

Properties of certain p -valently convex functions

Dinggong Yang and Shigeyoshi Owa

Abstract

A subclass $C_p(\lambda, \mu)$ ($p \in \mathbb{N}, 0 < \lambda < 1, -\lambda \leq \mu < 1$) of p -valently convex functions in the open unit disk $\mathbb{U}$ is introduced. The object of the present paper is to discuss some interesting properties of functions belonging to the class $C_p(\lambda, \mu)$.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $\mathcal{A}_p$ is said to be p -valently convex of order α if it satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > p\alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$). We denote by $\mathcal{K}_p(\alpha)$ the subclass of $\mathcal{A}_p$ consisting of functions which are p -valently convex of order α in $\mathbb{U}$. In particular, we denote by $\mathcal{K}_1(0) = \mathcal{K}$.

A function $f(z) \in \mathcal{A}_1$ is said to be uniformly convex in $\mathbb{U}$ if $f(z)$ is in the class $\mathcal{K}$ and has the property that the image arc $f(\gamma)$ is convex for every circular arc γ contained in $\mathbb{U}$ with center at $t \in \mathbb{U}$. We also denote by $\mathcal{UK}$ the subclass of $\mathcal{A}_1$ consisting of all uniformly convex functions in $\mathbb{U}$. Goodman [2] has introduced the class $\mathcal{UK}$ and given that $f(z) \in \mathcal{A}_1$ belongs to the class $\mathcal{UK}$ if and only if

$$\operatorname{Re} \left\{ 1 + (z-t) \frac{f''(z)}{f'(z)} \right\} \geq 0 \quad ((z, t) \in \mathbb{U} \times \mathbb{U}).$$

Ma and Minda [3] and Rønning [5] have showed a more applicable characterization for $\mathcal{UK}$. We state this as

Theorem A. Let $f(z) \in \mathcal{A}_1$. Then $f(z) \in \mathcal{UK}$ if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).$$

In view of Theorem A, Owa [4] considered a subclass $\mathcal{UK}(\mu)$ ($-1 < \mu < 1$) of $\mathcal{A}_1$. A function $f(z) \in \mathcal{A}_1$ is said to be a member of the class $\mathcal{UK}(\mu)$ ($-1 < \mu < 1$) if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \mu > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).$$

In the present paper we investigate the following subclass of $\mathcal{A}_p$.

Definition. A function $f(z) \in \mathcal{A}_p$ is said to be a member of the class $\mathcal{C}_p(\lambda, \mu)$ if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - p\mu > \lambda \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \quad (z \in \mathbb{U}) \quad (1)$$

for some λ ($0 < \lambda < 1$) and μ ($-\lambda \leq \mu < 1$).

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

In proving our results, we need the following lemmas.

Lemma 1.1. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \prec g(z)$$

and $g(z) \in \mathcal{K}$. Then $|a_n| \leq 1$ ($n = 1, 2, 3, \dots$).

We note that Lemma 1.1 can be seen in [1].

Lemma 1.2. *A function $f(z)$ in $\mathcal{A}_p$ belongs to the class $\mathcal{K}_p(\alpha)$ ($0 \leq \alpha < 1$) if*

$$\sum_{n=1}^{\infty} (p+n) \{n + p(1-\alpha)\} |a_{p+n}| \leq p^2(1-\alpha). \quad (2)$$

Proof. If the inequality (2) holds true, then we have that

$$\begin{aligned} \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| &= \left| \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^n} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}|}{p - \sum_{n=1}^{\infty} (p+n)|a_{p+n}|} \leq p(1-\alpha) \end{aligned} \quad (3)$$

for $z \in \mathbb{U}$. From (3), we easily seen that $f(z) \in \mathcal{K}_p(\alpha)$.

