

ONE-STEP EXTENSIONS OF SUBNORMAL 2-VARIABLE WEIGHTED SHIFTS

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(BASED ON JOINT WORK WITH R. CURTO AND J. YOON.)

1. INTRODUCTION

Consider the following reconstruction-of-the-measure problem:

Problem 1.1 (A). *Given two probability measures μ_1 and μ_2 on $\mathbb{R}_+^2$, find necessary and sufficeint conditions for the existence of a probability measure μ on $\mathbb{R}_+^2$ such that*

$$(1.1) \quad \frac{s \, d\mu(s, t)}{\int s \, d\mu(s, t)} = d\mu_1(s, t) \text{ and } \frac{t \, d\mu(s, t)}{\int t \, d\mu(s, t)} = d\mu_2(s, t).$$

Note that (1.1) implies that $td\mu_1(s, t) = \lambda sd\mu_2(s, t)$ for some $\lambda > 0$.

In this talk, we solve this interpolation problem using techniques from multivariable operator theory, namely the theory of 2-variable weighted shifts.

Definition 1.2. $T \in \mathcal{B}(\mathcal{H})$: normal if $T^*T = TT^*$,

subnormal if $T = N|_{\mathcal{H}}$, where N normal and $N(\mathcal{H}) \subseteq \mathcal{H}$,

hyponormal if $[T^*, T] := T^*T - TT^* \geq 0$.

Definition 1.3. $\mathbf{T} \equiv (T_1, \dots, T_n)$: hyponormal if

$$\begin{aligned} [\mathbf{T}^*, \mathbf{T}] &:= ([T_j^*, T_i])_{i,j=1}^k \\ &= \begin{pmatrix} [T_1^*, T_1] & [T_1^*, T_2] & \cdots & [T_1^*, T_n] \\ [T_2^*, T_1] & [T_2^*, T_2] & \cdots & [T_2^*, T_n] \\ \vdots & \vdots & \ddots & \vdots \\ [T_n^*, T_1] & [T_n^*, T_2] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0. \end{aligned}$$

Definition 1.4. The n -tuple $\mathbf{T} \equiv (T_1, T_2, \dots, T_n)$ is said to be normal if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal n -tuple to a common invariant subspace.

- Clearly, normal $\implies$ subnormal $\implies$ hyponormal.
- Normality(sub-, hypo-) of $\mathbf{T}$ is not affected by permuting of the operators T_i .
- If $(T_1, \dots, T_n)$ is normal(sub-, hypo-) then so is $(k_1 T_1, \dots, k_n T_n)$ for any $k_1, \dots, k_n \in \mathbb{C}$.
- If $(T_1, \dots, T_n)$ is normal(sub-, hypo-) then any operator in $LS\{T_1, \dots, T_n\}$ is normal(sub-, hypo-).

Problem 1.5 (Lifting Problem for Commuting Subnormals). *Find necessary and sufficient conditions for a pair of subnormal operators on a Hilbert space to admit commuting normal extensions i.e., to be subnormal.*

Necessary Conditions: Commuting

Sufficient Conditions: Doubly commuting, either T_1 or T_2 is normal, either T_1 or T_2 is isometry,...

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Besides their relevance for the construction of examples and counterexamples in Hilbert space operator theory, weighted shifts can also be used to detect properties such as subnormality, via the Lambert-Lubin Criterion([15, 17]):

Theorem 1.6 ([15]). *If $T \in \mathcal{B}(\mathcal{H})$ is one-one, then T is subnormal if and only if T_x is subnormal for all $x(\neq 0) \in \mathcal{H}$ where T_x is the weighted shift with weights $\{\frac{\|T^{n+1}x\|}{\|T^n x\|}\}_{n=0}^\infty$.*

Theorem 1.7 ([17]). *If $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are commuting and one-one, then $\mathbf{T} \equiv (T_1, T_2)$ is subnormal if and only if $\mathbf{T}_x$ is subnormal for all $x(\neq 0) \in \mathcal{H}$ where $\mathbf{T}_x$ is the 2-variable weighted shift with weights*

$$\alpha_{m,n} := \frac{\|T_1^{m+1}T_2^n x\|}{\|T_1^m T_2^n x\|} \text{ and } \beta_{m,n} := \frac{\|T_1^m T_2^{n+1} x\|}{\|T_1^m T_2^n x\|}.$$

Thus, to study the subnormality of commuting pairs, we focus on weighted shifts in the sequel.

