

A cone angle condition on strong convergence of hyperbolic 3-cone-manifolds

MICHIHIKO FUJII

藤井 道彦 (京都大・総合人間)

§0. Introduction.

By a hyperbolic 3-cone-manifold, we will mean an orientable riemannian 3-manifold C of constant sectional curvature -1 with cone-type singularity along simple closed geodesics Σ . To each component of the singularity Σ , is associated a cone angle. Kojima showed in [4] that for any values of cone angles, a non-singular part $C - \Sigma$ carries a complete hyperbolic structure C_{comp} of finite volume, and moreover that if the cone angles of C all are at most π , then there is an angle decreasing continuous family of deformations of C to the complete hyperbolic 3-manifold C_{comp} homeomorphic to $C - \Sigma$. The complete hyperbolic 3-manifold C_{comp} has torus cusps at the parts which correspond to the singularity Σ of C , and C_{comp} can be regarded as a hyperbolic 3-cone-manifold with cone angles all equal to zero.

Kojima proved the latter claim by using two machineries, the local rigidity by Hodgson-Kerckhoff [3] and the pointed Hausdorff-Gromov topology [2]. These machineries are fundamental when cone angles are $\leq 2\pi$. In particular, the local rigidity implies the practicability of deformations of a hyperbolic 3-cone-manifold with arbitrary small changes in the cone angles, in the case where the initial cone angles all are at most 2π . Then, if the cone angles of C all are at most π , one obtains deformations of C with decreasing the cone angles with arbitrary small amount. In [4], for extending such a small deformation globally, he analyzed phenomena which occur in the two cases, that is, in the case where tubular neighborhoods of the singularity Σ in the deformations are uniformly thick, and in the case where they collapse. For this analysis, he established three relative constants for hyperbolic 3-cone-manifolds which control the local geometry of cone-manifolds away from the singularity. Lemma 3.1.1 of [4] gives one of them, and is a key lemma to derive the other constants and also to analyze the phenomena above.

In this paper, we will show that the assumption “ $\leq \pi$ ” in Lemma 3.1.1 [4] about the cone angles can be improved to “ $< 2\pi$ ” (see Lemma 2), by using fundamental properties on Dirichlet domains of 3-cone-manifolds (see Lemma 1). Then, without changing the proof performed in the sections 3 and 5 of [4], it can be seen that, for each sequence $\{C_i\}_{i=1}^\infty$ consisting of deformations of C so that tubular neighborhoods of Σ in deformations C_i ($i \in \mathbb{N}$) are uniformly thick, if the cone angles of C_i ($i \in \mathbb{N}$) all are less than 2π , then there is a subsequence $\{C_{i_k}\}_{k=1}^\infty$ which converges strongly to a hyperbolic 3-cone-manifold C_* homeomorphic to C (see Theorem).

§1. Dirichlet polyhedra and a relative constant for hyperbolic 3-cone-manifolds.

Assume that the singular set Σ of any 3-cone-manifold C considered in this paper forms a link

$$\Sigma = \Sigma^1 \cup \dots \cup \Sigma^n$$

of n components. To each component Σ^j of Σ , associated is a cone angle $\alpha^j \in [0, \infty)$.

If C is hyperbolic and $\Sigma \neq \emptyset$, then $N := C - \Sigma$ has a non-singular but incomplete hyperbolic structure and C inherits a metric induced from a riemannian metric on N . We assume that C is complete with this metric. In particular, the metric completion of N is identical to the metric space C . We have a developing map of N from its universal covering space $\tilde{N}$,

$$\mathcal{D}_C : \tilde{N} \rightarrow \mathbf{H}^3,$$

and a holonomy representation

$$\rho_C : \pi_1(N) \rightarrow \mathrm{PSL}_2(\mathbf{C}).$$

They are called a developing map and a holonomy representation of a cone-manifold C .

Let L be a number with $L \leq -1$. Let $\mathcal{C}_{[L,0]}^{<\theta}$ be the set of pointed compact orientable 3-cone-manifolds of constant sectional curvature $K \in [L, 0]$ so that the cone angles all are less than θ . Let $\mathcal{C}_K^{<\theta}$ be a subset of $\mathcal{C}_{[L,0]}^{<\theta}$ consisting of cone-manifolds with a particular curvature constant K .

Now take a cone-manifold $C \in \mathcal{C}_K^{<2\pi}$ and a point $x \in C - \Sigma$. Then define the following subset of C ,

$$P_x := \{y \in C \mid y \text{ admits the unique shortest path to } x\},$$

and call it a Dirichlet fundamental domain of C about x .

Lemma 1. *The Dirichlet fundamental domain P_x of $C \in \mathcal{C}_K^{<2\pi}$ about x has the following properties.*

(1) P_x is isometrically realized as an interior of a star-shaped geodesic polyhedron in the simply connected 3-dimensional space $\mathbf{H}_K$ of constant curvature K . The closure is star-shaped geodesic polyhedron. We call this embedded compactified polyhedron a Dirichlet polyhedron of C about x , and denote it again by P_x .