2 Subordination properties

Our first result for properties of functions $f(z) \in \mathcal{A}_p$ is contained in

Theorem 2.1. *A function $f(z) \in \mathcal{C}_p(\lambda, \mu)$ if and only if*

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z)$$

with

$$h(z) = p + \frac{p(1-\mu)}{2\sin^2\beta} \left\{ \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^{\frac{2\beta}{\pi}} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}} \right)^{\frac{2\beta}{\pi}} - 2 \right\} \quad (\beta = \arccos\lambda). \quad (4)$$

Proof. Let $1 + \frac{zf''(z)}{f'(z)} = w$ and $w = u + iv$. Then the inequality (1) can be written as

$$u - p\mu > \lambda\sqrt{(u-p)^2 + v^2}. \quad (5)$$

By computing, we find that the inequality (5) is equivalent to

$$\left(u + \frac{p(\lambda^2 - \mu)}{1 - \lambda^2} \right)^2 - \frac{\lambda^2}{1 - \lambda^2} v^2 > \left(\frac{p\lambda(1 - \mu)}{1 - \lambda^2} \right)^2 \quad (6)$$

and

$$u > \frac{p(\lambda + \mu)}{1 + \lambda}. \quad (7)$$

Thus the domain of the values of $1 + \frac{zf''(z)}{f'(z)}$ for $z \in \mathbb{U}$ is

$$\mathbb{D} = \{w = u + iv : u \text{ and } v \text{ satisfy (6) with (7)}\}.$$

In order to prove our theorem, it suffices to show that the function $h(z)$ given by (4) maps $\mathbb{U}$ conformally onto the domain $\mathbb{D}$.

Consider the transformations

$$w_1 = \frac{1 - \lambda^2}{p(1 - \mu)} w + \frac{\lambda^2 - \mu}{1 - \mu}$$

and

$$t = \frac{1}{2} \left(w_2^{\frac{\pi}{\beta}} + w_2^{-\frac{\pi}{\beta}} \right),$$

where $\beta = \arccos\lambda$ and $w_2 = w_1 + \sqrt{w_1^2 - 1}$ is the inverse function of

$$w_1 = \frac{w_2 + \frac{1}{w_2}}{2}.$$

It is easy to verify that composite function $t = t(w)$ maps $\mathbb{D}^+$ defined by

$$\mathbb{D}^+ = \{w = u + iv : u \text{ and } v \text{ satisfy (6) with (7) and } v > 0\}$$

conformally onto the upper half plane $\text{Im}(t) > 0$ so that $w = p$ corresponds to $t = 1$ and $w = \frac{p(\lambda + \mu)}{1 + \lambda}$ to $t = -1$. With the help of the symmetry principle, this function $t = t(w)$ maps $\mathbb{D}$ conformally onto the domain

$$\mathbb{G} = \{t : |\arg(t + 1)| < \pi\}.$$

Since

$$t = 2 \left(\frac{1 + z}{1 - z} \right)^2 - 1$$

maps $\mathbb{U}$ onto $\mathbb{G}$, we see that

$$\begin{aligned} w &= p + \frac{p(1 - \mu)}{2(1 - \lambda^2)} \{ (t + \sqrt{t^2 - 1})^{\frac{p}{\pi}} + (t - \sqrt{t^2 - 1})^{-\frac{p}{\pi}} - 2 \} \\ &= p + \frac{p(1 - \mu)}{2\sin^2 \beta} \left\{ \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^{2\frac{p}{\pi}} + \left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right)^{2\frac{p}{\pi}} - 2 \right\} \\ &= h(z) \end{aligned}$$

maps $\mathbb{U}$ onto $\mathbb{D}$ with $h(0) = p$. Hence the proof of the theorem is completed. □

Theorem 2.1 gives the following corollaries.

Corollary 2.1. *If $f(z) \in \mathcal{C}_p(\lambda, \mu)$, then $f(z) \in \mathcal{K}_p\left(\frac{\lambda + \mu}{1 + \lambda}\right)$ and the order $\frac{\lambda + \mu}{1 + \lambda}$ is sharp with the extremal function*

$$f_0(z) = p \int_0^z \left(t_2^{p-1} \exp \int_0^{t_2} \frac{h(t_1) - p}{t_1} dt_1 \right) dt_2, \quad (8)$$

where $h(z)$ is given by (4).

Proof. Using (7) in the proof of Theorem 2.1 and noting that

$$\text{Re} \left(1 + \frac{zf_0''(z)}{f_0'(z)} \right) = \text{Re}(h(z)) \rightarrow p \frac{\lambda + \mu}{1 + \lambda}$$

as $z = \text{Re}(z) \rightarrow -1$, we have the corollary. □

Corollary 2.2. *If $f(z) \in \mathcal{C}_p(\lambda, \mu)$ and $-\lambda < \mu < \lambda < 1$, then*

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \arctan \left(\frac{1 - \mu}{\sqrt{\lambda^2 - \mu^2}} \right) \quad (z \in \mathbb{U}). \quad (9)$$

The bound in (9) is sharp with the extremal function $f_0(z)$ given by (8).

