

Degenerate parabolic equation derived from kinetic theory, revisited

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Abstract

We continue the study of a degenerate parabolic equation derived from the kinetic theory using Rényi-Tsallis' entropy, particularly, the quantized blowup mechanism for the critical mass exponent.

1 Introduction

The present paper studies the blowup mechanism for solutions to a degenerate parabolic equation in a kinetic theory describing the motion of a mean field of many self-interacting particles [2].

First, the particle density at $(x, t) \in \mathbf{R}^n \times (0, T)$ with the velocity v is denoted by $0 \leq f = f(x, v, t)$ which satisfies the kinetic equation

$$f_t + v \cdot \nabla_x f - \nabla \varphi \cdot \nabla_v f = -\nabla_v \cdot j \quad (1)$$

provided with the general dissipation flux term $-\nabla_v \cdot j$, where φ is the Newton potential generated by f . We have the density-pressure relation

$$p = p(\mu, \theta) \quad (2)$$

and the Poisson equation

$$\Delta \varphi = \mu, \quad (3)$$

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where p and θ stand for the pressure and the temperature, respectively.

The above flux term in (1) is determined by the maximum entropy production principle, so that f maximize the local entropy

$$S = \int_{\mathbf{R}^n} s(f(x, v, t)) dv$$

under the constraint

$$\begin{aligned}\mu(x, t) &= \int_{\mathbf{R}^n} f(x, v, t) dv \\ p(x, t) &= \frac{1}{n} \int_{\mathbf{R}^n} |v|^2 f(x, v, t) dv.\end{aligned}$$

Averaging f over the velocities $v \in \mathbf{R}^n$ and the passage to the limit of large friction or large times lead to

$$\mu_t = \nabla[D_* \cdot (\nabla p + \mu \nabla \varphi)], \quad (4)$$

that is a hydrodynamical limit of self-gravitating particles whereby the total mass

$$\lambda = \int_{\mathbf{R}^n} \mu(x, t) dx$$

is conserved during the evolution. We have, thus, several mean field equations according to the entropy function $s(f)$ subject to the law of partition of macroscopic states of particles into mezosopic states, that is the entropies of Boltzmann, Fermi-Dirac, Bose-Einstein, and so forth. System (2)-(4) is still under-determined, and there are several theories to prescribe the temperature θ . In the canonical statistics one takes the iso-thermal setting, and hence the temperature θ is a constant. In the micro-canonical statistics, on the other hand, θ is a function of t and the total energy

$$E = \frac{n}{2} \int_{\Omega} p dx + \frac{1}{2} \int_{\Omega} \mu \varphi dx$$

is prescribed independently of t .

If Rényi-Tsallis' entropy

$$S = \frac{-1}{q-1} \int_{\mathbf{R}^n} (f^q - f) dv$$

is adopted, then (2) becomes

$$p = \kappa \theta^{1-\frac{\gamma n}{2}} \mu^{1+\gamma},$$

where $\kappa > 0$ is a constant and $\frac{1}{\gamma} = \frac{1}{q-1} + \frac{n}{2}$, see [3, 1]. Normalizing physical constants, we can reduce (3)-(4) to the degenerate parabolic equation

$$u_t = \frac{m-1}{m} \Delta u^m - \nabla \cdot (u \nabla \Gamma * u), \quad u \geq 0 \quad \text{in } \mathbf{R}^n \times (0, T) \quad (5)$$

in the iso-thermal setting, where the new unknown u is a positive constant times μ , $\frac{1}{m-1} = \frac{1}{q-1} + \frac{n}{2}$, and

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)|x|^{n-2}} \quad (6)$$

with ω_{n-1} denoting the area of the boundary of the unit ball in $\mathbf{R}^n$.

When $n = 3$ and $q = \frac{5}{3}$, the case $m = 2 - \frac{2}{n} = \frac{4}{3}$ actually arises to (5). From the scaling invariance, see below, equation (5) of this exponent m is a higher-dimensional version of the Smoluchowski-Poisson equation associated with the Boltzmann entropy in two-space dimensions. This two-dimensional equation is given by

$$u_t = \Delta u - \nabla \cdot (u \nabla \Gamma * u), \quad u \geq 0 \quad \text{in } \mathbf{R}^2 \times (0, T) \quad (7)$$

defined for $\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$. It is thus a relative to the simplified system of chemotaxis and there arises the formation of collapse for the blowup solution in finite time similarly, that is

$$u(x, t) dx \rightarrow \sum_{x_0 \in \mathcal{S}} 8\pi \delta_{x_0}(dx) + f(x) dx \quad (8)$$

as $t \uparrow T$ in $\mathcal{M}(\mathbf{R}^n \cup \{\infty\})$ provided that $T < +\infty$ and

$$u_0 = u|_{t=0} \in X = L^1(\mathbf{R}^2, (1 + |x|^2)dx) \cap L^\infty(\mathbf{R}^2) \cap H^1(\mathbf{R}^2),$$

where T is the blowup time, $\mathbf{R}^2 \cup \{\infty\}$ is the one-point compactification of $\mathbf{R}^2$,

$$\begin{aligned} \mathcal{S} = \{x_0 \in \mathbf{R}^2 \cup \{\infty\} \mid & \text{there exist } x_k \rightarrow x_0 \text{ and } t_k \uparrow T \\ & \text{such that } u(x_k, t_k) \rightarrow +\infty\} \end{aligned} \quad (9)$$

the blowup set actually contained in $\mathbf{R}^2$, and $0 \leq f = f(x) \in L^1(\mathbf{R}^2) \cap C(\mathbf{R}^2 \setminus \mathcal{S})$, see [15, 18].