Example 1.8 (1-variable weighted shift). For a bounded sequence $a \equiv \{a_n\}_{n=0}^\infty$ of positive real numbers (called *weights*), let $W_a : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_a e_n := a_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$.

For a weighted shift W_a , the moments of a are given as

$$\gamma_k \equiv \gamma_k(a) := \begin{cases} 1, & \text{if } k = 0 \\ a_0^2 \cdots a_{k-1}^2, & \text{if } k \geq 1. \end{cases}$$

It is easy to see that W_a is never normal, and that it is hyponormal if and only if $a_0 \leq a_1 \leq \cdots$.

We shall often write $\text{shift}(a_0, a_1, \dots)$ to denote the weighted shift W_a .

Example 1.9 (2-variable weighted shift). For $\alpha \equiv \{\alpha_k\}, \beta \equiv \{\beta_k\} \in \ell^\infty(\mathbb{Z}_+^2)$, we define the 2-variable weighted shift $W_{(\alpha, \beta)} \equiv (W_\alpha, W_\beta)$ on $\ell^2(\mathbb{Z}_+^2)$ by

$$W_\alpha e_k := \alpha_k e_{k+\varepsilon_1} \text{ and } W_\beta e_k := \beta_k e_{k+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0), \varepsilon_2 := (0, 1)$ and $\{e_k : k \in \mathbb{Z}_+^2\}$ is the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+^2)$.

In an entirely similar way one can define multivariable weighted shifts.

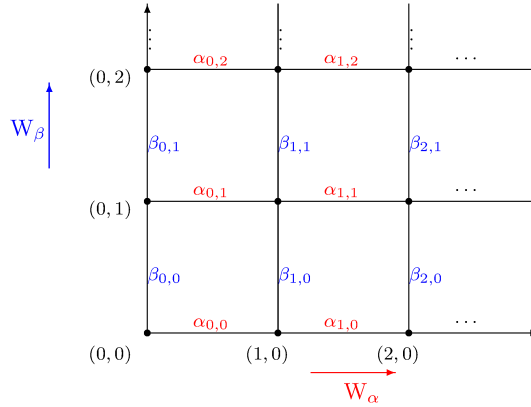
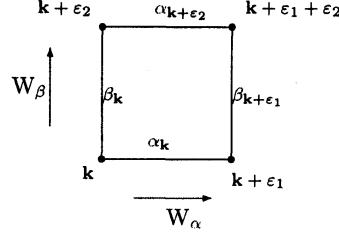


FIGURE 1. Weight diagram for 2-variable weighted shift $W_{(\alpha, \beta)}$

Clearly,

$$(1.2) \quad W_\alpha W_\beta = W_\beta W_\alpha \iff \alpha_k \beta_{k+\varepsilon_1} = \beta_k \alpha_{k+\varepsilon_2} \quad (\forall k \in \mathbb{Z}_+^2).$$



In the sequel, we assume that all 2-variable weighted shifts $W_{(\alpha, \beta)}$ are commuting, i.e., it satisfies condition (1.2).

Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moments $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$ of (α, β) of order $\mathbf{k}$ is defined by

$$\begin{cases} 1 & \text{if } \mathbf{k} \equiv (k_1, k_2) = (0, 0) \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (1.2), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to $\mathbf{k}$.

Question 1.10. Which weighted shifts are subnormal?

Theorem 1.11 (Berger's Theorem(1-variable)). W_a is subnormal if and only if there exists a probability measure ξ (called the Berger measure of W_a) supported in $[0, \|W_a\|^2]$ such that $\gamma_k(a) = \int s^k d\xi(s)$ ($k \geq 0$).

