frac{\delta^{ij} \delta^A_D \delta^C_B}{4\pi^2 |x-y|^2}, \quad (2.1.4)$$

where A, B, C, D are $\text{SU}(N)$ color indices. In practice, there exists a contribution reflecting the traceless nature of ϕ , but this contribution is suppressed in the large N limit.

The tree-level correlation function is obtained by contracting the scalar fields as illustrated in Figure 2.1. Each line in the figure represents a contraction (propagator). For planar diagrams such as Figures 2.1 (a) and 2.1 (b), the tree-level correlation functions are proportional to N^L , which arises from L factors of $\delta^A_A, \delta^{A'}_{A'} = N$. In contrast, for the non-planar diagram shown in Figure 2.1 (c), the correlation functions are proportional to N^{L-2} . Thus, in the large N limit, we can neglect the contributions of non-planar diagrams like (c) in comparison to planar diagrams such as (a) and (b). For the same reason, non-planar diagrams are ignored when calculating loop corrections. As a result, the tree-level correlation functions are given by:

$$\langle \mathcal{O}_{i_1 i_2 \dots i_L}(x) \overline{\mathcal{O}}^{j_1 j_2 \dots j_L}(y) \rangle_{\text{tree}} = \frac{1}{\sqrt{C_{i_1 i_2 \dots i_L} C_{j_1 j_2 \dots j_L}}} (\delta_{i_1}^{j_1} \dots \delta_{i_L}^{j_L} + \text{cycles}) \frac{1}{|x-y|^{2L}}, \quad (2.1.5)$$

where "cycles" refers to the $L-1$ cyclic permutations of the indices j_l .

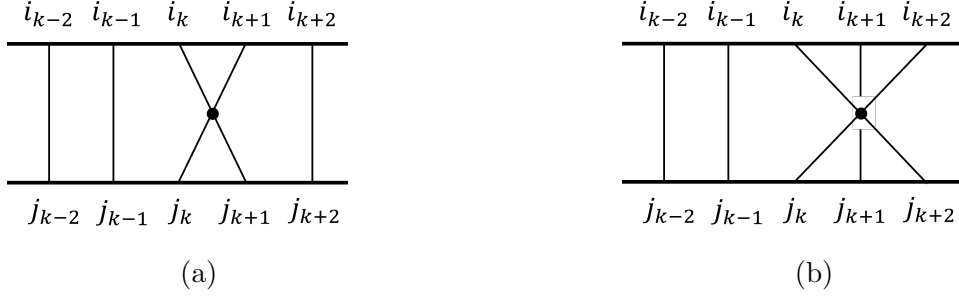


Figure 2.2: Feynman graphs containing scalar 4-point vertex. Solid lines are scalar propagator and dots are scalar 4-point vertex. (a) is planar and (b) is non-planar. Of course, there exist shift diagrams of (a) or (b), such as 2.1 (b)

Next, we consider the one-loop contributions to the two-point functions. The $\mathcal{N} = 4$ SYM action is given by:

$$S = \frac{1}{2g_{\text{YM}}^2} \int d^4x \text{Tr} \left(-\frac{1}{2} F^2 + \mathcal{D}_\mu \phi_i \mathcal{D}^\mu \phi^i - \sum_{i < j} [\phi_i, \phi_j]^2 + (\text{fermion}) \right), \quad (2.1.6)$$

where g_{YM} is the coupling constant, $F = dA$ is the gauge field strength, and $\mathcal{D}_\mu \phi_i = \partial_\mu \phi_i - [A_\mu, \phi_i]$ is the covariant derivative. The interactions involving scalar fields include a quartic term and interaction terms between scalars and gauge bosons, as well as terms involving fermions. While it is necessary to account for all possible one-loop corrections, it suffices to focus on Feynman diagrams containing scalar four-point vertices (Figure 2.2), owing to the constraints imposed by superconformal symmetry.¹

We rewrite the quartic term as:

$$\frac{g_{\text{YM}}^2}{4} \sum_{i,j} (\text{Tr} \phi_i \phi_i \phi_j \phi_j - \text{Tr} \phi_i \phi_j \phi_i \phi_j). \quad (2.1.7)$$

Next, we consider the subcorrelator shown in Figure 2.2 (a):

$$\begin{aligned} & \left\langle (\phi_{i_k} \phi_{i_{k+1}})^A_{C}(x) \left(\frac{ig_{\text{YM}}^2}{4} \int d^4z \sum_{i,j} (\text{Tr} \phi_i \phi_i \phi_j \phi_j(z) - \text{Tr} \phi_i \phi_j \phi_i \phi_j(z)) \right) (\phi^{j_k} \phi^{j_{k+1}})^{C'}_{A'}(y) \right\rangle \\ &= \frac{iN}{(4\pi^2)^4} \delta^A_{A'} \delta^{C'}_{C} \frac{Ng_{\text{YM}}^2}{4} \left(2\delta_{i_k}^{j_k} \delta_{i_{k+1}}^{j_{k+1}} + 2\delta_{i_k i_{k+1}} \delta^{j_k j_{k+1}} - 4\delta_{i_k}^{j_{k+1}} \delta_{i_{k+1}}^{j_k} \right) \int d^4z \frac{1}{(z-x)^4 (z-y)^4}. \end{aligned} \quad (2.1.8)$$

¹The remaining contributions can be computed indirectly.

The integral in (2.1.8) exhibits a logarithmic divergence as $z \rightarrow x$ or $z \rightarrow y$. To regularize this divergence, we introduce a UV cutoff μ . The integral over z , initially in Minkowski space, can be Wick rotated to Euclidean space, where $d^4z \rightarrow id^4z_E$. We then restrict the integral to regions satisfying $|z_E - x| \geq \mu^{-1}$ and $|z_E - y| \geq \mu^{-1}$. Under these conditions, the integral can be approximated as:

$$i \int \frac{d^4z_E}{(z-x)^4(z-y)^4} \approx \frac{2i}{|x-y|^4} \int^{|x-y|} \mu^{-1} \frac{d\xi d\Omega_3}{\xi} = \frac{2\pi^2 i}{|x-y|^4} \ln(\mu^2|x-y|^2). \quad (2.1.9)$$

Thus, we obtain:

$$\begin{aligned} & \left\langle \mathcal{O}_{i_1 i_2 \dots i_L}(x) \overline{\mathcal{O}}^{j_1 j_2 \dots j_L}(y) \right\rangle_{4\text{-point}} \\ &= \frac{\lambda}{16\pi^2} \frac{\ln(\mu^2|x-y|^2)}{(x-y)^{2L}} \sum_{k=1}^L (2P_{k,k+1} - K_{k,k+1} - 1) \frac{1}{\sqrt{C_{i_1 \dots i_L} C_{j_1 \dots j_L}}} \delta_{i_1}^{j_1} \dots \delta_{i_L}^{j_L} + \text{cycles}, \end{aligned} \quad (2.1.10)$$

where $\lambda = Ng_{\text{YM}}^2$, known as the 't Hooft coupling. Here, $P_{k,k+1}$ is the exchange operator, defined as the exchange of flavor indices between sites k and $k+1$:

$$P_{k,k+1} \delta_{i_1}^{j_1} \dots \delta_{i_k}^{j_k} \delta_{i_{k+1}}^{j_{k+1}} \dots \delta_{i_L}^{j_L} = \delta_{i_1}^{j_1} \dots \delta_{i_k}^{j_{k+1}} \delta_{i_{k+1}}^{j_k} \dots \delta_{i_L}^{j_L}. \quad (2.1.11)$$

$K_{k,k+1}$ is the trace operator, which contracts the flavor indices of the fields at sites k and $k+1$:

$$K_{k,k+1} \delta_{i_1}^{j_1} \dots \delta_{i_k}^{j_k} \delta_{i_{k+1}}^{j_{k+1}} \dots \delta_{i_L}^{j_L} = \delta_{i_1}^{j_1} \dots \delta_{i_k i_{k+1}}^{j_k j_{k+1}} \dots \delta_{i_L}^{j_L}. \quad (2.1.12)$$

Due to periodicity, $P_{L,L+1} = P_{1,L}$ and $K_{L,L+1} = K_{1,L}$.

We now calculate the remaining one-loop contributions. The relevant diagrams are shown in Figure 2.3. Gluons carry no flavor charge (R-charge), so the flavor indices remain unchanged during gluon exchange (Figure 2.3(a)). Naturally, self-energy contributions (Figures 2.3(b) and 2.3(c)) do not affect the flavor indices. Since these contributions maintain the same flavor structure as the tree-level result, they can be collectively represented by the constant C .

Adding the one-loop contributions to the tree-level correlation functions (2.1.5), we obtain:

$$\begin{aligned} & \left\langle \mathcal{O}_{i_1 i_2 \dots i_L}(x) \overline{\mathcal{O}}^{j_1 j_2 \dots j_L}(y) \right\rangle \\ &= \frac{1}{\sqrt{C_{i_1 \dots i_L} C_{j_1 \dots j_L}} (x-y)^{2L}} (1 - \ln(\mu^2(x-y)^2) \mathcal{D}^{(2)}) \delta_{i_1}^{j_1} \dots \delta_{i_L}^{j_L} + \text{cycles}, \end{aligned} \quad (2.1.13)$$

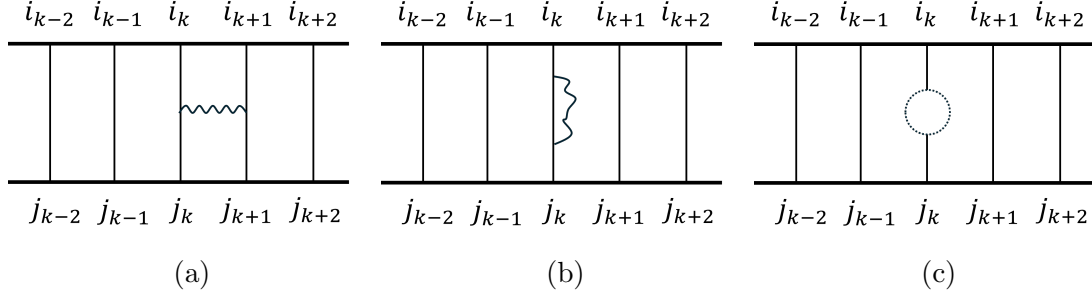


Figure 2.3: one-loop planar diagrams containing gluon or fermion. Solid lines indicate scalar propagator, wavy lines indicate gluon propagator and dotted lines indicate fermion propagator. (a) is gluon exchange contributions between neighboring scalars. (b) is scalar self-energy contributions from gluon. (c) is scalar self-energy contributions from a fermion loop.

where the operator $\mathcal{D}^{(2)}$ is expressed as:

$$\mathcal{D}^{(2)} = \frac{\lambda}{16\pi^2} \sum_{k=1}^L (1 - C - 2P_{k,k+1} + K_{k,k+1}). \quad (2.1.14)$$

Comparing (2.1.13) with (2.1.3), we see that the anomalous dimension γ corresponds to $\mathcal{D}^{(2)}$ at the one-loop level. The anomalous dimension γ is a real number, whereas $\mathcal{D}^{(2)}$ is a matrix. This is because the four-point interaction (2.1.7) induces operator mixing. As a result, the anomalous dimensions are obtained by diagonalizing $\mathcal{D}^{(2)}$.

Finally, we determine the constant C using a technical approach. Let us consider the chiral primary operator Ψ_L , defined as:

$$\Psi_L = \frac{(2\pi)^L}{\sqrt{L}N^{L/2}} \text{Tr } Z^L, \quad (2.1.15)$$

where $Z = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$. The anomalous dimension of the chiral primary (BPS) operator vanishes at all orders², implying $\mathcal{D}^{(2)}\Psi_L = 0$. Ψ_L consists only of the scalar field Z , it satisfies $P_{k,k+1}\Psi_L = \Psi_L$. Furthermore, as Ψ_L does not involve any conjugate field $\bar{Z}$, it also satisfies $K_{k,k+1}\Psi_L = 0$. Using these properties, we find that $C = -1$, and $\mathcal{D}^{(2)}$ simplifies to:

$$\mathcal{D}^{(2)} = \frac{\lambda}{8\pi^2} \sum_{k=1}^L \left(1 - P_{k,k+1} + \frac{1}{2}K_{k,k+1} \right). \quad (2.1.16)$$

²This follows from the non-renormalization theorem.