(2) Let y be a singular point, then there are two boundary faces of P_x both of which include y and whose dihedral angle equals to the cone angle at y . Moreover, the bisecting surface of these two faces contains x .

Proof. See Cooper-Hodgson-Kerckhoff [1]. $\square$

If $x \notin C - \Sigma$, the injectivity radius of C at x is to be the injectivity radius of $C - \Sigma$ at x . Denote it by $\text{inj}_x C$. The key lemma in this paper is the following:

Lemma 2. *Given positive numbers $D, I, R > 0$, and a curvature bound $L \leq -1$, there is a constant $U := U(D, I, R, L) > 0$ so that if $C \in \mathcal{C}_{[L,0]}^{<2\pi}$, $x \in C$ with $d(x, \Sigma) \geq D$ and $\text{inj}_x C \geq I$, then*

$$\text{inj}_y C \geq U$$

for any $y \in C$ with $d(y, \Sigma) \geq D$ and $d(y, x) \leq R$.

Proof. Suppose that there is not such a uniform bound U . Then, for some numbers $D, I, R > 0$ and $L \leq -1$, there exists a sequence of cone-manifolds $\{C_i\}_{i=1}^\infty \subset \mathcal{C}_{[L,0]}^{<2\pi}$ and points $x_i, y_i \in C_i$ such that

- (i) $d(x_i, \Sigma_i) \geq D, d(y_i, \Sigma_i) \geq D$,
- (ii) $\text{inj}_{x_i} C_i \geq I$,
- (iii) $d(y_i, x_i) \leq R$ and
- (iv) $\text{inj}_{y_i} C_i \leq 1/i$.

Take a Dirichlet polyhedron P_{y_i} of C_i about y_i in $\mathbf{H}_{K_i}$, where K_i is a curvature of C_i . There are points p_i, q_i on ∂P_{y_i} , which are identified in C_i and attain the shortest distance to y_i from ∂P_{y_i} . The union of these shortest paths $\overline{p_i y_i}, \overline{q_i y_i}$ forms a homotopically nontrivial shortest loop l_i in C_i based at y_i .

If i is large enough, p_i and q_i are on the interior of the faces of P_{y_i} respectively. Then by (i), (iv), and the properties of P_{y_i} described in Lemma 1, it can be seen that P_{y_i} is bounded by the extensions of the two faces.

Let $\phi_i(\leq \pi)$ be the angle between the segments $\overline{p_i y_i}$ and $\overline{q_i y_i}$ at y_i . If $\phi_i \rightarrow \pi$ as $i \rightarrow \infty$, then $\text{vol}(B_{R+I}(C_i, y_i)) \rightarrow 0$ by (iv). This is a contradiction since $B_I(C_i, x_i) \subset B_{R+I}(C_i, y_i)$ by (iii) and $\text{vol}(B_I(C_i, x_i)) > 0$ by (ii). Thus there is a number ϕ so that $\phi_i \leq \phi < \pi$. Therefore the loop l_i bends at y_i with angle uniformly away from π .

Let us lift l_i to a geodesic segment s_i in $\mathbf{H}_{K_i}$, based at y_i so that $p_i(= q_i)$ is its middle point. Let ρ_i be a holonomy representation of C_i ; $\rho_i : \pi_1(C_i - \Sigma_i) \rightarrow \text{PSL}_2(\mathbf{C})$. Then the action of $\rho_i(l_i)$ on $\mathbf{H}_{K_i}$ is either parabolic, loxodromic or elliptic. In any cases, the orbit of s_i by the action of a group generated by $\rho_i(l_i)$ forms a piecewise geodesic which bends with angle uniformly away from π , and the length of s_i goes to 0 when $i \rightarrow \infty$.

If there is a subsequence $\{k\} \subset \{i\}$ so that $\rho_k(l_k)$ all are parabolic, then the orbit of s_k goes to the ideal boundary of $\mathbf{H}_{K_k}$. This a contradiction, since the bending angle of the orbit of s_k should approaches π as $k \rightarrow \infty$ in the case where the orbit of s_k goes to ∞ and the length of s_k goes to 0 as $k \rightarrow \infty$.

If $\rho_i(l_i)$ is loxodromic, the orbit of s_i squeezes onto the axis of $\rho_i(l_i)$ since the length of s_i approaches 0 when $i \rightarrow \infty$. In particular, the axis of $\rho_i(l_i)$ becomes close to y_i when $i \rightarrow \infty$.

If there is a subsequence $\{k\} \subset \{i\}$ so that $\rho_k(l_k)$ all are loxodromic, the length of $\rho_k(l_k)$ goes to 0 when $k \rightarrow \infty$. If k is large enough, there is a very short simple closed geodesic in C_k near y_k . Then choose a new reference point z_k on this simple closed geodesic, take the Dirichlet polyhedron P_{z_k} about z_k , consider two hypersurfaces of $\mathbf{H}_{K_i}$ which bounds P_{z_k} and perform the same argument as before. This gives a contradiction.