The solution to (5) which we handle with is the weak solution formulated by [20]. First, given the initial value

$$0 \leq u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \quad \text{with} \quad u_0^m \in H^1(\mathbf{R}^n), \quad (10)$$

we take the approximate solution $u_\varepsilon = u_\varepsilon(x, t)$ satisfying

$$\begin{aligned} u_{\varepsilon t} &= \frac{m-1}{m} \Delta(u_\varepsilon + \varepsilon)^m - \nabla \cdot (u_\varepsilon \nabla \Gamma * u_\varepsilon) & \text{in } \mathbf{R}^n \times (0, T) \\ u|_{t=0} &= u_{0\varepsilon} & \text{in } \mathbf{R}^n \end{aligned}$$

for $0 < \varepsilon \ll 1$, where

$$\begin{aligned} 0 \leq u_{0\varepsilon} &\in L^1 \cap W^{2,p}(\mathbf{R}^n) & \text{for any } p \in [\frac{n}{n-1}, n+3] \\ \|u_{0\varepsilon}\|_p &\leq \|u_0\|_p, & \text{for any } p \in [1, \infty] \\ \|\nabla u_{0\varepsilon}^m\|_2 &\leq \|\nabla u_0^m\|_2 \\ u_{0\varepsilon} &\rightarrow u_0 & \text{strongly in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0 \text{ for some } p \in [\frac{n}{n-1}, \infty). \end{aligned}$$

Then we obtain the following theorem, passing to the limit $\varepsilon \downarrow 0$.

Theorem 1 *Assume that (10) holds. Then, there exists $0 < T \ll 1$ such that (5) has a weak solution in the sense that*

$$\iint_{\mathbf{R}^n \times [0, T]} \frac{m-1}{m} \nabla u^m \cdot \nabla \xi - u \nabla \Gamma * u \cdot \nabla \xi - u \xi_t \, dx dt = \int_{\mathbf{R}^n} u_0 \xi \, dx$$

provided with the properties

$$u \in C_*([0, T], L^p(\mathbf{R}^n)), \quad 1 < p \leq \infty,$$

regarding $L^p(\mathbf{R}^n) = L^{p'}(\mathbf{R}^n)'$, $\frac{1}{p'} + \frac{1}{p} = 1$,

$$\begin{aligned} u &\in L^\infty([0, T]; L^1(\mathbf{R}^n)) \cap L_{loc}^\infty([0, T]; L^\infty(\mathbf{R}^n)) \\ \nabla u^m &\in L^\infty(0, T; L^2(\mathbf{R}^n)) \\ \partial_t u^{\frac{m+1}{2}} &\in L^2(0, T; L^2(\mathbf{R}^n)) \\ \nabla \Gamma * u &\in L_{loc}^\infty([0, T]; L^2(\mathbf{R}^n)), \end{aligned} \tag{11}$$

and

$$\|u(t)\|_1 = \|u_0\|_1 \quad \text{for a.e. } t \in [0, T], \tag{12}$$

where $\xi \in H^1(0, T; L^2(\mathbf{R}^n)) \cap L^2(0, T; H^1(\mathbf{R}^n))$ is the test function satisfying $\xi(\cdot, t) = 0$ for $0 < T - t \ll 1$. Furthermore, it holds that

$$u_\varepsilon \xrightarrow{*} u \quad \text{in } L^\infty(0, T; L^q(\mathbf{R}^n)) \quad \text{for all } q \in (1, \infty], \tag{13}$$

regarding $L^\infty(0, T; L^q(\mathbf{R}^n)) = L^1(0, T; L^{q'}(\mathbf{R}^n))'$, $\frac{1}{q'} + \frac{1}{q} = 1$, for some subsequence of the approximate solutions. If the existence time of the weak solution u , denoted $T_{\max}$, is finite, then

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_\infty = +\infty. \tag{14}$$

Henceforth we put $T = T_{\max}$. We take the case

$$\int_{\mathbf{R}^n} |x|^2 u_0(x) dx < +\infty \quad (15)$$

to control the behavior of the solution at $x = \infty$. The next theorem assures a threshold of $\lambda = \|u_0\|_1$ for $T = +\infty$ to occur. The threshold value λ_* will be prescribed in the next section.

Theorem 2 *There is a constant $\lambda_* > 0$ determined by the dimension $n \geq 3$ such that if $u_0 = u_0(x)$ is the initial value satisfying (10), (15), and $\|u_0\|_1 < \lambda_*$, then $T = +\infty$ holds in (5) for $m = 2 - \frac{2}{n}$. Each $\lambda > \lambda_*$, on the other hand, takes $u_0 = u_0(x)$ such that (10), (15), $\|u_0\|_1 = \lambda$, and $T < +\infty$.*

The blowup set is now defined by $\mathcal{S} = \mathbf{R}^n \setminus \mathcal{B}$,

$$\mathcal{B} = \{x_0 \in \mathbf{R}^n \mid \text{there exists } r > 0 \text{ such that } \limsup_{t \uparrow T} \sup_{x \in B(x_0, r)} u(x, t) < +\infty\}$$

which is non-empty because the weak solution $u = u(x, t)$ satisfies the standard blowup criterion (14) for $T < +\infty$. Here and henceforth, we write $\sup_x$ for ess. sup_x . Next, we confirm the blowup rate. Thus we write (5) as

$$u_t = \frac{m-1}{m} \Delta u^m - \nabla u \cdot \nabla \Gamma * u + u^2,$$

and take the ODE part

$$\dot{\zeta} = \zeta^2.$$

It follows that

$$\zeta(t) = (T - t)^{-1} \quad (16)$$

and we see that the type I blowup rate is $O((T - t)^{-1})$. Then we say that $x_0 \in \mathcal{S}$ is type I if $\liminf_{t \uparrow T} (T - t) \|u(t)\|_{L^\infty(B(x_0, r_0))} < +\infty$ for some $r_0 > 0$ and type II in the other case. The next theorem assures the finiteness of type II blowup points.