Scalar single-trace operators in (2.1.1) can be interpreted as a one-dimensional spin chain with L sites, where each site hosts an $\text{SO}(6)$ vector "spin". By viewing $\mathcal{D}^{(2)}$ as the Hamiltonian of this spin chain, the problem of finding the anomalous dimensions becomes equivalent to solving the quantum mechanics of the spin chain.

2.2 $\text{SU}(2)$ sector dilatation operator: spin-chain Hamiltonian

To simplify the discussion, we consider a sub-sector of operators. Before doing so, let us review the symmetry structure of $\mathcal{N} = 4$ SYM theory. The $\mathcal{N} = 4$ superconformal group is $\text{PSU}(2, 2|4)$, which has a bosonic subgroup $\text{SU}(2, 2) \times \text{SU}(4)$. Here, $\text{SU}(2, 2) \sim \text{SO}(2, 4)$ represents the four-dimensional conformal group, while $\text{SU}(4) \sim \text{SO}(6)$ corresponds to the R-symmetry.

The conformal algebra consists of 15 generators: 4 generators of space-time translations P^μ , 6 generators of Lorentz transformations $M^{\mu\nu}$, 1 generator of dilatation D , and 4 generators of special conformal transformations K^μ . The dilatation generator and Lorentz generators commute. Consequently, the operators in this theory can be characterized by three charges: the scaling dimension Δ , and the spins S_1 and S_2 . The algebra of $\text{SO}(6)$ R-symmetry has three commuting generators in its Cartan subalgebra, with their associated charges denoted as J_i ($i = 1, 2, 3$). Overall, certain operators in this theory can be labeled by six charges: $(\Delta, S_1, S_2; J_1, J_2, J_3)$.

As an example, let us consider three complex scalar fields:

$$Z = \frac{1}{\sqrt{2}}(\phi^1 + \phi^2),, \quad W = \frac{1}{\sqrt{2}}(\phi^3 + \phi^4),, \quad X = \frac{1}{\sqrt{2}}(\phi^5 + \phi^6). \quad (2.2.1)$$

In four dimensions, scalar fields have a bare scaling dimension of 1 and are spinless ($S_1 = S_2 = 0$). The charges associated with Z, W, X are thus given by $(1, 0, 0; 1, 0, 0)$, $(1, 0, 0; 0, 1, 0)$, and $(1, 0, 0; 0, 0, 1)$, respectively. The chiral primary operator Ψ_L defined in (2.1.15) has charges $(L, 0, 0; L, 0, 0)$.

As discussed above, $\mathcal{N} = 4$ SYM theory has six mutually commuting generators, one of which is the dilatation operator. When diagonalizing the dilatation operator, the other charges are conserved. As a result, operator mixing occurs only among operators with identical charges. This gives rise to the existence of closed sectors.

For instance, single-trace operators composed of K Z -fields and $(L - K)$ W -fields carry the charges $(L, 0, 0; K, L - K, 0)$. Mixing occurs only between operators that have the same number of Z -fields and W -fields in all possible permutations. This restricted set of operators defines the $SU(2)$ sector.

From this point onward, we restrict our analysis to the $SU(2)$ closed sector. In this sector, the Z -field is labeled as spin-up ($\uparrow$), and the W -field as spin-down ($\downarrow$). Since the single-trace operators in the $SU(2)$ sector do not involve conjugate fields $\bar{Z}$ or $\bar{W}$, the contribution of $K_{k,k+1}$ is zero. The dilatation operator $\mathcal{D}^{(2)}$ is therefore expressed as:

$$\mathcal{D}^{(2)} = \frac{\lambda}{8\pi^2} \sum_{k=1}^L (1 - P_{k,k+1}), \quad (2.2.2)$$

where $P_{k,k+1}$ is the exchange operator, which swaps the spins at sites k and $k + 1$.

We introduce the spin operators S_i^a as follows:

$$S_i^x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_i^y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_i^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.2.3)$$

We define the spin interaction between adjacent sites as:

$$\vec{S}_k \cdot \vec{S}_{k+1} = \frac{1}{2} (S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+) + S_k^z S_{k+1}^z, \quad (2.2.4)$$

where $S_i^\pm = S_i^x \pm iS_i^y$ are ladder operators. The first term in (2.2.4) exchanges adjacent spins in opposite directions, while the second term contributes $+1/4$ if the spins are aligned and $-1/4$ otherwise. The exchange operator $P_{k,k+1}$ can be rewritten in terms of spin operators as:

$$P_{k,k+1} = \frac{1}{2} (1 - 4\vec{S}_k \cdot \vec{S}_{k+1}). \quad (2.2.5)$$

Substituting this into $\mathcal{D}^{(2)}$, we rewrite it as:

$$\mathcal{D}^{(2)} = \frac{\lambda}{8\pi^2} \sum_{k=1}^L \left(\frac{1}{2} - 2\vec{S}_k \cdot \vec{S}_{k+1} \right). \quad (2.2.6)$$

This is the Hamiltonian of the Heisenberg XXX spin chain with L sites. The XXX spin chain is a well-known quantum integrable system and can be solved analytically using the Bethe ansatz method [7].

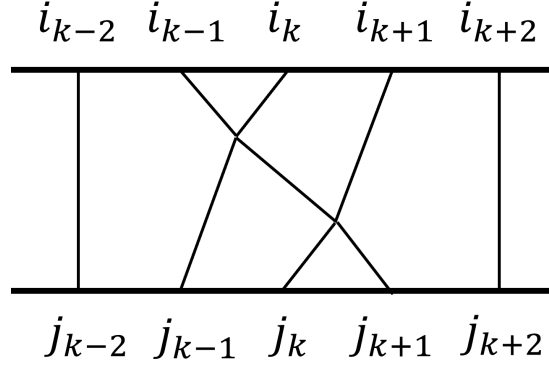


Figure 2.4: The example of a diagram with next-to-nearest neighbor contribution at two-loop level.

The dilatation operator at two-loop order and beyond has been studied in [61] (see [62, 63] for a review). At the two-loop level, the dilatation operator $\mathcal{D}^{(4)}$ includes a permutation of next-to-nearest neighbors. An example of a diagram illustrating such contributions is shown in Figure 2.4. The SU(2) sector dilatation operator at two-loop level is expressed as:

$$\mathcal{D}^{(4)} = \left(\frac{\lambda}{16\pi^2} \right)^2 \sum_{k=1}^L (-8 + 12P_{k,k+1} - 2(P_{k,k+1}P_{k+1,k+2} + P_{k+1,k+2}P_{k,k+1})). \quad (2.2.7)$$

Rewriting this in terms of spin operators, we have:

$$\mathcal{D}^{(4)} = \left(\frac{\lambda}{16\pi^2} \right)^2 \sum_{k=1}^L \left(-3 + 16\vec{S}_k \cdot \vec{S}_{k+1} - 4\vec{S}_k \cdot \vec{S}_{k+2} \right). \quad (2.2.8)$$

In addition to the terms corresponding to the XXX spin chain, the two-loop dilatation operator also includes next-to-nearest neighbor interactions.

Chapter 3

Quantum chaos and XXX spin-chain with NNN interactions

In this chapter, we introduce the concept of quantum chaos, focusing on its relevance to the motivation behind our research. Classical chaos is generally characterized by the behavior of orbits in phase space. However, in quantum mechanics, the notion of orbits is not applicable. Instead, alternative approaches for defining or identifying quantum chaos have been proposed, one of which is level-spacing statistics.

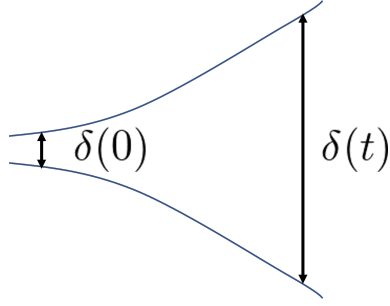
We begin by introducing classical chaos in Section 3.1, followed by a discussion on level-spacing statistics in Section 3.2. In Section 3.3, we explain the procedure for calculating level-spacing statistics and provide an example calculation. Finally, in Section 3.4, we discuss the chaos observed in the XXX spin chain with next-to-nearest neighbor (NNN) interactions, which is directly related to the motivation of our research.

3.1 Classical chaos

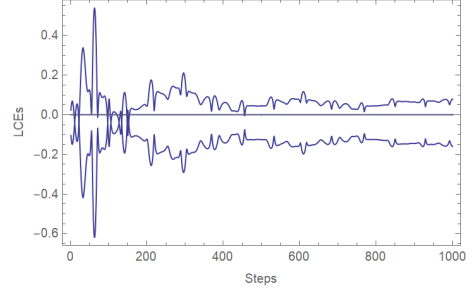
First of all, let us explain chaos in a classical dynamical system. In general, classical chaos is characterized by the following properties.

Non-linearity

First, non-linearity is a necessary condition for chaos. Linear differential equations are generally solvable and do not exhibit chaotic behavior. Chaotic behavior arises from



(a) Lyapunov exponent



(b) Lyapunov spectrum for the Duffing model

Figure 3.1: The Lyapunov exponent and spectrum as measures of sensitivity to initial conditions in chaotic systems.

deterministic dynamics, where randomness does not stem from external inputs or parameters but rather from the intrinsic non-linearity of the system. However, it is important to note that non-linearity alone does not guarantee chaos; for instance, integrable systems can also be non-linear.

Sensitivity to initial conditions

The most important property of chaos is sensitivity to initial conditions. This implies that even if two initial values are arbitrarily close, their trajectories in phase space will diverge exponentially over time. This behavior can be quantitatively evaluated using a measure known as the Lyapunov exponent.

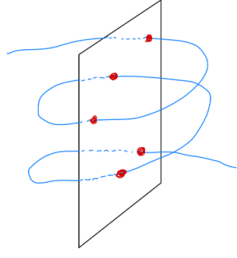
Let $\delta(t)$ represent the phase space distance between two trajectories at time t . For a chaotic system, $\delta(t)$ is approximately expressed as:

$$\|\delta(t)\| \sim \|\delta(0)\|e^{\lambda t}, \quad (3.1.1)$$

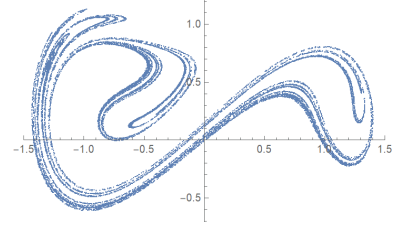
where λ is the Lyapunov exponent (Figure 3.1 (a)).

In a k -dimensional phase space, there are k Lyapunov exponents $(\lambda_1, \lambda_2, \dots, \lambda_k)$, collectively referred to as the Lyapunov spectrum (see Appendix A). The largest value, known as the maximal Lyapunov exponent, determines the system's sensitivity to initial conditions. If the maximal Lyapunov exponent is positive, the system exhibits chaotic behavior.

Figure 3.1 (b) illustrates an example of the Lyapunov spectrum for the Duffing model, a well-known chaotic system.



(a) Poincaré section



(b) Poincaré sections for the Duffing model

Figure 3.2: Poincaré sections illustrating the non-periodic nature of chaotic systems.

Non-periodicity

Chaotic systems do not converge to fixed points or asymptotically approach periodic orbits. Instead, their trajectories follow non-periodic paths. This behavior can be visualized using Poincaré sections, which show scattered points in phase space that do not form closed curves (Figure 3.2).

For instance, Figure 3.2(b) illustrates the Poincaré sections of the Duffing model, demonstrating its chaotic nature.