Theorem 1.12 (Berger's Theorem(2-variable)([14])). $W_{(\alpha, \beta)}$ is subnormal if and only if there is a probability measure μ (called the Berger measure of $W_{(\alpha, \beta)}$) supported in the 2-dimensional rectangle $R = [0, \|W_\alpha\|^2] \times [0, \|W_\beta\|^2]$ such that

$$\gamma_{\mathbf{k}}(\alpha, \beta) = \int_R s^{k_1} t^{k_2} d\mu(s, t), \forall \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2.$$

2. AUXILIARY LEMMAS

For a 2-variable weighted shift $W_{(\alpha, \beta)}$, we let $\mathcal{M}$ (resp. $\mathcal{N}$) be the invariant subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis vectors associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$ (resp. $k_1 \geq 1$ and $k_2 \geq 0$).

We consider the following problem:

Problem 2.1 (B). Assume that $W_{(\alpha, \beta)}|_{\mathcal{M}}$ and $W_{(\alpha, \beta)}|_{\mathcal{N}}$ are subnormal with the Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively. Find necessary and sufficient conditions on $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00} for the subnormality of $W_{(\alpha, \beta)}$.

Note that Problem (B) is equivalent to Problem (A).

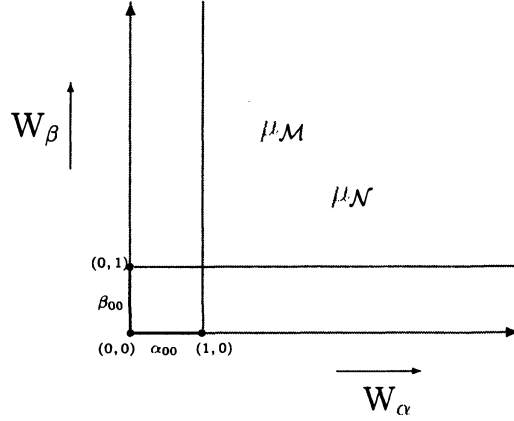
If W_a is subnormal with Berger measure ξ , and if we let for fixed $i \geq 1$,

$$\mathcal{L}_i := \bigvee \{e_n : n \geq i\}$$

then the Berger measure ξ_i of $W_a|_{\mathcal{L}_i}$ is $\frac{s^i}{\gamma_i} d\xi(s)$.

Lemma 2.2 (1-variable subnormal backward extension ([5])). If $W_a|_{\mathcal{L}_1}$ is subnormal with Berger measure ξ_1 then W_a is subnormal if and only if

$$(2.1) \quad \frac{1}{s} \in L^1(\xi_1) \quad \text{and} \quad a_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi_1)} \leq 1.$$

FIGURE 2. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Problem (B)

In this case, the Berger measure ξ of W_a is $d\xi(s) = \frac{a_0^2}{s} d\xi_1(s) + (1 - a_0^2 \|\frac{1}{s}\|_{L^1(\xi_1)}) d\delta_0(s)$.

- Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ if $\mu(E) \leq \nu(E)$ for each Borel subset $E \subseteq \mathbb{R}_+$.
- Let μ be a probability measure on $\mathbb{R}_+ \times \mathbb{R}_+$ and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $\mathbb{R}_+ \times \mathbb{R}_+$ is given by

$$d\mu_{ext}(s, t) := (1 - \delta_0(t)) \frac{1}{t \|\frac{1}{t}\|_{L^1(\mu)}} d\mu(s, t).$$

Here δ_0 denotes Dirac measure at 0.

- Given a measure μ on $X \times Y$, the *marginal measure* μ^X is a measure on X given by

$$\mu^X := \mu \circ \pi_X^{-1},$$

where $\pi_X : X \times Y \rightarrow X$ is the canonical projection onto X .

Lemma 2.3 (2-variable subnormal backward extension ([10])). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with the Berger measure $\mu_{\mathcal{M}}$ and that shift $(\alpha_{00}, \alpha_{10}, \dots)$ is subnormal with Berger measure ξ . Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*

- (i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
- (ii) $\beta_{00}^2 \leq (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
- (iii) $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \xi$.

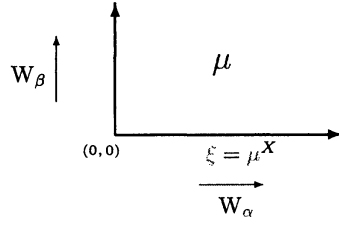
Moreover, if $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}_1})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \xi$.