Theorem 3 *Let $u_0 = u_0(x)$ be the initial value satisfying (10) and (15), and assume $T < +\infty$ for the above described weak solution $u = u(x, t)$ to (5) with $m = 2 - \frac{2}{n}$. Then, $\mathcal{S}$ is bounded and S_{II} is finite, where*

$$S_{II} = \left\{ x_0 \in \mathcal{S} \mid \lim_{t \uparrow T} (T - t) \|u(t)\|_{L^\infty(B(x_0, r_0))} = +\infty \text{ for any } r_0 > 0 \right\}.$$

In the case of the Smolchowski-Poisson equation in two-space dimensions (7), any $x_0 \in \mathcal{S}$ is type II. More strongly, it holds that

$$\lim_{t \uparrow T} (T - t) \|u(t)\|_{L^\infty(B(x_0, b(T-t)^{1/n}))} = +\infty \quad (17)$$

for any $b > 0$, see [11]. The finiteness of $\mathcal{S}_{II}$, and consequently that of $\mathcal{S}$, is also proven in this case, but the proof of Theorem 3 is quite different. This difference comes from essentially that of the roles of the second moment of u . We have, more precisely, $x \cdot \nabla \Gamma = -\frac{1}{2\pi}$ for $\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$, while $x \cdot \nabla \Gamma = -(n-2)\Gamma$ arises for (6) which results in (26) below.

This paper is composed of four sections. In §§2 and 3, we describe the proof of Theorems 2 and 3, respectively. In section 4, we argue related topics such as the formation of collapse, blowup rate, and mass quantization. We emphasize that the argument developed in this paper is formal.

2 Proof of Theorem 2

The first observation is that it is a *model B equation*, see [18], associated with the *free energy*

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2} \langle \Gamma * u, u \rangle. \quad (18)$$

In fact, we have

$$\delta \mathcal{F}(u)[v] = \left. \frac{d}{ds} \mathcal{F}(u + sv) \right|_{s=0} = \langle v, u^{m-1} - \Gamma * u \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product. Identifying $\mathcal{F}(u)$ with $u^{m-1} - \Gamma * u$, we can write (5) as

$$u_t = \nabla \cdot \left(\frac{m-1}{m} \nabla u^m - u \nabla \Gamma * u \right) = \nabla \cdot u \nabla \delta \mathcal{F}(u) \quad \text{in } \mathbf{R}^n \times (0, T).$$

From this form, we have the total mass conservation

$$\|u(t)\|_1 = \|u_0\|_1 = \lambda \quad (19)$$

and the decrease of the free energy

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(u) &= - \int_{\mathbf{R}^n} u |\nabla \delta \mathcal{F}(u)|^2 dx \\ &= - \int_{\mathbf{R}^n} u |\nabla (u^{m-1} - \Gamma * u)|^2 dx \leq 0. \end{aligned} \quad (20)$$

Regarding (19)-(20), we formulate the stationary state by

$$u^{m-1} - \Gamma * u = \text{constant in } \{u > 0\}, \quad \int_{\mathbf{R}^n} u dx = \lambda. \quad (21)$$

If the above constant is denoted by c , then $v = \Gamma * u + c$ satisfies

$$-\Delta v = v_+^q \quad \text{in } \mathbf{R}^n, \quad \int_{\mathbf{R}^n} v_+^q dx = \lambda, \quad (22)$$

where $m = 1 + \frac{1}{q}$. Problem (22) is invariant under the scaling transformation

$$v(x) \mapsto v_\mu(x) = \mu^\gamma v(\mu x) \quad (23)$$

if and only if $\gamma = n - 2$ and $q = \frac{1}{m-1} = \frac{n}{n-2}$, that is $m = 2 - \frac{2}{n}$, where $\mu > 0$ is a constant. If this exponent is the case, conversely, problem (23) admits a family of solutions each of which is necessarily radially symmetric and w_+^q has a compact support, see [21]. Then, we define the normalized solution $v_* = v_*(x)$ to (22) and the threshold $\lambda_* > 0$ of Theorem 2 by

$$-\Delta v_* = v_{*+}^q, \quad v_* \leq v_*(0) = 0 \quad \text{in } \mathbf{R}^n \quad \text{and} \quad \lambda_* = \int_{\mathbf{R}^n} v_{*+}^q dx,$$

respectively. The scaling property of the free energy

$$\mathcal{F}(u_\mu) = \mu^{n-2} \mathcal{F}(u) \quad (24)$$

now implies the following lemma.

Lemma 1 *It holds that*

$$j_* = \inf\{\mathcal{F}(u) \mid 0 \leq u \in L^m(\mathbf{R}^n), \int_{\mathbf{R}^n} u = \lambda_*\} = 0 \quad (25)$$

if $m = 2 - \frac{2}{n}$.

We can justify that the function

$$t \in [0, T) \mapsto \int_{\mathbf{R}^n} |x|^2 u(x, t) dx \in [0, +\infty)$$

is locally absolutely continuous and that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^n} |x|^2 u dx &= \frac{m-1}{m} \cdot 2n \int_{\mathbf{R}^n} u^m dx - (n-2) \langle \Gamma * u, u \rangle \\ &= 2(n-2) \mathcal{F}(u). \end{aligned} \quad (26)$$

Then the following lemma is proven [20].

Lemma 2 *If the initial value u_0 satisfies $\mathcal{F}(u_0) < 0$ and (15), then $T < +\infty$ arises.*

To show Theorem 2, first, we note that Wang-Ye's Trudinger-Moser inequality (25) is sharp. Thus it holds that

$$\inf\{\mathcal{F}(u) \mid u \geq 0, \text{supp } u \subset B_R, \int_{\mathbf{R}^n} u = \lambda\} = -\infty$$

for any $R > 0$ and $\lambda > \lambda_*$. Next if $\lambda = \|u_0\|_1 < \lambda_*$ is the case, we obtain

$$\sup_{t \in [0, T)} \|u(t)\|_m \leq C_1 \quad (27)$$

by (20) and (25). Then Moser's iteration scheme guarantees

$$\sup_{t \in [0, T)} \|u(t)\|_\infty < +\infty$$

and then $T = +\infty$ follows from (14).