Summary of classical chaos

To summarize, classical chaos is defined by three key properties: non-linearity, sensitivity to initial conditions, and non-periodicity ¹. These characteristics collectively describe the unpredictable and complex behavior of chaotic systems. Quantitative tools like Lyapunov exponents and visual representations such as Poincaré sections provide practical methods to identify and analyze chaos.

3.2 Quantum chaos: Level spacing statistics

The study of chaos theory emerged after the development of quantum mechanics. Naturally, this raises the question of how chaos can be defined and quantified within the framework of quantum mechanics. However, there are fundamental differences between classical and quantum mechanics that complicate this endeavor. In quantum mechanics,

¹There are many other properties.

the governing equations (Schrödinger equations) are inherently linear, and the concept of trajectories, central to classical chaos, is absent due to the uncertainty principle. As a result, it is not possible to directly translate the definitions and methods of classical chaos into the quantum realm. Instead, alternative approaches are required to identify and characterize chaos in quantum systems. One of the most widely used methods for analyzing quantum chaos is "level spacing statistics", which we will discuss in detail in this section.

Through extensive research on quantum chaos, various methods for characterizing and formulating quantum chaos have been developed. These include level spacing statistics, out-of-time-order correlators (OTOCs), spectral form factors, complexity, and others. In this paper, we focus on level statistics, which is the most traditional and widely studied approach to quantum chaos. The methodology for calculating level spacing statistics, as discussed in this paper, is primarily based on the review [64].

First, we introduce random matrix theory (RMT). This theory was originally developed by Wigner [65–67], Dyson [68], and others to understand the spectra of complex atomic nuclei. In large and complex quantum mechanical systems, determining individual energy levels and eigenstates is often intractable. Instead of focusing on exact solutions, RMT provides insights into the statistical properties of these systems by analyzing random matrices. This approach enables the study of universal features in energy spectra, which are independent of the specific details of the system.

We consider the Hamiltonian of a two-state system. This Hamiltonian is represented by a 2×2 matrix, with its entries being random numbers drawn from a Gaussian distribution:

$$H = \begin{pmatrix} \alpha & \gamma/\sqrt{2} \\ \gamma/\sqrt{2} & \beta \end{pmatrix}, \quad (3.2.1)$$

where α, β, γ are real random variables drawn from a Gaussian distribution with zero mean and variance σ^2 . The probability density function for these variables is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \quad (3.2.2)$$

The physical quantities of interest are determined by taking statistical averages over the

random variables. The eigenvalues of the Hamiltonian (3.2.1) are given by:

$$E_{1,2} = \frac{\alpha + \beta}{2} \pm \frac{1}{2} \sqrt{(\alpha - \beta)^2 + 2\gamma^2}. \quad (3.2.3)$$

We compute the level spacing statistics, defined as the probability distribution $P(E_1 - E_2 = \omega) \equiv P(\omega)$. The calculation proceeds as follows:

$$\begin{aligned} P(\omega) &= \frac{1}{(2\pi)^{3/2} \sigma^3} \int d\alpha d\beta d\gamma \delta\left(\sqrt{(\alpha - \beta)^2 + 2\gamma^2} - \omega\right) \exp\left(-\frac{\alpha^2 + \beta^2 + \gamma^2}{2\sigma^2}\right) \quad (3.2.4) \\ &= \frac{\sqrt{2}}{(2\pi)^{3/2} \sigma^3} \int dx dy d\gamma \delta\left(\sqrt{2x^2 + 2\gamma^2} - \omega\right) \exp\left(-\frac{(y + \frac{1}{\sqrt{2}}x)^2}{\sigma^2}\right) \exp\left(-\frac{x^2 + \gamma^2}{2\sigma^2}\right) \end{aligned} \quad (3.2.5)$$

$$= \frac{1}{(2\pi)\sigma} \int dx d\gamma \delta\left(\sqrt{2(x^2 + \gamma^2)} - \omega\right) \exp\left(-\frac{x^2 + \gamma^2}{2\sigma^2}\right) \quad (3.2.6)$$

$$= \frac{1}{2\pi\sigma^2} \int dr d\theta \delta\left(\sqrt{2}r - \omega\right) r \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad (3.2.7)$$

$$= \frac{\omega}{2\sigma^2} \exp\left(-\frac{\omega^2}{4\sigma^2}\right). \quad (3.2.8)$$

In (3.2.5), we made a change of variables, $\alpha = \sqrt{2}x + y$ and $\beta = y$. In (3.2.6), the integral over y was performed, which is a Gaussian integral. In (3.2.7), we introduced polar coordinates, $x = r \cos \theta$ and $\gamma = r \sin \theta$. Finally, integrating over r and θ gives the result in (3.2.8).

Strictly speaking, the level spacing statistics is often written in the form:

$$P(\omega) = A_\beta \omega^\beta \exp(-B_\beta \omega^2), \quad (\text{Wigner surmise}). \quad (3.2.9)$$

Here, the parameter β is determined by the symmetry of the Hamiltonian. If the system possesses time-reversal symmetry, the Hamiltonian is a real symmetric matrix, and the level spacing statistics is given by (3.2.8), where $\beta = 1$. When normalized such that the average level spacing is 1, the normalized distribution is:

$$P(\omega) = \frac{\pi}{2} \omega \exp\left(-\frac{\pi}{4} \omega^2\right). \quad (3.2.10)$$

This is known as the Wigner-Dyson distribution.

If we consider a large system, the Hamiltonian is represented as an $N \times N$ matrix. The ensemble of such Hamiltonians is described by the following probability distribution:

$$f(H) \sim \exp(-A \text{Tr}(H^2)), \quad (3.2.11)$$

where A is a normalization constant. By integrating over the ensemble of Hamiltonians H , it is possible to derive results about the level-spacing statistics. However, the exact expressions are generally too complex to be written in a simple closed form. Remarkably, despite this complexity, the level-spacing statistics can be well-approximated by the Wigner surmise given in (3.2.10).

RMT has been remarkably successful in calculating the spectral properties of complex nuclei. This naturally raises the question: what types of complex quantum systems can be described by RMT? In 1984, Bohigas, Giannoni, and Schmit investigated the quantum Sinai's billiard [69]. In the classical case, the motion of a point particle in Sinai's billiard is chaotic. Their study revealed that the level-spacing statistics of the quantum Sinai's billiard follows the Wigner-Dyson distribution. Based on this result, it is conjectured that the level-spacing statistics of a quantum system, whose classical counterpart exhibits chaos, can be described by RMT. This conjecture is now known as the Bohigas-Giannoni-Schmit (BGS) conjecture. Although the BGS conjecture has not been rigorously proven, extensive numerical and experimental tests have confirmed its validity in a wide range of systems². As a result, the definition of quantum chaos is often taken to mean that the level-spacing statistics of the system can be described by Wigner-Dyson statistics.

On the other hand, let us consider systems that are not quantum chaotic, particularly integrable systems. In such cases, the behavior of level-spacing statistics is described by the Berry-Tabor conjecture [70]. The Berry-Tabor conjecture states that for any quantum system that is classically integrable, the energy levels are uncorrelated, and their values are determined independently of each other. As a result, the level-spacing statistics follows a Poisson distribution:

$$P(\omega) = \exp(-\omega). \quad (3.2.12)$$

This distribution reflects the lack of correlations between energy levels, a characteristic feature of integrable systems.

Finally, we summarize the characteristics of the Wigner-Dyson distribution (3.2.8) and the Poisson distribution (3.2.12), highlighting the differences between them. These distributions are depicted in Figure 3.3. In the Wigner-Dyson distribution (Fig. 3.3 (a)), the value at $\omega = 0$ vanishes, and the maximum value is attained at $\omega = \sqrt{2}\sigma$. This

²There are, however, a few counterexamples.

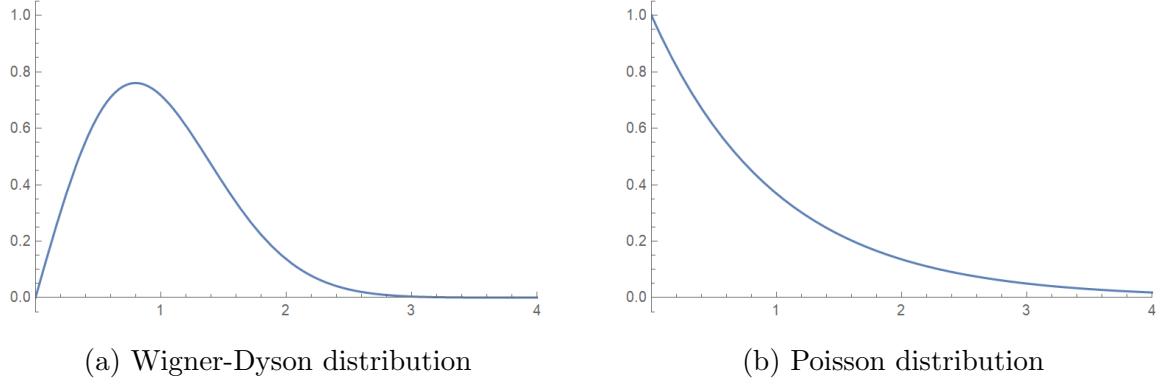


Figure 3.3: Level-spacing statistics $P(\omega)$ based on two conjectures.

indicates that chaotic systems exhibit energy level repulsion, where adjacent energy levels tend to avoid each other. On the other hand, in the Poisson distribution (Fig. 3.3 (b)), $P(\omega)$ reaches its maximum at $\omega = 0$. In integrable systems, the presence of sufficient conserved charges leads to the division of states into symmetry sectors, resulting in many degenerate states. As a result, level repulsion does not occur. By examining the differences between these distributions, we can determine whether a quantum mechanical system is chaotic or integrable³.

3.3 Example of level spacing statistics

In this section, we calculate the Wigner-Dyson distribution using a quantum chaotic system as an example. The spin-chain Hamiltonian, represented as a finite-dimensional matrix, allows for numerical computation without the need for an explicit cutoff⁴.

As a specific example, we consider a random spin-chain Hamiltonian. The Hamiltonian is given by:

$$H_{\text{ran}} = - \sum_{k=1}^L \sum_{a=x,y,z} J_{ka} \sigma_k^a \sigma_{k+1}^a, \quad (3.3.1)$$

where $(\sigma_k^x, \sigma_k^y, \sigma_k^z)$ are the Pauli matrices acting on the spin at the k -th site and as the identity operator on all other sites. The coefficients J_{ka} are uniformly random numbers

³This approach defines quantum chaos in terms of level-spacing statistics, regardless of whether the system has a classical counterpart.

⁴In practice, some additional processing or approximation may still be required.

sampled from the interval $[0, 1]$. Unlike in random matrix theory (RMT), we do not take a statistical average over many realizations of the Hamiltonian. This Hamiltonian is composed of nearest-neighbor interactions, similar to the XXX model, but it is expected to exhibit chaotic behavior⁵.

Before numerically calculating the eigenvalues of (3.3.1), we must de-symmetrize the Hamiltonian. If a quantum system possesses symmetries (conserved charges), its energy eigenvalues are generally degenerate. This leads to an increase in the level-spacing distribution at $\omega = 0$. Physically, states with different charges cannot mix, causing the Hamiltonian to become block-diagonal, with eigenvalues in different symmetry sectors being uncorrelated.

For example, in the case of (3.3.1), the system has parity symmetry, corresponding to the invariance under simultaneous flipping of spins at all sites. This parity transformation is represented as:

$$\mathcal{P} : |\uparrow\uparrow\downarrow\uparrow\downarrow\rangle \rightarrow |\downarrow\downarrow\uparrow\downarrow\uparrow\rangle . \quad (3.3.2)$$

The parity operator $\mathcal{P}$ can be expressed using the Pauli matrices as:

$$\mathcal{P} = \bigotimes_{i=1}^L \sigma_i^x . \quad (3.3.3)$$

From an algebraic perspective, the Hamiltonian satisfies the commutation relations:

$$[H_{\text{ran}}, \sigma^x] = [H_{\text{ran}}, \sigma^y] = [H_{\text{ran}}, \sigma^z] = 0, \quad \sigma^a = \bigotimes_{i=1}^L \sigma_i^a . \quad (3.3.4)$$

This implies that H_{ran} and σ^a can be simultaneously diagonalized. Furthermore, if the number of sites L is even, the operators σ^x , σ^y , and σ^z commute with each other.