Therefore $\rho_i(l_i)$ all but finitely many exceptions are elliptic. Take a subsequence $\{j\} \subset \{i\}$ so that $\rho_j(l_j)$ all are elliptic. The orbit of s_j rounds around a geodesic which is an extension of a lift of a component of Σ_j . Since the length of s_j goes 0 when $i \rightarrow \infty$, y_j approaches the geodesic. This contradicts (i). $\square$

§2. Strong convergence of hyperbolic 3-cone-manifolds.

Let C be a compact orientable hyperbolic 3-cone-manifold with singularity Σ . The singular set Σ has been assumed to form a link

$$\Sigma = \Sigma^1 \cup \dots \cup \Sigma^n$$

of n components. Let $\mathcal{T}$ be the maximal tube about Σ , that is, a union of open tubular neighborhoods $\mathcal{T}^j$'s which has the following properties,

- (a) each component $\mathcal{T}^j$ is an equidistant tubular neighborhood to the j -th component Σ^j of Σ ,

- (b) among ones having the property (a), the set of radii arranged in order of magnitude from the smallest one is maximal in lexicographical order.

Let us denote by $\partial\mathcal{T}^j$ an abstract boundary of $\mathcal{T}^j$. The actual boundary $\partial\mathcal{T}$ of $\mathcal{T}$ in C is a union of isometrically embedded tori with a finite number of contact points. The first contact point on $\partial\mathcal{T}$ is the point which admits two shortest paths to Σ from $\partial\mathcal{T}$. The finest point on $\partial\mathcal{T}$ is the point on $\partial\mathcal{T}$ which attains the minimum among $\{\text{inj}_x(C) | x \in \partial\mathcal{T}\}$.

A deformation of a hyperbolic 3-cone-manifold C is a hyperbolic 3-cone-manifold C_a together with a reference homeomorphism $\xi_a : (C, \Sigma) \rightarrow (C_a, \Sigma_a)$.

Now take a sequence $\{C_i\}_{i=1}^\infty$ of compact orientable hyperbolic 3-cone-manifolds with the following properties,

- (1) each C_i is a deformation of C with a reference homeomorphism $\xi_i : C \rightarrow C_i$,
- (2) $\alpha_i^j < 2\pi$ for all $1 \leq j \leq n$ and any $i \in \mathbf{N}$, where α_i^j is a cone angle along the component Σ_i^j ,
- (3) $\{\alpha_i^j\}_{i=1}^\infty$ converges to a number $\beta^j \in [0, 2\pi]$ for all $1 \leq j \leq n$.

Theorem. Let $\{C_i\}_{i=1}^\infty$ be a sequence of compact orientable hyperbolic 3-cone-manifolds as above. Suppose that there is a constant $D_1 > 0$ such that $D_1 \leq \text{radius } T_i^j$ for any $1 \leq j \leq n$ and any $i \in \mathbf{N}$. Then there is a subsequence $\{C_{i_m}\}_{m=1}^\infty$ which converges strongly to a hyperbolic 3-cone-manifold C_* homeomorphic to C , where the notion “converge strongly” is defined as follows; the sequence $\{C_{i_m}\}_{m=1}^\infty$ converges geometrically to the cone-manifold C_* homeomorphic to C and a sequence $\{\rho_{i_m}\}_{i_m}^\infty$ of their holonomy representations converges algebraically to the holonomy representation ρ_* of C_* with respect to the identification by ξ_{i_m} .

Remark. The property (2) induces the following one,

- (4) there is a constant $V_{\max}$ such that $\text{vol}(C_i) \leq V_{\max}$.

Remark. By the argument on geometric convergence due to Gromov [2], it can be shown that the following property is satisfied,

- (5) the sequence $\{(C_i, c_i)\}_{i=1}^\infty$ has a subsequence $\{(C_{i_k}, c_{i_k})\}_{k=1}^\infty$ which converges geometrically to a complete metric space.

Proof. Take a subsequence $\{i_k\} \subset \{i\}$ which satisfies the properties (1), ..., (5). By choosing a further subsequence, we may assume that the sequence $\{C_{i_k}\}_{k=1}^\infty$ satisfies the following properties also,

- (6) c_{i_k} lies on a component $\partial\mathcal{T}_{i_k}^c$ with a constant reference number c , and
- (7) f_{i_k} lies on a component on a component $\partial\mathcal{T}_{i_k}^f$ with a constant reference number f .

Then the sequence $\{c_{i_k}\}_{k=1}^\infty$ has the same property as in Kojima [4,section 4], except for the condition on the range of the cone angles.

By following the arguments described in section 3 and section 5 of [4], we can verify that Corollary 5.1.4 of [4] holds with replacing the cone angle condition " $\alpha_i^j \leq \pi$ " with " $\alpha_i^j < 2\pi$ ", if Lemma 3.1.1 of [4] holds with the cone angle condition " $< 2\pi$ ". Lemma 2 is exactly such a version of Lemma 3.1.1 of [4]. Then Corollary 5.1.4 of [4] with the cone angle condition " $\alpha_i^j < 2\pi$ " holds. This is what we need. $\square$

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Faculty of Integrated Human Studies

Kyoto University

Sakyo-ku

Kyoto 606-8501

JAPAN

E-mail address: mfujii@math.h.kyoto-u.ac.jp