3 Proof of Theorem 3

The first step to prove Theorem 3 is the ε -regularity stated below [20]. It is done by a standard argument of the localization of Lemma 1.

Theorem 4 *We have $\varepsilon_0 > 0$ and $C_2 > 0$ independent of $x_0 \in \mathbf{R}^n$ and $0 < R \ll 1$ such that*

$$\limsup_{t \uparrow T} \int_{B(x_0, R)} u(x, t) dx < \varepsilon_0$$

implies

$$\limsup_{t \uparrow T} \|u(t)\|_{L^\infty(B(x_0, R/2))} \leq C_2.$$

For the proof, we note

$$\begin{aligned} v &= \Gamma * u = v_1 + v_2 \\ v_1(x, t) &= \int_{|y-x| \geq 1} \Gamma(x-y) u(y, t) dy \\ v_2(x, t) &= \int_{|y-x| < 1} \Gamma(x-y) u(y, t) dy. \end{aligned}$$

Since

$$\|v_2\|_q \leq \|u\|_1 \|\Gamma \cdot \chi_B\|_q$$

for $B = B(0, 1)$ we obtain

$$\|v_1\|_\infty + \|v_2\|_q \leq C_3(q) \|u\|_1$$

for $1 \leq q < \frac{n}{n-2}$. Next, we introduce $\tilde{v}$ by

$$-\Delta \tilde{v} + \tilde{v} = v$$

and obtain $\tilde{v} = \tilde{v}_1 + \tilde{v}_2$ with

$$\begin{aligned} \|\tilde{v}_2\|_{W^{2,q}(\mathbf{R}^n)} &\leq C_4(q) && \text{for } 1 < q < \frac{n}{n-2} \\ \|\tilde{v}_1\|_{W^{2,r}(B(0,R))} &\leq C_5(R, r) && \text{for any } R > 0 \text{ and } 1 < r < \infty, \end{aligned} \quad (28)$$

using the above v_i , $i = 1, 2$. Then, $v = \tilde{v} + w$ holds with w solving

$$-\Delta w + w = u.$$

For this w we can apply the estimates of [14]. Thus we obtain Lemma 4 because estimate (28) is applicable to $\tilde{v}$.

Lemm 4 implies the boundedness of the blowup set S .

Lemma 3 *It holds that*

$$\limsup_{t \uparrow T} \|u(t)\|_{L^\infty(|x| > R)} \leq C_6 \quad (29)$$

for $R \gg 1$.

Proof: We have

$$\int_{\mathbf{R}^n} |x|^2 u(x, t) dx \leq C_7(T, u_0) \quad (30)$$

for

$$C_7(T, u_0) = 2(n-2)T\mathcal{F}(u_0) + \int_{\mathbf{R}^n} |x|^2 u_0 dx,$$

and hence it follows that

$$\sup_{t \in [0, T]} \int_{|x| > R} u(x, t) dx \leq \frac{1}{R^2} C_7(T, u_0).$$

Taking $R \gg 1$ as $C_7(T, u_0)R^{-2} < \varepsilon_0$, we obtain (29) by Lemma 3. ■

Given $x_0 \in \mathcal{S}$ and $0 < R \ll 1$, we take $0 \leq \varphi = \varphi_{x_0, R}(x) \in C_0^\infty(\mathbf{R}^n)$ satisfying $\text{supp } \varphi \subset \overline{B(x_0, 2R)}$ and $\varphi = 1$ on $B(x_0, R)$ and put

$$A(t) = \int_{\mathbf{R}^n} \varphi(x) u(x, t) dx.$$

We justify the formal calculation

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbf{R}^n} \varphi u dx \right|^2 &= \left| \int_{\mathbf{R}^n} u \nabla(u^{m-1} - \Gamma * u) \cdot \nabla \varphi dx \right|^2 \\ &\leq \int_{\mathbf{R}^n} u |\nabla(u^{m-1} - \Gamma * u)|^2 dx \cdot \int_{\mathbf{R}^n} u |\nabla \varphi|^2 dx \\ &\leq -\|\nabla \varphi\|_\infty^2 \lambda \frac{d}{dt} \mathcal{F}(u) \end{aligned} \quad (31)$$

which means

$$(A')^2 \leq -\frac{\|\nabla \varphi\|_\infty^2 \lambda}{2(n-2)} H''. \quad (32)$$

If

$$\lim_{t \uparrow T} \mathcal{F}(u(t)) > -\infty \quad (33)$$

is the case, therefore, it follows that

$$\int_0^T \left| \frac{d}{dt} \int_{\mathbf{R}^n} \varphi u dx \right| dt \leq T^{1/2} \left\{ \int_0^T \left| \frac{d}{dt} \int_{\mathbf{R}^n} \varphi u dx \right|^2 dt \right\}^{1/2} < +\infty$$

and hence

$$\lim_{t \uparrow T} A(t) = \lim_{t \uparrow T} \int_{\mathbf{R}^n} \varphi(x) u(x, t) dx \quad (34)$$

exists. Since Lemma 3 guarantees

$$\liminf_{t \uparrow T} A(t) = \limsup_{t \uparrow T} A(t) \geq \limsup_{t \uparrow T} \int_{B(x_0, R)} u(x, t) dx \geq \varepsilon_0,$$

we obtain

$$\liminf_{t \uparrow T} \int_{B(x_0, R)} u(x, t) dx \geq \varepsilon_0$$

for any $x_0 \in \mathcal{S}$, and hence the finiteness of $\mathcal{S}$ by the total mass conservation (19).