To de-symmetrize the Hamiltonian, we define projection operators as:

$$P_{\pm}^a = \frac{1}{2}(\sigma^a \pm 1) . \quad (3.3.5)$$

These projection operators are used to project the Hilbert space into a specific symmetry sector. By applying the projection, we restrict the Hamiltonian to a particular de-symmetrized subspace and then calculate the energy eigenvalues $\{E_i\}$.

⁵There is a small chance that the system could be integrable, depending on the specific random coefficients.

Before calculating the level-spacing statistics, we must "unfold the spectrum." This process removes the overall energy dependence and normalizes the spectrum so that the average level spacing is set to one. Unfolding ensures that the level-spacing statistics reflect only the fluctuations in the energy spectrum, without being influenced by the density of states or other global features of the spectrum. There is no universally fixed method for unfolding the spectrum, but here we describe a commonly used approach based on the cumulative level number:

$$n(E) = \sum_{i=1}^m \Theta(E - E_i), \quad (3.3.6)$$

where m is the total number of states in the sector, and $\Theta(x)$ is the Heaviside step function. We decompose $n(E)$ into an average part and a fluctuation part:

$$n(E) = n_{\text{av}}(E) + n_{\text{fl}}(E). \quad (3.3.7)$$

The average part $n_{\text{av}}(E)$ is determined by fitting the spectrum to a polynomial of sufficiently high order. Using this average part, the unfolded spectrum is defined as:

$$\varepsilon_i = n_{\text{av}}(E_i). \quad (3.3.8)$$

Finally, we compute the level spacings as:

$$s_i = \varepsilon_{i+1} - \varepsilon_i, \quad (3.3.9)$$

and create a histogram of the level spacings with a chosen bin width.

The results for the random spin-chain Hamiltonian (3.3.1) are shown in Figure 3.4. It is evident from the figure that the level-spacing distribution closely follows the Wigner-Dyson distribution (3.2.10).

3.4 Chaotic spin chain in AdS/CFT

In this section, we review the calculation of the XXX spin-chain Hamiltonian with next-to-nearest neighbor (NNN) interactions, based on [53]. For certain parameter values, this calculation was performed using the methods outlined in [71, 72].

The planar $\mathcal{N} = 4$ super Yang-Mills (SYM) theory is widely believed to be integrable at all orders in the coupling constant, with substantial evidence supporting this integrability.

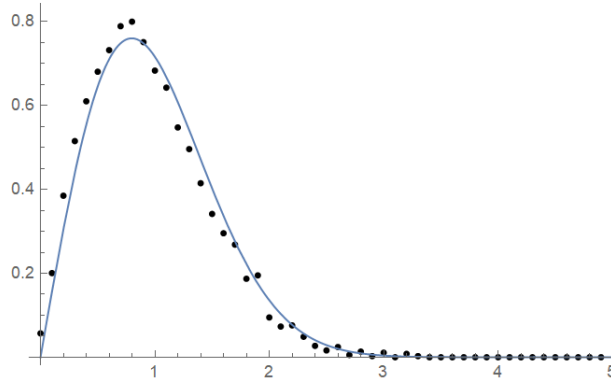


Figure 3.4: Level-spacing statistics of (3.3.1) with $L = 14$. Black dots: numerical result. Solid line: Wigner-Dyson distribution (3.2.10).

However, it has been demonstrated that the dilatation operator, when truncated at the two-loop level, exhibits quantum chaotic behavior. This result challenges the notion of integrability in $\mathcal{N} = 4$ SYM theory when finite-order approximations are considered.

We consider the XXX spin-chain Hamiltonian with next-to-nearest neighbor (NNN) interactions, which corresponds to the one- and two-loop order dilatation operator in $\mathcal{N} = 4$ SYM theory:

$$H = \epsilon_0 \sum_{i=1}^L 1 + \lambda_1 \sum_{i=1}^L \vec{S}_i \cdot \vec{S}_{i+1} + \lambda_2 \sum_{i=1}^L \vec{S}_i \cdot \vec{S}_{i+2}, \quad (3.4.1)$$

where the parameters are given by:

$$\epsilon_0 = 1 - 3g^2, \quad \lambda_1 = -4(1 - 4g^2), \quad \lambda_2 = -4g^2, \quad g^2 = \frac{\lambda}{16\pi^2}. \quad (3.4.2)$$

The unperturbed XXX spin-chain, corresponding to the nearest-neighbor interaction term (λ_1 term), is a well-known integrable system. To determine whether the extended Hamiltonian (3.4.1) with NNN interactions exhibits chaotic behavior, we numerically compute the level-spacing distribution in the following analysis.

The Hamiltonian (3.4.1) possesses several symmetries. Before performing numerical calculations, it is necessary to de-symmetrize the Hamiltonian by restricting it to a specific symmetry sector. The relevant symmetries and corresponding conserved charges are as follows:

1. Total Spin: We fix the total spin $S^z = \sum_{i=1}^L S_i^z$, which is equivalent to fixing the impurity number $M = L/2 - S^z$.

2. Translation Invariance: Since the Hamiltonian is translationally invariant, we also fix the total momentum of the system.

3. Parity Charge: The system is symmetric under the parity transformation (3.3.2). Therefore, we consider only the positive-parity states.

4. Global SU(2) Symmetry: The Hamiltonian has a global SU(2) symmetry. To simplify the computation, we restrict our attention to the lowest-weight states, defined by $S^- |\Psi\rangle = 0$.

By imposing these constraints, the Hamiltonian is effectively reduced to a smaller subspace, allowing for more efficient numerical analysis.

Due to limitations in reproducing the results, only the findings are presented here. For detailed calculations and figures, see [53]. The authors of [53] performed calculations for the sector of positive-parity states with $L = 20$ and $M = 7$. To evaluate the results quantitatively, they introduced a generalized Brody distribution, defined as:

$$P_B(s) = A_B(\alpha, \omega) s^{\alpha\omega} e^{-B_B(\alpha, \omega) s^{1+\omega}}, \quad (3.4.3)$$

$$A_B(\alpha, \omega) = (1 + \omega) \frac{\Gamma\left(\frac{2+\alpha\omega}{1+\omega}\right)^{1+\alpha\omega}}{\Gamma\left(\frac{1+\alpha\omega}{1+\omega}\right)^{2+\alpha\omega}}, \quad (3.4.4)$$

$$B_B(\alpha, \omega) = \frac{\Gamma\left(\frac{2+\alpha\omega}{1+\omega}\right)^{1+\omega}}{\Gamma\left(\frac{1+\alpha\omega}{1+\omega}\right)^{1+\omega}}. \quad (3.4.5)$$

This distribution interpolates between the Wigner-Dyson distribution (3.2.10) at $\omega = 1$ and the Poisson distribution (3.2.12) at $\omega = 0$. For $g^2 = 0$ (the unperturbed XXX spin-chain) and assuming $\alpha = 1$, the authors found that the distribution fits well with $\omega = 0.03$. This result is consistent with the integrable nature of the XXX spin-chain, where the level-spacing statistics follow a Poisson distribution. For $g^2 = 0.2$ and assuming $\alpha = 1$, the distribution fits well with $\omega = 0.88$. Alternatively, assuming $\omega = 1$, the best fit was obtained with $\alpha = 0.82$. When both parameters were allowed to vary simultaneously, the optimal fit was achieved with $\alpha = 1.1$ and $\omega = 0.81$. In all cases, the value of ω is close to unity, indicating a level-spacing distribution consistent with quantum chaotic behavior.

Chapter 4

Chaotic string motion in a near pp-wave limit

In this chapter, we numerically analyze the classical string motion in a near pp-wave limit of $\text{AdS}_5 \times S^5$, based on our work [60].

As discussed in Chapter 2, the dilatation operator acting on scalar single-trace operators in $\mathcal{N} = 4$ SYM theory corresponds to a spin-chain Hamiltonian. In particular, the two-loop $\text{SU}(2)$ sector dilatation operator corresponds to the XXX spin-chain with next-nearest-neighbor interactions. Moreover, as shown in Chapter 3, this spin-chain system exhibits chaotic behavior, as demonstrated by computing the level-spacing statistics.

On the other hand, let us now consider the string theory side. The string sigma model, whose target spacetime is $\text{AdS}_5 \times S^5$, is known to be classically integrable. A natural question arises: what happens to the chaotic behavior observed at the two-loop level on the gauge theory side when viewed from the perspective of string theory?

In this study, we focus on the near pp-wave limit of $\text{AdS}_5 \times S^5$. An intriguing question is whether the world-sheet theory in this limit remains classically integrable. This problem was previously studied in [18], where no evidence of classical chaos was found. However, that analysis was based on a point-particle ansatz and was thus limited in scope. In this chapter, we employ a stringy ansatz to extend the analysis and demonstrate that the motion is chaotic. This is achieved by computing Poincaré sections and the Lyapunov spectrum (Sections 4.2 and 4.3).

Our findings are strikingly analogous to the Hénon-Heiles model, which emerges as a

third-order truncation of a three-particle periodic Toda chain. While the Toda chain is integrable, the Hénon-Heiles model is not, providing a useful analogy to understand the loss of integrability in the near pp-wave limit (Section 4.1).

4.1 Breaking classical integrability by truncation

It is interesting to see a simple example that classical integrability is broken via a finite-order truncation of an integrable system. Here, let us consider a three-particle periodic Toda chain and derive a non-integrable Henon-Heiles model as a finite-order truncation.

4.1.1 Three-particle periodic Toda chain

The Hamiltonian of a three-particle periodic Toda chain is given by

$$H = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + e^{-(Q_2-Q_1)} + e^{-(Q_3-Q_2)} + e^{-(Q_1-Q_3)} - 3. \quad (4.1.1)$$

This system is classically integrable in the sense of Liouville. Hamilton's equations are

$$\frac{dQ_1}{dt} = P_1, \quad \frac{dQ_2}{dt} = P_2, \quad \frac{dQ_3}{dt} = P_3, \quad (4.1.2)$$

$$\frac{dP_1}{dt} = e^{Q_3-Q_1} - e^{Q_1-Q_2}, \quad \frac{dP_2}{dt} = e^{Q_1-Q_2} - e^{Q_2-Q_3}, \quad \frac{dP_3}{dt} = e^{Q_2-Q_3} - e^{Q_3-Q_1}. \quad (4.1.3)$$

The number of degrees of freedom is three and hence three conserved charges (in involution) ensure the classical integrability.

Let us see that the system (4.1.1) has three conserved charges below. The first one is the total momentum $I_1 \equiv P_1 + P_2 + P_3$. By using (4.1.3), it is conserved as

$$\frac{dI_1}{dt} = \frac{d}{dt}(P_1 + P_2 + P_3) = 0. \quad (4.1.4)$$

Next, in order to derive the second one, let us evaluate the time derivative of $P_1 P_2$ as follows:

$$\begin{aligned} \frac{d(P_1 P_2)}{dt} &= P_1 \frac{dP_2}{dt} + P_2 \frac{dP_1}{dt} \\ &= \frac{dQ_1}{dt} (e^{Q_1-Q_2} - e^{Q_2-Q_3}) + \frac{dQ_2}{dt} (e^{Q_3-Q_1} - e^{Q_1-Q_2}) \\ &= \frac{d}{dt} e^{Q_1-Q_2} - \frac{dQ_1}{dt} e^{Q_2-Q_3} + \frac{dQ_2}{dt} e^{Q_3-Q_1}. \end{aligned} \quad (4.1.5)$$

Similar results can also be derived as

$$\frac{d(P_2P_3)}{dt} = \frac{d}{dt}e^{Q_2-Q_3} - \frac{dQ_2}{dt}e^{Q_3-Q_1} + \frac{dQ_3}{dt}e^{Q_1-Q_2}, \quad (4.1.6)$$

$$\frac{d(P_3P_1)}{dt} = \frac{d}{dt}e^{Q_3-Q_1} - \frac{dQ_3}{dt}e^{Q_1-Q_2} + \frac{dQ_1}{dt}e^{Q_2-Q_3}. \quad (4.1.7)$$

Then, by summing up the above three expressions (4.1.5)-(4.1.7), one can see that

$$\frac{d}{dt}(P_1P_2 + P_2P_3 + P_3P_1 - e^{Q_1-Q_2} - e^{Q_2-Q_3} - e^{Q_3-Q_1}) = 0 \quad (4.1.8)$$

and find the second conserved charge

$$I_2 \equiv P_1P_2 + P_2P_3 + P_3P_1 - e^{Q_1-Q_2} - e^{Q_2-Q_3} - e^{Q_3-Q_1}. \quad (4.1.9)$$

The Hamiltonian (4.1.1) can be expressed as $H = I_1^2/2 - I_2 + 3$ and thus I_2 is equivalent to the total energy.