Proof. Let the function $h(z)$ be defined by (4). Then $h(\mathbb{U}) = \mathbb{D}$ and an easy calculation yields that

$$\min\{\theta : |\arg(h(z))| < \theta \ (z \in \mathbb{U})\} = \arctan \left(\frac{1 - \mu}{\sqrt{\lambda^2 - \mu^2}} \right)$$

for $-\lambda < \mu < \lambda < 1$. Therefore the corollary follows immediately from Theorem 2.1. $\square$

Next we derive

Theorem 2.2. Let $f(z) \in \mathcal{C}_p(\lambda, \mu)$ and $h(z)$ be defined by (4). Then

$$\frac{f'(z)}{pz^{p-1}} \prec \exp \int_0^z \frac{h(t) - p}{t} dt \quad (10)$$

and

$$\left| \frac{f'(z)}{pz^{p-1}} \right| < \exp \int_0^1 \frac{h(\rho) - p}{\rho} d\rho \quad (z \in \mathbb{U}). \quad (11)$$

The bound in (11) is sharp with the extremal function $f_0(z)$ given by (8).

Proof. Since the function $h(z) - p$ is univalent and starlike (with respect to the origin), by Theorem 2.1 and the result due to Suffridge [6, Theorem 3], we have

$$\log \left(\frac{f'(z)}{pz^{p-1}} \right) = \int_0^z \left(\frac{f''(t)}{f'(t)} - \frac{p-1}{t} \right) dt \prec \int_0^z \frac{h(t) - p}{t} dt, \quad (12)$$

which implies the subordination (10).

Furthermore, noting that the univalent function $h(z)$ maps the disk $|z| < \rho$ ($0 < \rho \leq 1$) onto the domain which is convex and symmetric with respect to the real axis, we deduce that

$$\operatorname{Re} \int_0^z \frac{h(t) - p}{t} dt = \int_0^1 \frac{\operatorname{Re}\{h(\rho z) - p\}}{\rho} d\rho < \int_0^1 \frac{h(\rho) - p}{\rho} d\rho \quad (13)$$

for $z \in \mathbb{U}$. Thus the inequality (11) follows from (12) and (13). $\square$

Remark. If we let $\beta = \frac{\pi}{4}$ and $x = \left(\frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}} \right)^{\frac{1}{2}}$ ($0 \leq \rho < 1$), then

$$\begin{aligned} & \int_0^1 \left\{ \left(\frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}} \right)^{2\frac{\beta}{\pi}} + \left(\frac{1 - \sqrt{\rho}}{1 + \sqrt{\rho}} \right)^{2\frac{\beta}{\pi}} - 2 \right\} \frac{d\rho}{\rho} \\ &= 8 \int_1^{+\infty} \left(\frac{x}{x^2 + 1} - \frac{1}{x + 1} \right) dx = 4 \log 2. \end{aligned}$$

Thus, as the special case of Theorem 2.2, we have that if $f(z) \in \mathcal{C}_p(\frac{1}{\sqrt{2}}, \mu)$ ($-\frac{1}{\sqrt{2}} \leq \mu < 1$), then

$$\left| \frac{f'(z)}{pz^{p-1}} \right| < 16^{p(1-\mu)} \quad (z \in \mathbb{U}),$$

and the result is sharp.

3 Coefficient inequalities

Theorem 3.1. *If*

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

belongs to $\mathcal{C}_p(\lambda, \mu)$, then

$$|a_{p+1}| \leq \frac{8p^2(1-\mu)}{p+1} \left(\frac{\beta}{\pi \sin \beta} \right)^2 \quad (\beta = \arccos \lambda). \quad (14)$$

The result is sharp.