In the other case of

$$\lim_{t \uparrow T} \mathcal{F}(u(t)) = -\infty, \quad (35)$$

we have $\mathcal{F}(u(t_0)) < 0$ for some $t_0 \in [0, T)$. We may assume $t_0 = 0$ without loss of generality. Inequality (26) then implies

$$\frac{dH}{dt} < 0 \quad (36)$$

for

$$H(t) = \int_{\mathbf{R}^n} |x|^2 u(x, t) dx$$

and hence there is $H(T) = \lim_{t \uparrow T} H(t) \geq 0$. If $H(T) = 0$ is the case, then

$$\lim_{t \uparrow T} \int_{|x| > \varepsilon} u(x, t) dx = 0$$

for any $\varepsilon > 0$ which implies $\mathcal{S} \subset \{0\}$ by Lemma 2. Thus we may assume $H(T) > 0$ furthermore.

Lemma 4 *It holds that*

$$\sup_{t' \in [t, \frac{t+T}{2}]} A(t') \leq A(t) + C_8(H(t) - H(T))^{1/2}. \quad (37)$$

Proof: Inequality (32) implies

$$\int_t^{t'} (t' - s) A'(s)^2 ds \leq \frac{\|\nabla \varphi\|_\infty^2 \lambda}{2(n-2)} (H(t) - H(t'))$$

for $0 \leq t \leq t' < T$ by $H'(t) \leq 0$, and therefore, it holds that

$$\begin{aligned} \left| A\left(\frac{t+t'}{2}\right) - A(t) \right|^2 &= \left| \int_t^{\frac{t+t'}{2}} A'(s) ds \right|^2 \\ &\leq \int_t^{\frac{t+t'}{2}} (t' - s)^{-1} ds \cdot \int_t^{t'} (t' - s) A'(s)^2 ds \\ &\leq \frac{\log 2}{2} \cdot \frac{\|\nabla \varphi\|_\infty^2}{n-2} \cdot \lambda \cdot (H(t) - H(t')) \\ &\leq \frac{\log 2}{2} \cdot \frac{\|\nabla \varphi\|_\infty^2}{n-2} \cdot \lambda \cdot (H(t) - H(T)) \end{aligned}$$

for $t' \in [t, T)$. This implies

$$A\left(\frac{t+t'}{2}\right) \leq A(t) + C_8(H(t) - H(T))^{1/2}$$

for $t' \in [t, T)$ and hence (37). ■

In the following proof the scaling property of (5), $m = 2 - \frac{2}{n}$ takes a role. In fact, if $u = u(x, t)$ is a solution, then $u_\mu(x, t) = \mu^n u(\mu x, \mu^n t)$ satisfies

$$\begin{aligned} u_{\mu t} &= \frac{m-1}{m} \Delta u_\mu^m - \nabla \cdot (u_\mu \nabla \Gamma * u_\mu), \quad u_\mu \geq 0 \quad \text{in } \mathbf{R}^n \times (0, T_\mu) \\ \int_{\mathbf{R}^n} u_\mu dx &= \int_{\mathbf{R}^n} u dx \quad \text{for } t \in [0, T_\mu), \end{aligned} \quad (38)$$

where $\mu > 0$ is a constant and $T_\mu = \mu^{-n} T$.

Lemma 5 *There is $t_0 \in [0, T)$ and $C_9 > 0$ such that if*

$$\int_{B(x_0, 4(T-t_1)^{1/n})} u(x, t_1) dx < \varepsilon_0/2$$

then it follows that

$$\sup_{B(x_0, (T-t_1)^{1/n}) \times [t_1 + \frac{1}{8}(T-t_1), t_1 + \frac{3}{8}(T-t_1)]} (T-t)u(x, t) \leq C_9, \quad (39)$$

where $x_0 \in \mathbf{R}^n$ and $t_1 \in [t_0, T)$.

Proof: We have

$$A(t_1) < \varepsilon_0/2$$

for

$$A(t) = \int_{\mathbf{R}^n} \varphi_{x_0, 4(T-t_1)^{1/n}}(x) u(x, t) dx$$

from the assumption and hence

$$\sup_{t' \in [t_1, \frac{T+t_1}{2}]} A(t') < \varepsilon_0 \quad (40)$$

if $0 < T - t_0 \ll 1$, $t_1 \in [t_0, T)$ by Lemma 4.

Here we use the scaling property (38) and put

$$\tilde{u}(x, t) = \mu^n u(\mu x + x_0, \mu^n t + t_1), \quad \mu^n + t_1 = \frac{T + t_1}{2}.$$

It holds that

$$\tilde{u}_t = \frac{m-1}{m} \Delta \tilde{u}^m - \nabla \cdot (\tilde{u} \nabla \Gamma * \tilde{u}), \quad \tilde{u} \geq 0, \quad \text{in } \mathbf{R}^n \times (0, 1) \quad (41)$$

with

$$\mu^n = \frac{T - t_1}{2} \quad (42)$$

and

$$\sup_{t \in (0,1)} \|\tilde{u}(t)\|_{L^1(B(0,2))} < \varepsilon_0 \quad (43)$$

by (40). In this case, we can argue similarly to [12] using the parabolic regularity concerning the local L^r norm uniformly in $r \geq 1$ and Moser's iteration scheme. The analogous result to Lemma 4,

$$\sup_{t \in [1/4, 3/4]} \|\tilde{u}(t)\|_{L^\infty(B(0,1))} \leq C_{10},$$

is obtained. This inequality means

$$\sup_{B(x_0, (T-t_1)^{1/n}) \times [t_1 + \frac{1}{8}(T-t_1), t_1 + \frac{3}{8}(T-t_1)]} (T - t_1)u(x, t) \leq C_{10}$$

and hence (39) for $C_9 = \frac{3}{4}C_{10}$. ■

Proof of Theorem 3: The finiteness of $\mathcal{S}$ will follow from

$$\inf_{x_0 \in \mathcal{S}_{II}} \lim_{r \downarrow 0} \liminf_{t \uparrow T} \int_{B(x_0, r)} u(x, t) dx \geq \varepsilon_0/2$$

because of the total mass conservation (19). Assuming the contrary, we have $x_0 \in \mathcal{S}_{II}$, $r_0 > 0$, and $t_j \uparrow T$ such that