The last is to find out the third one. As explained in Appendix A in detail, after some algebra, the time derivative of $P_1P_2P_3$ can be rewritten as follows:

$$\frac{d}{dt}(P_1P_2P_3) = \frac{d}{dt}(P_1e^{Q_2-Q_3} + P_2e^{Q_3-Q_1} + P_3e^{Q_1-Q_2}). \quad (4.1.10)$$

Thus $I_3 = P_1P_2P_3 - P_1e^{Q_2-Q_3} - P_2e^{Q_3-Q_1} - P_3e^{Q_1-Q_2}$ is the third conserved charge. As one can easily check, I_1 , I_2 and I_3 are in involution. Hence this system is classically integrable in the sense of Liouville.

4.1.2 Derivation of the Henon-Heiles model

Here, let us derive the Henon-Heiles model as a third-order truncation of the Toda chain introduced in the previous subsection.

Assume that Q_i 's are small and expand the exponential potentials in the Hamiltonian (4.1.1) in third order of Q_i . Then the resulting Hamiltonian is given by

$$H = H_0 + H_1, \quad (4.1.11)$$

$$H_0 \equiv \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \frac{1}{2}\{(Q_1 - Q_2)^2 + (Q_2 - Q_3)^2 + (Q_3 - Q_1)^2\}, \quad (4.1.12)$$

$$H_1 \equiv \frac{1}{6}\{(Q_1 - Q_2)^3 + (Q_2 - Q_3)^3 + (Q_3 - Q_1)^3\}, \quad (4.1.13)$$

where H_0 is the part of coupled harmonic oscillators and H_1 is the interaction term.

It is convenient to perform a rotation represented by

$$Q_i = \sum_{j=1}^3 A_{ij} \zeta_j, \quad P_i = \sum_{j=1}^3 A_{ij} \eta_j, \quad (4.1.14)$$

where the constant orthogonal matrix A_{ij} ($A^T A = I$) is defined as

$$A \equiv \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (4.1.15)$$

Then the resulting Hamiltonian is given by

$$H = \frac{1}{2}(\eta_1^2 + \eta_2^2 + \eta_3^2 + 3\zeta_1^2 + 3\zeta_2^2) + \frac{3}{2\sqrt{2}}\left(\zeta_2\zeta_1^2 - \frac{1}{3}\zeta_2^3\right). \quad (4.1.16)$$

Since the total momentum $P_1 + P_2 + P_3 = \sqrt{3}\eta_3$ is conserved, we can drop off η_3 by setting that $\eta_3 = 0$. Then Hamilton's equations are rewritten in terms of ζ_1 , ζ_2 , η_1 and η_2 as follows:

$$\frac{d\zeta_1}{dt} = \eta_1, \quad \frac{d\zeta_2}{dt} = \eta_2, \quad \frac{d\eta_1}{dt} = -3\zeta_1 - \frac{3}{\sqrt{2}}\zeta_1\zeta_2, \quad \frac{d\eta_2}{dt} = -3\zeta_2 - \frac{3}{2\sqrt{2}}(\zeta_1^2 - \zeta_2^2). \quad (4.1.17)$$

After rescaling the variables like

$$q_1 = \frac{1}{2\sqrt{2}}\zeta_1, \quad q_2 = \frac{1}{2\sqrt{2}}\zeta_2, \quad p_1 = \frac{1}{2\sqrt{6}}\eta_1, \quad p_2 = \frac{1}{2\sqrt{6}}\eta_2, \quad \tau = \sqrt{3}t, \quad (4.1.18)$$

Hamilton's equations in (4.1.17) and the resulting Hamiltonian are given by, respectively,

$$\frac{dq_1}{d\tau} = p_1, \quad \frac{dq_2}{d\tau} = p_2, \quad \frac{dp_1}{d\tau} = -q_1 - 2q_1q_2, \quad \frac{dp_2}{d\tau} = -q_2 - q_1^2 + q_2^2, \quad (4.1.19)$$

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + q_2q_1^2 - \frac{1}{3}q_2^3. \quad (4.1.20)$$

The system described by the above H is known as the Henon-Heiles model.

This model is well-known as a non-integrable model. A typical Poincaré section is depicted in Fig. 4.1, where $E = 1/6$, $q_1 = 0$ and $p_1 > 0$. Different colors mean different trajectories corresponding to different initial values. This result indicates that the motion becomes chaotic for almost all of the initial values.

Note here that the total momentum I_1 and the total energy (which is equivalent to I_2) remain to be conserved charges by the invariance of the system under spatial and time translations, while the third conserved charge does not exist. Actually, the expanded form of I_3 is not conserved. For the detail of this point, see Appendix 4.A.

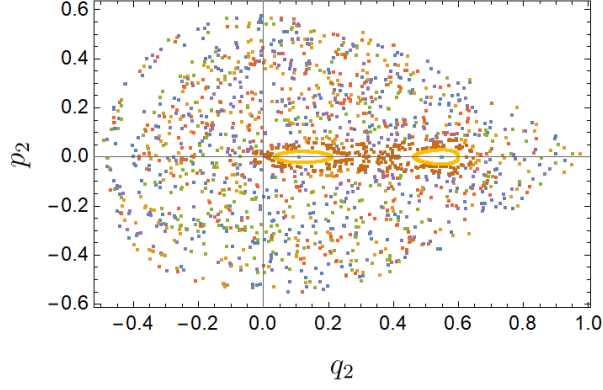


Figure 4.1: Poincaré section of the Henon-Heiles model with $E = 1/6$

4.2 The Hamiltonian of a near pp-wave string

In this section, we present the light-cone Hamiltonian of a bosonic string theory propagating on a near pp-wave limit of the $\text{AdS}_5 \times \text{S}^5$ background by following the pioneering work [59].

4.2.1 A near pp-wave limit of $\text{AdS}_5 \times \text{S}^5$

Let us start from the metric of $\text{AdS}_5 \times \text{S}^5$ with the global coordinates:

$$ds^2 = R^2(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \cos^2 \theta d\phi^2 + d\theta^2 + \sin^2 \theta d\Omega_3'^2). \quad (4.2.1)$$

Here $d\Omega_3^2$ and $d\Omega_3'^2$ are the metrics of S^3 's, and R is the common radius of AdS_5 and S^5 .

Then the metric (4.2.1) can be rewritten as

$$ds^2 = R^2 \left(-\left(\frac{1 + \tilde{z}^2/4}{1 - \tilde{z}^2/4} \right) dt^2 + \left(\frac{1 - \tilde{y}^2/4}{1 + \tilde{y}^2/4} \right) d\phi^2 + \frac{d\tilde{z}^2 + \tilde{z}^2 d\Omega_3^2}{(1 - \tilde{z}^2/4)^2} + \frac{d\tilde{y}^2 + \tilde{y}^2 d\Omega_3'^2}{(1 + \tilde{y}^2/4)^2} \right) \quad (4.2.2)$$

by using the new coordinates $\tilde{z}$ and $\tilde{y}$ defined as

$$\cosh \rho \equiv \frac{1 + \tilde{z}^2/4}{1 - \tilde{z}^2/4}, \quad \cos \theta \equiv \frac{1 - \tilde{y}^2/4}{1 + \tilde{y}^2/4}. \quad (4.2.3)$$

By taking the light-cone coordinates as

$$\tilde{x}^+ = t, \quad \tilde{x}^- = -t + \phi, \quad (4.2.4)$$

and rescaling the coordinates as

$$\tilde{x}^+ \longrightarrow x^+, \quad \tilde{x}^- \longrightarrow \frac{x^-}{R^2}, \quad \tilde{z} \longrightarrow \frac{z}{R}, \quad \tilde{y} \longrightarrow \frac{y}{R}, \quad (4.2.5)$$

one may take the $R \rightarrow \infty$ limit. This is nothing but the Penrose limit [54, 55]. The resulting metric with the subleading correction [59] is given by

$$ds^2 = ds_0^2 + \frac{1}{R^2} ds_2^2 + \mathcal{O}\left(\frac{1}{R^4}\right), \quad (4.2.6)$$

$$ds_0^2 \equiv 2dx^+ dx^- - (z^2 + y^2)(dx^+)^2 + dz^2 + z^2 d\Omega_3^2 + dy^2 + y^2 d\Omega_3'^2, \quad (4.2.7)$$

$$ds_2^2 \equiv -2y^2 dx^+ dx^- + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2}z^2(dz^2 + z^2 d\Omega_3^2) - \frac{1}{2}y^2(dy^2 + y^2 d\Omega_3'^2). \quad (4.2.8)$$

The leading term (4.2.7) is the metric part of the maximally supersymmetric pp-wave background [56, 57].

4.2.2 Light-cone Hamiltonian

Next, we will derive the light-cone Hamiltonian of a string theory on the near pp-wave background (4.2.6). For simplicity, we focus on the bosonic sector. The subscripts μ, ν denote the ten-dimensional indices $+, -, 1, \dots, 8$ and I, J denote only the transverse directions $1, \dots, 8$.

The classical string action is given by

$$S = \int d\tau d\sigma \mathcal{L} = \frac{1}{2} \int d\tau d\sigma \sqrt{-\det(h_{ab})} h^{ab} \partial_a x^\mu \partial_b x^\nu g_{\mu\nu}, \quad (4.2.9)$$

where $g_{\mu\nu}$ is the background metric (4.2.6) and x^μ are the target-space coordinates. The world-sheet metric is given by h_{ab} ($a, b = \tau, \sigma$). The equation of motion for h_{ab} imposes that the stress-energy tensor T_{ab} on the world-sheet should vanish and lead to the Virasoro constraints:

$$T_{ab} = \partial_a x^\mu \partial_b x^\nu g_{\mu\nu} - \frac{1}{2} h_{ab} h^{cd} \partial_c x^\mu \partial_d x^\nu g_{\mu\nu} = 0. \quad (4.2.10)$$