Proof. It can be easily verified that

$$1 + \frac{zf''(z)}{f'(z)} = p + \left(1 + \frac{1}{p}\right) a_{p+1} z + \dots \quad (15)$$

and

$$\begin{aligned} h(z) &= p + \frac{p(1-\mu)}{2\sin^2 \beta} \left(\frac{8\beta}{\pi} + \frac{8\beta}{\pi} \left(\frac{2\beta}{\pi} - 1 \right) \right) z + \dots \\ &= p + 8p(1-\mu) \left(\frac{\beta}{\pi \sin \beta} \right)^2 z + \dots, \end{aligned} \quad (16)$$

where $h(z)$ is given by (4). Since

$$f(z) = z^p + a_{p+1} z^{p+1} + \dots \in \mathcal{C}_p(\lambda, \mu),$$

it follows from (15), (16) and Theorem 2.1 that

$$\begin{aligned} \frac{\pi^2}{8p(1-\mu)} \left(\frac{\sin \beta}{\beta} \right)^2 \left(1 + \frac{zf''(z)}{f'(z)} - p \right) &= \frac{p+1}{8p^2(1-\mu)} \left(\frac{\pi \sin \beta}{\beta} \right)^2 a_{p+1} z + \dots \\ &< \frac{\pi^2}{8p(1-\mu)} \left(\frac{\sin \beta}{\beta} \right)^2 (h(z) - p). \end{aligned}$$

In view of

$$\frac{\pi^2}{8p(1-\mu)} \left(\frac{\sin \beta}{\beta} \right)^2 (h(z) - p) \in \mathcal{K},$$

we get (14) by using Lemma 1.1. Also the bound in (14) is sharp for the function $f_0(z)$ given by (8).

□

Next we see

Theorem 3.2. *If the function*

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

belonging to the class $\mathcal{A}_p$ satisfies

$$\sum_{n=1}^{\infty} (p+n) \{n(1+\lambda) + p(1-\mu)\} |a_{p+n}| \leq p^2(1-\mu), \quad (17)$$

then $f(z)$ belongs to the class $\mathcal{C}_p(\lambda, \mu)$.

Proof. Applying the inequality (17), we deduce that

$$\begin{aligned} & \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) - p\mu - \lambda \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \\ & \geq p(1-\mu) - (1+\lambda) \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \\ & = p(1-\mu) - (1+\lambda) \left| \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^n} \right| \\ & \geq p(1-\mu) - (1+\lambda) \left(\frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}|}{p - \sum_{n=1}^{\infty} (p+n)|a_{p+n}|} \right) \\ & \geq 0, \end{aligned}$$

which shows that $f(z) \in \mathcal{C}_p(\lambda, \mu)$. □

By using Theorem 3.2 and Corollary 2.1, we easily have

Corollary 3.1. *Let*

$$f(z) = z^p + \sum_{n=1}^{\infty} (-1)^{n+1} |a_{p+n}| z^{p+n}$$

be in the class $\mathcal{A}_p$. Then $f(z)$ belongs to the class $\mathcal{C}_p(\lambda, \mu)$ if and only if $f(z) \in \mathcal{K}_p \left(\frac{\lambda+\mu}{1+\lambda} \right)$.

Finally, we derive

Theorem 3.3. *A function $f(z) = z^p + a_{p+n}z^{p+n}$ ($n \in \mathbb{N}$) is in the class $\mathcal{C}_p(\lambda, \mu)$ if and only if*

$$|a_{p+n}| \leq \frac{p^2(1-\mu)}{(p+n)\{n(1+\lambda) + p(1-\mu)\}}. \quad (18)$$

Proof. In view of Theorem 3.2, it suffices to show the only if part. Let us suppose that $f(z) \in \mathcal{C}_p(\lambda, \mu)$. Then

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) - p\mu - \lambda \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|$$

$$= p(1 - \mu) + \operatorname{Re} \left(\frac{n(p+n)a_{p+n}z^n}{p + (p+n)a_{p+n}z^n} \right) - \lambda \left| \frac{n(p+n)a_{p+n}z^n}{p + (p+n)a_{p+n}z^n} \right| > 0. \quad (19)$$

Writing $a_{p+n} = |a_{p+n}|e^{i\theta}$ ($\neq 0$) and letting $z \rightarrow e^{i\frac{x-\theta}{n}}$ ($z \in \mathbb{U}$), we have $a_{p+n}z^n \rightarrow -|a_{p+n}|$ and it follows from (19) that

$$p(1 - \mu) - (1 + \lambda) \frac{n(p+n)|a_{p+n}|}{p - (p+n)|a_{p+n}|} \geq 0,$$

which implies the inequality (18).

□

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Dinggong Yang
Department of Mathematics
Suzhou University
Suzhou, Jiangsu 215006
People's Republic of China

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502