$$\int_{B(x_0, 2r_0)} u(x, t_j) dx < \varepsilon_0/2$$

for $j = 1, 2, \dots$. Then we obtain

$$\sup_{y \in B(x_0, r_0)} \int_{B(y, 4(T-t_j)^{1/n})} u(x, t_j) dx < \varepsilon_0/2$$

for j sufficiently large, and, therefore,

$$\sup_{B(y, (T-t_j)^{1/n}) \times [t_j + \frac{1}{8}(T-t_j), t_j + \frac{3}{8}(T-t_j)]} (T - t)u(x, t) \leq C_9$$

by Lemma 5, where $y \in B(x_0, r_0)$ is arbitrary. Then, it follows that

$$\sup_{B(x_0, r_0) \times [t_j + \frac{1}{8}(T-t_j), t_j + \frac{3}{8}(T-t_j)]} (T - t)u(x, t) \leq C_9$$

and hence $\liminf_{t \uparrow T} (T - t) \|u(t)\|_{L^\infty(B(x_0, r_0))} < +\infty$, a contradiction. ■

4 Further Discussions

Using a compactness property of a solution sequence, we are able to show another aspect of the finiteness of type II blowup points. It is obvious that $S_* \subset S$ for S_* defined in the following theorem. This theorem may be compared with a non-degeneracy of the blowup point concerning the semilinear parabolic equation with sub-critical nonlinearity [5].

Theorem 5 *The set*

$$S_* = \{x_0 \in \mathbf{R}^n \mid \liminf_{t \uparrow T} \inf_{B(x_0, b(T-t)^{1/n})} (T-t)u(\cdot, t) > 0 \text{ for any } b > 0\}$$

is finite.

Proof: If

$$\inf_{x_0 \in S_*} \lim_{r \downarrow 0} \liminf_{t \uparrow T} \int_{B(x_0, r)} u(x, t) dx > 0$$

is not the case, we have $x_k \in S_*$, $r_k > 0$, $0 < T - t_{jk} < \frac{1}{j_k}$, $j, k = 1, 2, \dots$ such that

$$\int_{B(x_k, 2r_k)} u(x, t_{jk}) dx < \min \left\{ \frac{\varepsilon_0}{2}, \frac{1}{2k} \right\}. \quad (44)$$

Given k , we have j_k such that

$$\sup_{y \in B(x_k, r_k)} \int_{B(y, 4(T-t_{jk})^{1/n})} u(x, t_{jk}) dx < \frac{\varepsilon_0}{2}$$

for $j \geq j_k$ which implies

$$\sup_{B(x_k, r_k) \times [t_{jk} + \frac{1}{8}(T-t_{jk}), t_{jk} + \frac{3}{8}(T-t_{jk})]} (T-t)u(x, t) \leq C_9 \quad (45)$$

by Lemma 5 with j_k replaced larger if necessary. We obtain, also,

$$\sup_{t \in [t_{jk}, \frac{1}{2}(T+t_{jk})]} \|u(t)\|_{L^1(B(x_k, 2r_k))} < \frac{1}{k} \quad (46)$$

by (44) and Lemma 4 under the same agreement.

Inequalities (45)-(46) imply

$$\sup_{B(x_k, r_k) \times [\frac{1}{4}, \frac{3}{4}]} \|u_{jk}(t)\|_{L^\infty(B(0, \mu_{jk}^{-1} r_k))} \leq C_{11}$$

$$\sup_{B(x_k, r_k) \times [0, 1]} \|u_{jk}(t)\|_{L^1(B(0, 2\mu_{jk}^{-1} r_k))} < \frac{1}{k}$$

for

$$\begin{aligned}\mu_{jk} &= \frac{1}{2}(T - t_{jk}) \\ u_{jk}(x, t) &= \mu_{jk}^n u(\mu_{jk}x + x_k, \mu_{jk}^n t + t_{jk}).\end{aligned}$$

Then passing to a subsequence of $\{j\}$ denoted by the same symbol, we have

$$u_{jk} \rightarrow u_k \quad \text{locally uniformly in } \mathbf{R}^n \times [\tfrac{3}{8}, \tfrac{5}{8}] \quad (47)$$

as $j \rightarrow \infty$ for $k = 1, 2, \dots$ by a diagonal argument and a parabolic regularity, where $u_k = u_k(x, t)$ is a solution to (5) satisfying

$$\begin{aligned}\sup_{t \in [\frac{3}{8}, \frac{5}{8}]} \|u_k(t)\|_{L^\infty(\mathbf{R}^n)} &\leq C_{11} \\ \sup_{t \in [\frac{3}{8}, \frac{5}{8}]} \|u_k(t)\|_{L^1(\mathbf{R}^n)} &\leq \frac{1}{k}.\end{aligned}$$

This relation implies

$$u_k \rightarrow 0 \quad \text{locally uniformly in } \mathbf{R}^n \times [\tfrac{3}{8}, \tfrac{5}{8}]$$

as $k \rightarrow \infty$. Given $b > 0$ and $\eta > 0$, therefore, we have

$$\|u_k\|_{L^\infty(B(0, 2b) \times [\frac{3}{8}, \frac{5}{8}])} < \frac{\eta}{2}$$

for a k sufficiently large, and, then, we have $j_{b, \eta, k}$ such that

$$\|u_{jk}\|_{L^\infty(B(0, 2b) \times [\frac{3}{8}, \frac{5}{8}])} < \eta \quad (48)$$

for any $j \geq j_{b, \eta, k}$. This inequality implies

$$\sup_{B(x_k, \mu_{jk}b) \times [t_{jk} + \frac{3}{16}(T - t_{jk}), t_{jk} + \frac{5}{16}(T - t_{jk})]} (T - t)u(x, t) < \eta$$

and hence

$$\liminf_{t \uparrow T} (T - t) \|u(t)\|_{L^\infty(B(x_k, b(T-t)^{1/n}))} \leq \eta,$$

by sending $j \rightarrow \infty$, so that

$$\liminf_{t \uparrow T} (T - t) \|u(t)\|_{L^\infty(B(x_k, b(T-t)^{1/n}))} = 0$$

because $\eta > 0$ is arbitrary. This relation contradicts $x_k \in \mathcal{S}_*$. ■

We say that $x(t) \in \mathbf{R}^n$ attains a positive local maximum if $u(\cdot, t)$ is positive in a neighborhood of $x(t)$ and $x(t)$ is its local maximizer.