The canonical momenta and their inversion in terms of $\dot{x}^\mu$ are given by

$$p_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\tau x^\mu)} = h^{\tau a} \partial_a x^\nu g_{\mu\nu}, \quad \dot{x}^\mu = \frac{1}{h^{\tau\tau}} g^{\mu\nu} p_\nu - \frac{h^{\tau\sigma}}{h^{\tau\tau}} x'^\mu, \quad (4.2.11)$$

where $\dot{x}^\mu \equiv \partial x^\mu / \partial \tau$ and $x'^\mu \equiv \partial x^\mu / \partial \sigma$. Then the Virasoro constraints (4.2.10) can be rewritten as

$$p_\mu p_\nu g^{\mu\nu} + x'^\mu x'^\nu g_{\mu\nu} = 0, \quad (4.2.12)$$

$$p_\mu x'^\mu = 0. \quad (4.2.13)$$

Here, let us employ the light-cone gauge:

$$x^+ = \tau, \quad p_- = \text{const}. \quad (4.2.14)$$

Then the light-cone Hamiltonian is defined as

$$\mathcal{H}_{\text{lc}} \equiv -p_+. \quad (4.2.15)$$

By solving the equation (4.2.12) in terms of $\mathcal{H}_{\text{lc}}$, the light-cone Hamiltonian is expressed as

$$\mathcal{H}_{\text{lc}} = \frac{p^- g^{+-}}{g^{++}} - \frac{1}{g^{++}} \sqrt{p_-^2 g - g^{++} \left(g_{--} \left(\frac{p_I x'^I}{p_-^2} \right)^2 + p_I p_J g^{IJ} + x'^I x'^J g_{IJ} \right)}, \quad (4.2.16)$$

where

$$g \equiv (g^{+-})^2 - g^{++} g^{--}. \quad (4.2.17)$$

Finally, we write down the light-cone Hamiltonian on the near pp-wave background (4.2.6). Let $(g_0)_{\mu\nu}$ and $(g_2)_{\mu\nu}$ denote the leading term (4.2.7) and the sub-leading term (4.2.8), respectively. Then the background metric is expressed as

$$g_{\mu\nu} = (g_0)_{\mu\nu} + \frac{1}{R^2} (g_2)_{\mu\nu} + \mathcal{O}\left(\frac{1}{R^4}\right), \quad g^{\mu\nu} = (g'_0)^{\mu\nu} + \frac{1}{R^2} (g'_2)^{\mu\nu} + \mathcal{O}\left(\frac{1}{R^4}\right), \quad (4.2.18)$$

where $g'_0 = g_0^{-1}$ and $g'_2 = -g_0^{-1} g_2 g_0^{-1}$. By substituting the expanded metric (4.2.18) for (4.2.16), the light-cone Hamiltonian [59] is evaluated as

$$\mathcal{H}_{\text{lc}} = \mathcal{H}_0 + \frac{1}{R^2} \mathcal{H}_{\text{int}} + \mathcal{O}\left(\frac{1}{R^4}\right), \quad (4.2.19)$$

$$\mathcal{H}_0 = \frac{1}{2} (p_I p_J (g'_0)^{IJ} + x'^I x'^J (g_0)_{IJ} + y^2 + z^2), \quad (4.2.20)$$

$$\begin{aligned} \mathcal{H}_{\text{int}} = & \frac{1}{8} \left((y^2 + z^2)^2 - (p_I p_J (g'_0)^{IJ} + x'^I x'^J (g_0)_{IJ})^2 \right) + \frac{1}{2} (p_I x'^I) \\ & + \frac{1}{4} (z^2 - y^2) (p_I p_J (g'_0)^{IJ} + x'^I x'^J (g_0)_{IJ}) + \frac{1}{2} (p_I p_J (g'_2)^{IJ} + x'^I x'^J (g_2)_{IJ}). \end{aligned} \quad (4.2.21)$$

Here we have set $p_- = 1$. In the following, we will not consider the higher-order terms with $\mathcal{O}(1/R^4)$. The leading term $\mathcal{H}_0$ is a sum of harmonic oscillators and hence this part is obviously integrable. The subleading correction in the Penrose limit gives rise to the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ which contains the fourth-order terms of the canonical variables (x^I, p_I) . Thus the question we want to study below is whether this interacting dynamics is integrable or not.

4.3 Chaotic dynamics

In this section, let us answer the question, whether this interacting dynamics described by the Hamiltonian (4.2.19) is integrable or not. Actually, it was considered in the preceding work [18], where a point-particle ansatz was considered and chaotic dynamics was not found. In the following, we will take another stringy ansatz and show classical chaos directly by computing Poincaré sections and Lyapunov spectrum.

4.3.1 A stringy ansatz

There are two S^3 parts in the metric (4.2.6). We suppose that the string is sitting at a point in S^3 from AdS_5 and hence the terms with $d\Omega_3^2$ are ignored below.

It is convenient to parametrize S^3 from S^5 by using the Hopf coordinates (η, ξ_1, ξ_2) :

$$d\Omega_3'^2 = d\eta^2 + \sin^2 \eta d\xi_1^2 + \cos^2 \eta d\xi_2^2. \quad (4.3.1)$$

Note that p_{ξ_1} and p_{ξ_2} are conserved and hence we will set $p_{\xi_1} = p_{\xi_2} = 0$ below.

In order to study the chaotic dynamics, it is necessary to impose an ansatz so as to reduce the system from a two-dimensional field theory to a mechanical system. In the following, we employ a winding string ansatz :

$$\begin{aligned} z = 0, \quad p_z = 0, \quad y = y(\tau), \quad p_y = p_y(\tau), \\ \xi_1 = a_1 \sigma, \quad \xi_2 = a_2 \sigma \quad \eta = \eta(\tau), \quad p_\eta = p_\eta(\tau). \end{aligned} \quad (4.3.2)$$

Here a_i ($i = 1, 2$) are integers due to the periodicity of ξ_i . Note that this ansatz is consistent with the full equation of motion derived from $\mathcal{H}_{\text{lc}}$ in (4.2.19). This ansatz

leads to the reduced equations of motion

$$\dot{y} = p_y - \frac{1}{2R^2} \left[P_y^3 + \frac{P_y P_\eta^2}{y^2} + y^2 (a_1^2 \sin^2 \eta + a_2^2 \cos^2 \eta) \right], \quad (4.3.3)$$

$$\dot{\eta} = \frac{p_\eta}{y^2} - \frac{1}{2R^2} \left[\frac{p_\eta^3}{y^4} + \frac{p_y^2 p_\eta}{y^2} + (a_1^2 \sin^2 \eta + a_2^2 \cos^2 \eta) \right], \quad (4.3.4)$$

$$\begin{aligned} \dot{p}_y = & \frac{p_\eta^2}{y^3} - y (a_1^2 \sin^2 \eta + a_2^2 \cos^2 \eta + 1) \\ & + \frac{1}{2R^2} \left[-5y^3 - 2yp_y^2 - \frac{p_\eta^4}{y^5} - \frac{p_y^2 p_\eta^2}{y^3} - \frac{p_y^4}{4y} + y^3 \left(a_1^2 \sin^2 \eta + a_2^2 \cos^2 \eta + \frac{p_y^2 + 4y^2}{2y^2} \right)^2 \right], \end{aligned} \quad (4.3.5)$$

$$\begin{aligned} \dot{p}_\eta = & \frac{1}{2} (a_2^2 - a_1^2) y^2 \sin 2\eta \\ & - \frac{1}{8R^2} (a_2^2 - a_1^2) \sin 2\eta \left[(a_2^2 - a_1^2) y^4 \cos 2\eta + (a_1^2 + a_2^2 + 4) y^4 + 2p_\eta^2 + 2y^2 p_y^2 \right]. \end{aligned} \quad (4.3.6)$$

These can also be reproduced from the reduced light-cone Hamiltonian :

$$\mathcal{H}_{\text{lc}} = \mathcal{H}_0 + \frac{1}{R^2} \mathcal{H}_{\text{int}}, \quad (4.3.7)$$

$$\mathcal{H}_0 = \frac{1}{2} \left(p_y^2 + \frac{p_\eta^2}{y^2} + y^2 + y^2 (a_1^2 \sin^2 \eta + a_2^2 \cos^2 \eta) \right), \quad (4.3.8)$$

$$\begin{aligned} \mathcal{H}_{\text{int}} = & y^2 p_y^2 + p_\eta^2 + y^4 (1 - a_1^2 \sin^2 \eta - a_2^2 \cos^2 \eta) \\ & - \frac{(y^2 p_y^2 + p_\eta^2 + y^4 (1 + a_1^2 \sin^2 \eta + a_2^2 \cos^2 \eta))^2}{2y^4}. \end{aligned} \quad (4.3.9)$$

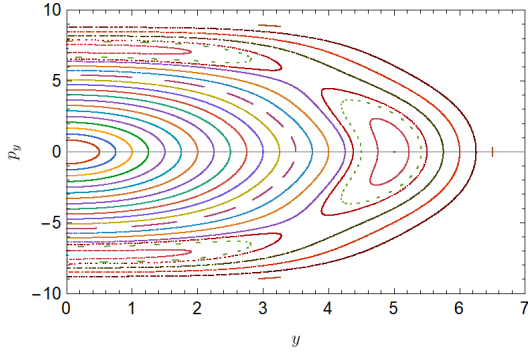
When $a_1^2 = a_2^2$, the reduced system is independent of η . In the following analysis, we set $a_1 = 1$ and $a_2 = 2$ so as to keep the contribution of η . Moreover, the AdS radius R is set to $R = 5.0$ for simplicity.

4.3.2 Poincaré sections and Lyapunov spectrum

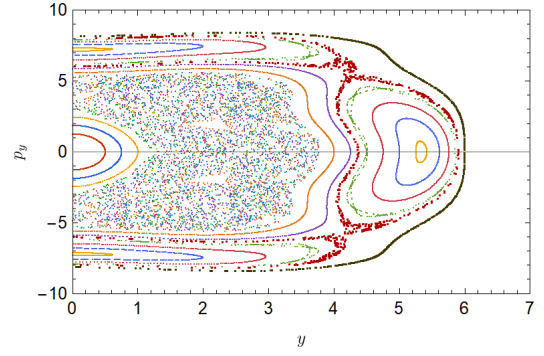
We compute Poincaré sections and Lyapunov spectrum for the set of equations of motion (4.3.3)–(4.3.6). Poincaré sections are taken at $\eta = \pi/2$ with $p_\eta > 0$ and plotted for $E = 5.0, 11.0, 13.0$ as depicted in Figs. 4.2(a)–(c), respectively. Different colors indicate different initial values satisfying the energy constraint. When $E = 5.0$, there are no chaotic motions and only quasi-periodic orbits exist [Fig. 4.2(a)]. These periodic orbits are called the Kolmogorov-Arnold-Moser (KAM) tori [73–75]. When $E = 11.0$, some KAM tori have collapsed and classical chaos appears [Fig. 4.2(b)]. When $E = 13.0$, almost all orbits become chaotic [Fig. 4.2(c)].

Figure 4.2(d) is a Lyapunov spectrum for $E = 13.0$. The initial values are taken as $y = 2.0$, $\eta = \pi/2$ and $p_y = 1.0$. Then the maximum Lyapunov exponent is not zero definitely and indicates classical chaos.

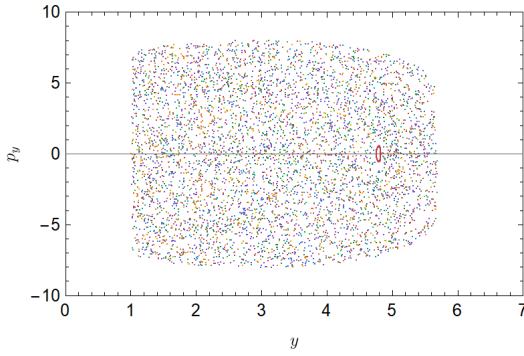
Finally, it should be remarked that the finite-size effect of a string has played a significant role in generating classical chaos. The stringy ansatz (4.3.2) reflects the finite size of a string while it has not been taken into account in the ansatz employed in [18]. This observation is also consistent with the result of [53] in the Landau-Lifshitz limit [76,77]. It is because the spatial direction of the string world-sheet is taken to be infinite in this limit and the finite size effect of a string is dropped off. Hence classical chaos does not appear in this limit as discussed in [53]. On the other hand, the weak chaos in the next-nearest-neighbor interacting spin chain has been discussed before taking the thermodynamic limit and the finite-size effect has been included in this analysis.