Lemma 2.4 ([11]). *Let μ be the Berger measure of a subnormal 2-variable weighted shift $W_{(\alpha,\beta)}$, and let ξ be the Berger measure of the associated 0-th horizontal 1-variable shift $(\alpha_{00}, \alpha_{10}, \dots)$. Then $\xi = \mu^X$.*

3. MAIN RESULT AND APPLICATION

We provides a concrete solution of Problem (B) in terms of $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00} .

Theorem 3.1 (Main Theorem). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with associated Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively, and let $c := \frac{\int s d\mu_{\mathcal{M}}}{\int t d\mu_{\mathcal{N}}} \equiv \frac{\alpha_{01}^2}{\beta_{10}^2}$. Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following*

FIGURE 3. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Lemma 2.4

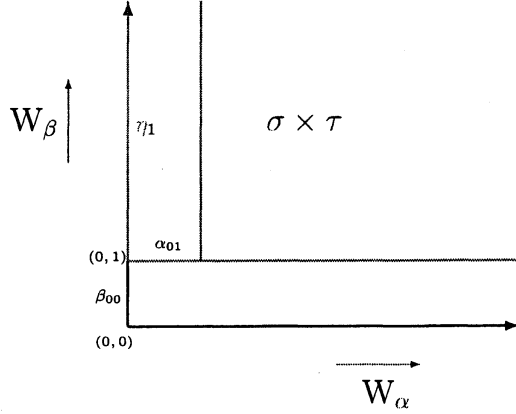
conditions hold:

- (i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ and $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;
- (ii) $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$;
- (iii) $\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \delta_0 - \frac{c}{s} (\mu_{\mathcal{N}})^X \right\} \leq \delta_0$.

For a measure μ with $\frac{1}{s} \in L^1(\mu)$, we write $d\tilde{\mu}(s) := \frac{1}{s \left\| \frac{1}{s} \right\|_{L^1(\mu)}} d\mu(s)$.

Lemma 3.2 ([8]). *Let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift given in Figure 4. Then $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal if and only if $\psi := \eta_1 - \alpha_{01}^2 \left\| \frac{1}{s} \right\|_{L^1(\sigma)} \tau$ is a positive measure. In this case, the Berger measure of $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is*

$$\mu_{\mathcal{M}} = \alpha_{01}^2 \left\| \frac{1}{s} \right\|_{L^1(\sigma)} \tilde{\sigma} \times \tau + \delta_0 \times \psi.$$

FIGURE 4. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Lemma 3.2

As a special case of Main Theorem, we have:

Theorem 3.3 (The case when $W_{(\alpha,\beta)}$ has a core of tensor form). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with associated Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively, and let $\rho := \mu_{\mathcal{M}}^X$. Also assume that $\mu_{\mathcal{M} \cap \mathcal{N}} = \sigma \times \tau$ for some 1-variable probability measures σ and τ . Then $\rho = \mu_{\mathcal{M}}^X = (\mu_{\mathcal{M}})_{ext}^X$, and hence $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*

- (i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ and $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;

- (ii) $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$;
 (iii) $\left(\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \right) \rho \leq \xi$, where ξ is the Berger measure of shift $(\alpha_{00}, \alpha_{10}, \dots)$.

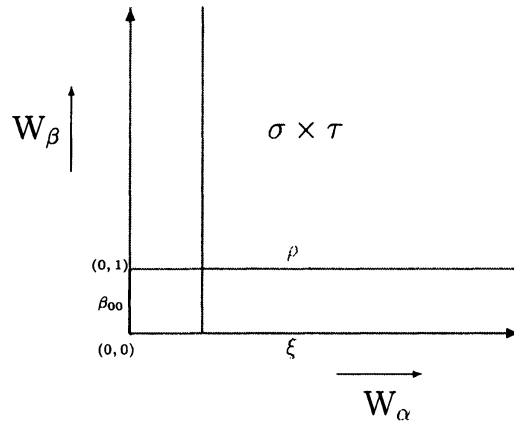


FIGURE 5. 2-variable weighted shift $W_{(\alpha, \beta)}$ in Theorem 3.3

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