Theorem 6 *If $\#\mathcal{S} = +\infty$, there are infinite number of $x_0 \in \mathcal{S}$ satisfying that each $b > 0$ admits $t_0 \in [0, T)$ such that $x(t) \notin B(x_0, b(T-t)^{1/n})$ for any $t \in [t_0, T)$, provided that $x(t)$ attains a positive local maximum of $u(\cdot, t)$ such that*

$$\limsup_{t \uparrow T} u(x(t), t) = +\infty \quad (49)$$

and

$$\liminf_{t \uparrow T} \inf_{B(x_0, b(T-t)^{1/n})} u(\cdot, t) > 0 \quad (50)$$

for any $b > 0$.

Proof: If $\#\mathcal{S} = +\infty$ is the case, there are infinite number of $x_0 \in \mathcal{S} \setminus \mathcal{S}_*$. Since $x(t)$ attains a positive local maximum of $u(\cdot, t)$, it follows that

$$\dot{m} \leq m^2$$

for $m(t) = u(x(t), t)$, see [4], and hence

$$m(t) = u(x(t), t) \geq (T - t)^{-1}$$

holds by (49). From this inequality if $|x(t_k) - x_0| \leq C(T - t_k)^{1/n}$, $t_k \uparrow T$, then we have $x_0 \in \mathcal{S}_*$, a contradiction. The proof is complete. ■

A natural question evoked by Theorem 6 is the existence a radially symmetric shock wave concentrating toward a blowup point. A formal dimension analysis of [7] applies to formulate such a solution. Thus we assume a radially symmetric bulk moving to the origin of which distance from the origin, the height, and the thickness are $R(t)$, $h(t)$, and $\mu(t)$, respectively, provided with the property $0 < \mu(t) \ll R(t) = o(1)$. At this bulk we have $\frac{\partial}{\partial r} \sim \frac{1}{\mu}$, $r \sim R$, and $u \sim h$ so that $|\frac{u_r}{r}| \sim \frac{h}{\mu R} \ll |u_{rr}| \sim \frac{h}{\mu^2}$. We have $|v_r| \ll |v_{rr}|$ similarly. Then (5) is reduced to

$$u_t = u_{rr}^m - (uv_r)_r, \quad -v_{rr} = u \quad (51)$$

which implies

$$-v_{rrt} = u_{rr}^m + \frac{1}{2}(v_r^2)_{rr},$$

and, hence

$$(v_r)_t + [u^m + \frac{1}{2}v_r^2]_r = 0. \quad (52)$$

Since $r = R(t)$ is regarded as a wavefront of u , the propagation speed of this wave is formulated by $c = \dot{R}(t)$. Then the Rankine-Hugoniot condition to equation (52) describing a conservation law reads

$$c[v_r]_{R(t)} = [u^m + \frac{1}{2}v_r^2]_{R(t)} = [\frac{1}{2}v_r^2]_{R(t)},$$

where

$$[\zeta]_{R(t)} = \lim_{r \downarrow R(t)} \zeta(r) - \lim_{r \uparrow R(t)} \zeta(r) = \zeta(R(t)^+) - \zeta(R(t)^-),$$

see [13]. Using the second equation of (43) assumed for $u = \frac{M}{\omega_{n-1}R^{n-1}}\chi_{r=R}$ with $M = \|u(t)\|_1$, we can readily derive

$$v_r(R(t)^+, t) = -\frac{M}{\omega_{n-1}R(t)^{n-1}}, \quad v_r(R(t)^-, t) = 0.$$

Therefore, it follows that

$$\dot{R}(t) = -\frac{M}{2\omega_{n-1}R(t)^{n-1}}$$

and hence

$$R(t) \sim (T - t)^{1/n}, \quad t \uparrow T.$$

Next, we plug in $u \sim h$ and $r \sim \mu$ to (43), which should be valid at the bulk. Then we obtain $v \sim \mu^2 h$ from the second equation, and hence $(uv_r)_r \sim h^2$. Now the first equation assures

$$\frac{h^m}{\mu^2} \sim h^2$$

and hence $h \sim \mu^{-n}$ by $m = 2 - \frac{2}{n}$. We have, on the other hand,

$$\omega_{n-1}R^{n-1}\mu h \sim M.$$

Therefore, it follows that

$$\mu \sim (T - t)^{1/n}, \quad h \sim (T - t)^{-1}, \quad t \uparrow T.$$

This case contradicts the ansatz $0 < \mu \ll R$. What we actually do is to replace $\frac{u_r}{r}$ by u_{rr} since $\mu \sim R$. Then all the above formal asymptotic rates are justified.

This blowup rate is type I and hence $x_0 \notin \mathcal{S}_{II}$. Whether $x_0 \in \mathcal{S}_*$ or not is delicate due to the condition (50). Actually this condition is violated

in the case of the higher-dimensional Smoluchowski-Poisson equation, see [7]. In this connection, we remind that the non-existence of the non-trivial (backward) self-similar solution with a finite mass to (5) is proven similarly to [9].

The type II blowup point, on the other hand, will be realized using the stationary state provided with the quantized mass. Such a blowup pattern is also examined in the higher-dimensional Smoluchowski-Poisson equation [8]. According to these study, we expect also that these type I and type II blowup patterns will be stable and unstable, respectively.

Any local mass is of bounded variation in time around the above described type I and type II blowup points, so that will be totally finite by the ε -regularity. Thus it seems to be difficult to realize an infinite number of blowup points to (5) by a combination of essentially radially symmetric blowup profiles.