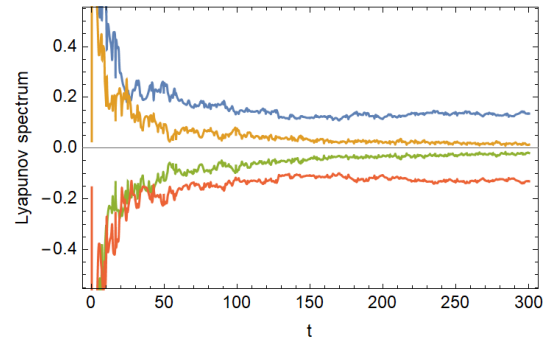
(a) Poincaré section for $E = 5.0$



(b) Poincaré section for $E = 11.0$



(c) Poincaré section for $E = 13.0$



(d) Lyapunov spectrum for $E = 13.0$

Figure 4.2: Poincaré sections and Lyapunov spectrum

Appendix 4.A The third conserved charge I_3

Let us present here the third conserved charge in a periodic three-particle Toda chain.

In order to figure out the speciality of the exponential potential in the Toda chain, it is instructive to start from more general setup as follows:

$$H = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + V_1(Q_3 - Q_2) + V_2(Q_1 - Q_3) + V_3(Q_2 - Q_1). \quad (4.A.1)$$

Here the potential V_i ($i = 1, 2, 3$) are arbitrary real scalar potentials depending only on the difference of dynamical variables Q_i 's by supposing the translational invariance. A special case where

$$V_1 = e^{-(Q_3 - Q_2)} - 1, \quad V_2 = e^{-(Q_1 - Q_3)} - 1, \quad V_3 = e^{-(Q_2 - Q_1)} - 1 \quad (4.A.2)$$

corresponds to the Toda chain (4.1.1).

Then, Hamilton's equations are

$$\frac{dQ_1}{dt} = P_1, \quad \frac{dQ_2}{dt} = P_2, \quad \frac{dQ_3}{dt} = P_3, \quad (4.A.3)$$

$$\frac{dP_1}{dt} = -\left(\frac{\partial V_2}{\partial Q_1} + \frac{\partial V_3}{\partial Q_1}\right), \quad \frac{dP_2}{dt} = -\left(\frac{\partial V_1}{\partial Q_2} + \frac{\partial V_3}{\partial Q_2}\right), \quad \frac{dP_3}{dt} = -\left(\frac{\partial V_1}{\partial Q_3} + \frac{\partial V_2}{\partial Q_3}\right). \quad (4.A.4)$$

The total momentum and energy are conserved charge.

It is the turn to consider the third conserved charge. Let us introduce the following quantity I_3 :

$$I_3 \equiv P_1 P_2 P_3 - P_1 V_1 - P_2 V_2 - P_3 V_3. \quad (4.A.5)$$

Then it is useful to evaluate the time derivative of $P_1 P_2 P_3$ as

$$\begin{aligned} \frac{d}{dt}(P_1 P_2 P_3) &= +P_2 P_3 \frac{dP_1}{dt} + P_1 P_3 \frac{dP_2}{dt} + P_1 P_2 \frac{dP_3}{dt} \\ &= -P_2 P_3 \left(\frac{\partial V_2}{\partial Q_1} + \frac{\partial V_3}{\partial Q_1}\right) - P_1 P_3 \left(\frac{\partial V_1}{\partial Q_2} + \frac{\partial V_3}{\partial Q_2}\right) - P_1 P_2 \left(\frac{\partial V_1}{\partial Q_3} + \frac{\partial V_2}{\partial Q_3}\right) \\ &= P_2 P_3 \left(\frac{\partial V_2}{\partial Q_3} + \frac{\partial V_3}{\partial Q_2}\right) + P_1 P_3 \left(\frac{\partial V_1}{\partial Q_3} + \frac{\partial V_3}{\partial Q_1}\right) + P_1 P_2 \left(\frac{\partial V_1}{\partial Q_2} + \frac{\partial V_2}{\partial Q_1}\right) \\ &= P_1 \frac{dV_1}{dt} + P_2 \frac{dV_2}{dt} + P_3 \frac{dV_3}{dt} \\ &= \frac{d}{dt}(P_1 V_1 + P_2 V_2 + P_3 V_3) - \left(\frac{dP_1}{dt} V_1 + \frac{dP_2}{dt} V_2 + \frac{dP_3}{dt} V_3\right). \end{aligned} \quad (4.A.6)$$

To make I_3 a conserved charge, the remaining part must vanish like

$$\frac{dP_1}{dt}V_1 + \frac{dP_2}{dt}V_2 + \frac{dP_3}{dt}V_3 = 0. \quad (4.A.7)$$

By using Hamilton's equations in (4.A.4), it can be rewritten as

$$\left(\frac{\partial V_2}{\partial Q_1} + \frac{\partial V_3}{\partial Q_1}\right)V_1 + \left(\frac{\partial V_1}{\partial Q_2} + \frac{\partial V_3}{\partial Q_2}\right)V_2 + \left(\frac{\partial V_1}{\partial Q_3} + \frac{\partial V_2}{\partial Q_3}\right)V_3 = 0. \quad (4.A.8)$$

Since the exponential potential (4.A.2) in the Toda chain satisfies (4.A.8), I_3 becomes the third conserved charge. However, when the exponential potential is truncated to a finite order, the relation (4.A.8) is not satisfied any more.

Chapter 5

Conculusion and discussion

In this paper, we have reconsidered string motion in a near pp-wave limit of $\text{AdS}_5 \times \text{S}^5$ and shown the presence of classical chaos under a stringy ansatz by computing Poincaré sections and Lyapunov spectrum. As a result, the string theory in this limit is not classically integrable, while it is integrable both on the original $\text{AdS}_5 \times \text{S}^5$ and the exact pp-wave background. This result is quite analogous to the Henon-Heiles model a a truncation of a periodic three particle Toda chain.

Here we have studied classical chaos appearing in the middle process of the Penrose limit. What is the significance of this classical chaos? The near pp-wave geometry is just an approximation of the full background. No classical chaos appears in the string theory on the full $\text{AdS}_5 \times \text{S}^5$ background as in the Toda chain. In this sense, this chaos may be seen as a perturbative artifact.

However, it might be possible to consider the gauge-theory counterpart by expanding the dilatation operator perturbatively as discussed in [53]. It would be interesting to check the AdS/CFT correspondence for the intermediate chaotic strings. The near pp-wave geometry is still a solution to type IIB supergravity [78], on which the world-sheet beta function vanishes. It is also nice to try to reveal the relation between a near pp-wave string state and a truncated dilatation operator in the $\mathcal{N} = 4$ SYM theory at quantum level.

Our study provides new insights into the interplay between integrability and chaos in string theory and its gauge theory dual, enriching the understanding of the AdS/CFT correspondence. We hope this work encourages further exploration of chaotic dynamics

in holographic contexts.

Acknowledgment

I would like to take this opportunity to thank everyone who supported me in completing this thesis. First, I am very grateful to Shigeki Sugimoto, my Ph.D. supervisor, for their constant guidance and encouragement throughout my research. I also want to thank Kentaroh Yoshida, my collaborator and mentor, for their helpful advice and support during this work. I would like to thank Takaaki Ishii, my collaborator, for working with me and for the valuable discussions that helped me a lot in my research. I am also thankful to the people I met at Kyoto University during my Ph.D. course. My lab members, professors, and friends provided a supportive and inspiring environment. I would further like to express my gratitude to "ChatGPT", which assisted me with proofreading and improving my English writing throughout this thesis. Finally, I would like to thank my family and friends for always being there for me and supporting me.

Appendix A

Calculation of Lyapunov spectrum

We consider the differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)). \quad (\text{A.0.1})$$

as classical equation of motions. We consider two trajectories (A,B). Let $\mathbf{x}(t)$ be the position of A at time t , and $\delta\mathbf{x}(t)$ be the difference between A and B, then the position of B is $\mathbf{x}(t) + \delta\mathbf{x}(t)$. The differential equation A.0.1 for B yields the variational equation given by

$$\dot{\mathbf{x}}(t) + \delta\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t) + \delta\mathbf{x}(t)) \sim \mathbf{f}(\mathbf{x}(t)) + D\mathbf{f}(\mathbf{x}(t))\delta\mathbf{x}(t), \quad (\text{A.0.2})$$

$$\delta\dot{\mathbf{x}}(t) = D\mathbf{f}(\mathbf{x}(t)) \cdot \delta\mathbf{x}(t), \quad (\text{A.0.3})$$

where $D\mathbf{f}$ is Jacobian matrix. We solve this equation and get

$$\delta\mathbf{x}(t) = U_{\mathbf{x}(0)}^t \delta\mathbf{x}(0), \quad (\text{A.0.4})$$

where $U_{\mathbf{x}(t)}^t$ satisfies

$$U_{\mathbf{x}(0)}^{t+s} = U_{\mathbf{x}(s)}^t U_{\mathbf{x}(0)}^s. \quad (\text{A.0.5})$$

To calculate the Lyapunov exponents in each direction, the variations $\delta\mathbf{x}(0)$ are expanded in orthonormal basis, and the change in magnitude of each basis vector with time evolution is defined as Lyapunov:

$$\lambda_i := \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|U_{\mathbf{x}(0)}^t \mathbf{e}_i^0\|}{\|\mathbf{e}_i^0\|}. \quad (\text{A.0.6})$$

We obtain a set of Lyapunov exponents $(\lambda_1, \lambda_2, \dots, \lambda_N)$, which we call the Lyapunov spectrum. The existence of this limit is shown by [79].

In practice, the following algorithm is used for numerical calculations of Lyapunov exponents. (We use Mathematica package [80].)

1. Preparing orthonormal basis $\{\mathbf{e}_i^0\}$.
2. With a time evolution of Δt , each basis vector transforms as $\{\mathbf{v}_i = U^t \mathbf{e}_i^0\}$.
3. Orthonormalization (Gram–Schmidt orthonormalization).

$$\begin{aligned}
\mathbf{u}_1^1 &= \mathbf{v}_1, \\
\mathbf{u}_2^1 &= \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1, \\
\mathbf{u}_3^1 &= \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2, \\
&\dots, \\
\mathbf{e}_i^1 &= \mathbf{u}_i / \|\mathbf{u}_i\|.
\end{aligned}$$

4. Repeat (1-3) to obtain $\{\mathbf{e}_i^1, \mathbf{e}_i^2, \dots\}$.
5. Define $\alpha_i^k = \|\mathbf{u}_i^k\|$, Lyapunov spectrum is given by

$$\lambda_k = \lim_{n \rightarrow \infty} \frac{1}{n \Delta t} \sum_{i=1}^n \alpha_k^i. \quad (\text{A.0.7})$$

Numerically, at finite n , we draw a graph with $n \delta t$ on the horizontal axis and λ_k on the vertical axis, and finally check for convergence to a constant value, which is the Lyapunov spectrum.

Bibliography

- [1] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.
- [2] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, 428:105–114, 1998.
- [3] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.
- [4] Niklas Beisert et al. Review of AdS/CFT Integrability: An Overview. *Lett. Math. Phys.*, 99:3–32, 2012.
- [5] Gautam Mandal, Nemani V. Suryanarayana, and Spenta R. Wadia. Aspects of semiclassical strings in AdS(5). *Phys. Lett. B*, 543:81–88, 2002.
- [6] Iosif Bena, Joseph Polchinski, and Radu Roiban. Hidden Symmetries of the $\text{AdS}_5 \times S^5$ Superstring. *Phys. Rev. D*, 69:046002, 2004.
- [7] J. A. Minahan and K. Zarembo. The Bethe ansatz for N=4 superYang-Mills. *JHEP*, 03:013, 2003.
- [8] David Eliecer Berenstein, Juan Martin Maldacena, and Horatiu Stefan Nastase. Strings in flat space and pp waves from N=4 superYang-Mills. *JHEP*, 04:013, 2002.
- [9] Leopoldo A. Pando Zayas and Cesar A. Terrero-Escalante. Chaos in the Gauge / Gravity Correspondence. *JHEP*, 09:094, 2010.
- [10] Pallab Basu, Diptarka Das, and Archisman Ghosh. Integrability Lost. *Phys. Lett. B*, 699:388–393, 2011.