The next theorem, see [16] for the proof, shows that any blowup point is type II if the free energy is bounded. A similar fact is shown to the semilinear parabolic equation with critical Sobolev growth, see [17]. We mention also that the Herrero-Velázquez solution [6] for the two-dimensional Smoluchowski-Poisson equation (7) has the same profile, boundedness of the free energy and type II blowup rate.

Theorem 7 *If (33) holds, then each $x_0 \in \mathcal{S}$ is type II. We have, more precisely, the formation of collapse*

$$u(x, t)dx \rightarrow \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \quad (53)$$

as $t \uparrow T$ in $\mathcal{M}(\mathbf{R}^n)$ with $m(x_0) > 0$, $x_0 \in \mathcal{S}$ and $0 \leq f = f(x) \in L^1(\mathbf{R}^n) \setminus C(\mathbf{R}^n \setminus \mathcal{S})$ and also (17) for any $b > 0$.

The scaling (38) induces the backward self-similar transformation

$$\begin{aligned} v(y, s) &= (T - t)u(x, t) \\ y &= (x - x_0)/(T - t)^{1/n} \\ s &= -\log(T - t). \end{aligned} \quad (54)$$

It follows that

$$\begin{aligned} v_s &= \frac{m-1}{m} \Delta v^m - \nabla \cdot v \nabla (\Gamma * v + \frac{|y|^2}{2n}), \quad v \geq 0 \\ &\text{in } \mathbf{R}^n \times (-\log T, +\infty). \end{aligned} \quad (55)$$

Then there arises the decrease of the free energy and its recursive relation between the second moment. They are, formally, given by

$$\begin{aligned} \frac{d}{ds} \hat{\mathcal{F}}(v) &= - \int_{\mathbf{R}^n} v \left| \nabla \left(v^{m-1} - \Gamma * v - \frac{|y|^2}{2n} \right) \right|^2 dy \leq 0 \\ \frac{d}{ds} \int_{\mathbf{R}^n} |y|^2 v dy &= 2(n-2) \hat{\mathcal{F}}(v) + \int_{\mathbf{R}^n} |y|^2 v dy, \end{aligned} \quad (56)$$

where

$$\hat{\mathcal{F}}(v) = \left\{ \int_{\mathbf{R}^n} \left(\frac{v^m}{m} - \frac{|y|^2}{2n} v \right) dy - \frac{1}{2} \langle \Gamma * v, v \rangle \right\}.$$

Equation (55) is actually written as

$$v_t = \nabla \cdot v \nabla \delta \hat{\mathcal{F}}(v) \quad \text{in } \mathbf{R}^n \times (-\log T, +\infty)$$

and hence the first equality of (56) reads

$$\frac{d}{ds} \hat{\mathcal{F}}(v) = - \int_{\mathbf{R}^n} v \left| \nabla \delta \hat{\mathcal{F}}(v) \right|^2 dy.$$

Relation (56) now implies

$$\frac{d}{ds} \int_{\mathbf{R}^n} |y|^2 v dy \leq 2(n-2) \hat{\mathcal{F}}(v_0) + \int_{\mathbf{R}^n} |y|^2 v dy, \quad (57)$$

The assumption

$$2(n-2) \hat{\mathcal{F}}(v_0) + \int_{\mathbf{R}^n} |y|^2 v_0 dy < 0$$

induces the contradiction, $\int_{\mathbf{R}^n} |y|^2 v dy < 0$ for $s \gg 1$ by (57). Thus, it holds that

$$2(n-2) \hat{\mathcal{F}}(v_0) + \int_{\mathbf{R}^n} |y|^2 v_0 dy \geq 0,$$

which must be translated in s

$$\frac{d\hat{H}}{ds} = 2(n-2) \hat{\mathcal{F}}(v) + \hat{H} \geq 0, \quad s > -\log T, \quad (58)$$

where $\hat{H} = \int_{\mathbf{R}^n} |y|^2 v(y, s) dy$. Inequality (58) means

$$\frac{d}{dt} \log \left\{ \frac{H(t)}{(T-t)^{2/n}} \right\} \geq 0,$$

see [16] for a direct proof.

The notion of regular blowup points arises in accordance with the mass quantization, $m(x_0) = \lambda_*$ in (53). First, we shall show the estimate of collapse mass from below. A blowup point x_0 is called isolated if $\mathcal{S} \cap B(x_0, R) = \{x_0\}$ and non-degenerate if

$$\liminf_{t \uparrow T} \inf_{x \in B(x_0, R)} u(x, t) > 0,$$

where $0 < R \ll 1$. The following lemma is proven in [16].

Lemma 6 *If $T < +\infty$ occurs to (5) and $x_0 \in \mathcal{S}$ is an isolated non-degenerate blowup point, then it holds that*

$$\limsup_{t \uparrow T} \mathcal{F}(\varphi^{1/m} u(t)) < +\infty, \quad (59)$$

where $\varphi = \varphi_{x_0, R}$ with $0 < R \ll 1$.

Theorem 8 *If (53) holds and $x_0 \in \mathcal{S}$ is isolated and non-degenerate, then we obtain $m(x_0) \geq \lambda_*$ in (53).*

Proof: Since $x_0 \in \mathcal{S}$ is non-degenerate, we have $0 < R \ll 1$ and $0 \leq f = f(x) \in L^1(B(x_0, 2R)) \cap C(B(x_0, 2R) \setminus \{x_0\})$ such that any $t_k \uparrow T$ admits $\{t'_k\} \subset \{t_k\}$ and $m(x_0) \geq 0$ satisfying

$$u(x, t'_k) dx \rightharpoonup m(x_0) \delta_{x_0}(dx) + f(x) dx.$$

If $m(x_0) < \lambda_*$ is the case, we obtain

$$\|u(t'_k)\|_{L^m(B(x_0, R))} \leq C_{11}.$$

Using (53), we follow the argument of [11] and obtain the result. ■

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