- [11] Pallab Basu and Leopoldo A. Pando Zayas. Chaos rules out integrability of strings on $\text{AdS}_5 \times T^{1,1}$. *Phys. Lett. B*, 700:243–248, 2011.
- [12] Pallab Basu and Leopoldo A. Pando Zayas. Analytic Non-integrability in String Theory. *Phys. Rev. D*, 84:046006, 2011.
- [13] A. Stepanchuk and A. A. Tseytlin. On (non)integrability of classical strings in p-brane backgrounds. *J. Phys. A*, 46:125401, 2013.
- [14] Yuri Chervonyi and Oleg Lunin. (Non)-Integrability of Geodesics in D-brane Backgrounds. *JHEP*, 02:061, 2014.
- [15] Yuhma Asano, Hideki Kyono, and Kentaroh Yoshida. Melnikov’s method in String Theory. *JHEP*, 09:103, 2016.
- [16] Pallab Basu, Diptarka Das, Archisman Ghosh, and Leopoldo A. Pando Zayas. Chaos around Holographic Regge Trajectories. *JHEP*, 05:077, 2012.
- [17] Dimitrios Giataganas and Konstadinos Sfetsos. Non-integrability in non-relativistic theories. *JHEP*, 06:018, 2014.
- [18] Yuhma Asano, Daisuke Kawai, Hideki Kyono, and Kentaroh Yoshida. Chaotic strings in a near Penrose limit of $\text{AdS}_5 \times T^{1,1}$. *JHEP*, 08:060, 2015.
- [19] Koji Hashimoto, Keiju Murata, and Kentaroh Yoshida. Chaos in chiral condensates in gauge theories. *Phys. Rev. Lett.*, 117(23):231602, 2016.
- [20] Pallab Basu, Pankaj Chaturvedi, and Prasant Samantray. Chaotic dynamics of strings in charged black hole backgrounds. *Phys. Rev. D*, 95(6):066014, 2017.
- [21] Takaaki Ishii, Keiju Murata, and Kentaroh Yoshida. Fate of chaotic strings in a confining geometry. *Phys. Rev. D*, 95(6):066019, 2017.
- [22] Dimitrios Giataganas and Konstantinos Zoubos. Non-integrability and Chaos with Unquenched Flavor. *JHEP*, 10:042, 2017.
- [23] Dibakar Roychowdhury. Analytic integrability for strings on η and λ deformed backgrounds. *JHEP*, 10:056, 2017.

- [24] Jan de Boer, Eva Llabrés, Juan F. Pedraza, and David Vegh. Chaotic strings in AdS/CFT. *Phys. Rev. Lett.*, 120(20):201604, 2018.
- [25] Carlos Núñez, José Manuel Penín, Dibakar Roychowdhury, and Jeroen Van Gorsel. The non-Integrability of Strings in Massive Type IIA and their Holographic duals. *JHEP*, 06:078, 2018.
- [26] Kostas Filippas. Non-integrability on AdS₃ supergravity backgrounds. *JHEP*, 02:027, 2020.
- [27] Konstantinos S. Rigatos. Non-integrability in AdS₃ vacua. *JHEP*, 02:032, 2021.
- [28] Koji Hashimoto, Keiju Murata, and Norihiro Tanahashi. Chaos of Wilson Loop from String Motion near Black Hole Horizon. *Phys. Rev. D*, 98(8):086007, 2018.
- [29] Carlos Nunez, Dibakar Roychowdhury, and Daniel C. Thompson. Integrability and non-integrability in $\mathcal{N} = 2$ SCFTs and their holographic backgrounds. *JHEP*, 07:044, 2018.
- [30] Tetsuya Akutagawa, Koji Hashimoto, Keiju Murata, and Toshihiro Ota. Chaos of QCD string from holography. *Phys. Rev. D*, 100(4):046009, 2019.
- [31] Da-Zhu Ma, Dan Zhang, Guoyang Fu, and Jian-Pin Wu. Chaotic dynamics of string around charged black brane with hyperscaling violation. *JHEP*, 01:103, 2020.
- [32] Kostas Filippas. Nonintegrability of the Ω deformation. *Phys. Rev. D*, 101(4):046025, 2020.
- [33] Jitendra Pal, Arnab Mukherjee, Arindam Lala, and Dibakar Roychowdhury. Analytic (non)integrability of Arutyunov-Bassi-Lacroix model. *Phys. Lett. B*, 820:136496, 2021.
- [34] Dibakar Roychowdhury. Non-integrability for $\mathcal{N} = 1$ SCFTs in 5d. *Phys. Rev. D*, 104(8):086010, 2021.
- [35] Koji Hashimoto and Norihiro Tanahashi. Universality in Chaos of Particle Motion near Black Hole Horizon. *Phys. Rev. D*, 95(2):024007, 2017.

- [36] Mihailo Čubrović. The bound on chaos for closed strings in Anti-de Sitter black hole backgrounds. *JHEP*, 12:150, 2019.
- [37] Dimitrios Giataganas. Chaotic Motion near Black Hole and Cosmological Horizons. *Fortsch. Phys.*, 70(1):2200001, 2022.
- [38] Dimitrios Giataganas, Leopoldo A. Pando Zayas, and Konstantinos Zoubos. On Marginal Deformations and Non-Integrability. *JHEP*, 01:129, 2014.
- [39] Takaaki Ishii, Shodai Kushiro, and Kentaroh Yoshida. Chaotic string dynamics in deformed $T^{1,1}$. *JHEP*, 05:158, 2021.
- [40] Patrick Hayden and John Preskill. Black holes as mirrors: Quantum information in random subsystems. *JHEP*, 09:120, 2007.
- [41] Yasuhiro Sekino and Leonard Susskind. Fast Scramblers. *JHEP*, 10:065, 2008.
- [42] Nima Lashkari, Douglas Stanford, Matthew Hastings, Tobias Osborne, and Patrick Hayden. Towards the Fast Scrambling Conjecture. *JHEP*, 04:022, 2013.
- [43] Stephen H. Shenker and Douglas Stanford. Black holes and the butterfly effect. *JHEP*, 03:067, 2014.
- [44] Stephen H. Shenker and Douglas Stanford. Multiple Shocks. *JHEP*, 12:046, 2014.
- [45] Stephen H. Shenker and Douglas Stanford. Stringy effects in scrambling. *JHEP*, 05:132, 2015.
- [46] Juan Maldacena, Stephen H. Shenker, and Douglas Stanford. A bound on chaos. *JHEP*, 08:106, 2016.
- [47] Tristan McLoughlin, Raul Pereira, and Anne Spiering. One-loop non-planar anomalous dimensions in super Yang-Mills theory. *JHEP*, 10:124, 2020.
- [48] Tristan McLoughlin, Raul Pereira, and Anne Spiering. Quantum chaos in perturbative super-Yang-Mills Theory. *SciPost Phys.*, 14(3):049, 2023.
- [49] D. Szász-Schagrín, B. Pozsgay, and G. Takács. Weak integrability breaking and level spacing distribution. *SciPost Phys.*, 11:037, 2021.

- [50] Didina Serban and Matthias Staudacher. Planar $N=4$ gauge theory and the Inozemtsev long range spin chain. *JHEP*, 06:001, 2004.
- [51] N. Beisert, V. Dippel, and M. Staudacher. A Novel long range spin chain and planar $N=4$ super Yang-Mills. *JHEP*, 07:075, 2004.
- [52] Niklas Beisert and Matthias Staudacher. Long-range $psu(2,2|4)$ Bethe Ansatz for gauge theory and strings. *Nucl. Phys. B*, 727:1–62, 2005.
- [53] Tristan McLoughlin and Anne Spiering. Chaotic spin chains in AdS/CFT. *JHEP*, 09:240, 2022.
- [54] Roger Penrose. *Any Space-Time has a Plane Wave as a Limit*, pages 271–275. Springer Netherlands, Dordrecht, 1976.
- [55] Rahmi Gueven. Plane wave limits and T duality. *Phys. Lett. B*, 482:255–263, 2000.
- [56] Matthias Blau, Jose M. Figueroa-O’Farrill, Christopher Hull, and George Papadopoulos. A New maximally supersymmetric background of IIB superstring theory. *JHEP*, 01:047, 2002.
- [57] Matthias Blau, Jose M. Figueroa-O’Farrill, Christopher Hull, and George Papadopoulos. Penrose limits and maximal supersymmetry. *Class. Quant. Grav.*, 19:L87–L95, 2002.
- [58] R. R. Metsaev. Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background. *Nucl. Phys. B*, 625:70–96, 2002.
- [59] Curtis G. Callan, Jr., Hok Kong Lee, Tristan McLoughlin, John H. Schwarz, Ian Swanson, and Xinkai Wu. Quantizing string theory in $AdS(5) \times S^5$: Beyond the pp wave. *Nucl. Phys. B*, 673:3–40, 2003.
- [60] Shodai Koshiro and Kentaroh Yoshida. Chaotic string motion in a near pp-wave limit. *JHEP*, 01:065, 2023.
- [61] N. Beisert, C. Kristjansen, and M. Staudacher. The Dilatation operator of conformal $N=4$ superYang-Mills theory. *Nucl. Phys. B*, 664:131–184, 2003.

- [62] C. Sieg. Review of AdS/CFT Integrability, Chapter I.2: The spectrum from perturbative gauge theory. *Lett. Math. Phys.*, 99:59–84, 2012.
- [63] Adam Rej. Review of AdS/CFT Integrability, Chapter I.3: Long-range spin chains. *Lett. Math. Phys.*, 99:85–102, 2012.
- [64] Luca D’Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol. From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Adv. Phys.*, 65(3):239–362, 2016.
- [65] Eugene P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics*, 62(3):548–564, 1955.
- [66] Eugene P. Wigner. Characteristics vectors of bordered matrices with infinite dimensions ii. *Annals of Mathematics*, 65(2):203–207, 1957.
- [67] Eugene P. Wigner. On the distribution of the roots of certain symmetric matrices. *Annals of Mathematics*, 67(2):325–327, 1958.
- [68] Freeman J. Dyson. Statistical Theory of the Energy Levels of Complex Systems. III. *Journal of Mathematical Physics*, 3(1):166–175, January 1962.
- [69] O. Bohigas, M. J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Phys. Rev. Lett.*, 52:1–4, Jan 1984.
- [70] Berry Michael Victor and Tabor M. Level clustering in the regular spectrum. *Proc. R. Soc. Lond. A*, 356:375–394, September 1977.
- [71] Theodore C. Hsu and J. C. Angle’s d’Auriac. Level repulsion in integrable and almost-integrable quantum spin models. *Phys. Rev. B*, 47:14291–14296, Jun 1993.
- [72] D. Poilblanc, T. Ziman, J. Bellissard, F. Mila, and G. Montambaux. Poisson vs. goe statistics in integrable and non-integrable quantum hamiltonians. *Europhysics Letters*, 22(7):537, jun 1993.
- [73] Academician A. N. Kolmogorov. On conservation of conditionally periodic motions for a small change in hamilton’s function. *Proceedings of the USSR Academy of Sciences*, 98:527–530, 1954.

- [74] Vladimir I Arnol'd. Small denominators and problems of stability of motion in classical and celestial mechanics. *Russian Mathematical Surveys*, 18(6):85, dec 1963.
- [75] J. Moser. *On invariant Curves of area-preserving mappings of an annulus*. Nachrichten der Akademie der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse. Vandenhoeck & Ruprecht, 1962.
- [76] Martin Kruczenski. Spin chains and string theory. *Phys. Rev. Lett.*, 93:161602, 2004.
- [77] B. Stefanski, Jr. and Arkady A. Tseytlin. Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations. *JHEP*, 05:042, 2004.
- [78] Yastoshi Takayama and Kentaroh Yoshida. Bubbling 1/2 BPS geometries and Penrose limits. *Phys. Rev. D*, 72:066014, 2005.
- [79] V. I. Oseledets. A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trans. Mosc. Math. Soc.*, 19:197–231, 1968.
- [80] Marco Sandri. Numerical Calculation of Lyapunov Exponents. *The Mathematica Journal*, 6:78, 1996.