

N-Fractional Calculus Operator Method to Some Second Order homogeneous Euler's Equation

Tsuyako Miyakoda and Katsuyuki Nishimoto

Abstract

In this article, the solutions to homogeneous second order Euler's equation

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot az + \varphi \cdot b = 0, \quad (z \neq 0)$$

where

$$\varphi_0 = \varphi = \varphi(z), \quad \varphi_\alpha = \frac{d^\alpha \varphi}{dz^\alpha} \text{ (for } \alpha > 0)$$

are discussed by means of N- fractional calculus operator.

We have the following fractional differintegrated forms as particular solutions;

$$(i)\varphi = (z^{-(2\gamma+a)})_{-(1+\gamma)} \equiv \varphi_{[1](a,b)} \quad (\text{denote})$$

$$(\gamma = \frac{1}{2}(1 - a + \sqrt{p}), \quad p \neq 0)$$

$$(ii)\varphi = (z^{-(2\delta+a)})_{-(1+\delta)} \equiv \varphi_{[2](a,b)}$$

$$(\delta = \frac{1}{2}(1 - a - \sqrt{p}), \quad p \neq 0)$$

and

$$(iii)\varphi = (z^{-1})_{\frac{1}{2}(a-3)} \equiv \varphi_{[3](a,b)} \quad (p = 0)$$

where $p = (a - 1)^2 - 4b$.

1 Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, where C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ . (Here D contains the points over the curve C). And, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \quad (\nu \notin \mathbb{Z}^-), \end{aligned} \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C_-, \quad 0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in \mathbb{R}, \quad \Gamma; \text{ Gamma function,}$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

Notice that (1) is reduced to Goursat's integral for $\nu = n (n \in \mathbb{Z}^+)$ and is reduced to the famous Cauchy's integral for $\nu = 0$. that is, (1) is an extension of Cauchy's integral and of Goursat's integral, consequently, Cauchy and Goursat's integrals are special cases of (1).

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu f(z) = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}^-), \quad (\text{Refer to [1]}) \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$(N^\beta \circ N^\alpha) f = (N^\beta N^\alpha) f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| \leq \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) Lemma. We have [1]

(i)

$$((z - c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \alpha} \quad (|\frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(\log(z - c))_\alpha = -e^{i\pi\alpha} \Gamma(\alpha) (z - c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z - c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z - c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z - c \neq 0$ in (i), and $z - c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} u_{\alpha - k} v_k \quad (u = u(z), v = v(z))$$

2 Solutions to some homogeneous Euler's equation

Theorem 1. Let $\varphi \in F = \{\varphi : 0 \neq |\varphi_\nu| < \infty, \nu \in R\}$ then the second order homogeneous Euler's equation

$$L(\varphi; z; a, b) := \varphi_2 \cdot z^2 + \varphi_1 \cdot az + \varphi \cdot b = 0 \quad (z \neq 0) \quad (1)$$

$$(\varphi_\alpha = d^\alpha \varphi / dz^\alpha \quad \text{for } \alpha > 0, \varphi_0 = \varphi = \varphi(z))$$

has particular solutions of the forms in fractional differintegrate form as follows;

$$(i) \quad \varphi = (z^{-(2\gamma+a)})_{-(1+\gamma)} \equiv \varphi_{[1](a,b)} \quad (\text{denote}) \quad (2)$$

$$(\gamma = \frac{1}{2}\{-(a-1) + \sqrt{(a-1)^2 - 4b}\}, (a-1)^2 - 4b \neq 0)$$

$$(ii) \quad \varphi = (z^{-(2\delta+a)})_{-(1+\delta)} \equiv \varphi_{[2](a,b)} \quad (3)$$

$$(\delta = \frac{1}{2}\{-(a-1) - \sqrt{(a-1)^2 - 4b}\}, (a-1)^2 - 4b \neq 0)$$

and

$$(iii) \quad \varphi = (z^{-1})_{\frac{1}{2}(a-3)} \equiv \varphi_{[3](a,b)} \quad (4)$$

$$((a-1)^2 - 4b = 0).$$

Proof

We operate the N-fractional calculus operator of order α ($\alpha \notin Z^-$) N^α to the both sides of equation (1), then

$$(\varphi_2 \cdot z^2)_\alpha + (\varphi_1 \cdot az)_\alpha + (\varphi \cdot b)_\alpha = 0. \quad (5)$$

From the index low we notice

$$N^\alpha \varphi_m = (\varphi_m)_\alpha = \varphi_{m+\alpha} \quad (m = 2, 1) \quad (6)$$

And by Lemma (iv) we have

$$\begin{aligned} N^\alpha(\varphi_2 \cdot z^2) &= (\varphi_2 \cdot z^2)_\alpha \\ &= \sum_{k=0}^2 \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (\varphi_2)_{\alpha-k} (z^2)_k \end{aligned} \quad (7)$$

$$= \varphi_{2+\alpha} \cdot z^2 + \varphi_{1+\alpha} \cdot 2\alpha z + \varphi_\alpha \cdot \alpha(\alpha-1), \quad (8)$$

$$N^\alpha(\varphi_1 \cdot az) = (\varphi_1 \cdot az)_\alpha = \sum_{k=0}^1 \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (\varphi_1)_{\alpha-k} \cdot (az)_k \quad (9)$$

$$= \varphi_{1+\alpha} \cdot az + \varphi_\alpha \cdot a\alpha \quad (10)$$

and

$$N^\alpha(\varphi \cdot b) = \varphi_\alpha \cdot b. \quad (11)$$

Therefore we obtain

$$\varphi_{2+\alpha} \cdot z^2 + \varphi_{1+\alpha} \cdot (2\alpha + a)z + \varphi_\alpha \cdot \{\alpha^2 + \alpha(a-1) + b\} = 0. \quad (12)$$

We choose α such that

$$\alpha^2 + \alpha(a-1) + b = 0, \quad (13)$$

that is

$$\alpha = \frac{1}{2}\{-(a-1) + \sqrt{p}\} \equiv \gamma, \quad \alpha = \frac{1}{2}\{-(a-1) - \sqrt{p}\} \equiv \delta \quad (14)$$

where $p = (a-1)^2 - 4b$.

When $\alpha = \gamma$ and $p \neq 0$, we have

$$\varphi_{2+\gamma} \cdot z^2 + \varphi_{1+\gamma} \cdot z(2\gamma + a) = 0 \quad (15)$$

from (12) by applying (14). Setting

$$\psi = \psi(z) = \varphi_{1+\gamma}, \quad (\varphi = \psi_{-(1+\gamma)}) \quad (16)$$

and we obtain

$$\psi_1 \cdot z^2 + \psi \cdot z(2\gamma + a) = 0. \quad (17)$$

Then a particular solution to this equation is given by

$$\psi(z) = z^{-(2\gamma+a)} \quad (18)$$

Therefore we obtain

$$\varphi(z) = (z^{-(2\gamma+a)})_{-(1+\gamma)} = \varphi_{[1](a,b)}. \quad (19)$$

Inversely the function shown by (19) satisfy the equation (1) clearly. Since

$$\varphi_{1+\nu} = (z^{-(2\gamma+a)})_{-(1+\gamma)+1+\gamma} = z^{-(2\gamma+a)}, \quad (20)$$

and

$$\varphi_{2+\nu} = (z^{-(2\gamma+a)})_{-(1+\gamma)+2+\gamma} = (z^{-(2\gamma+a)})_1, \quad (21)$$

we have

$$LHS \text{ of } (15) = (z^{-(2\gamma+a)})_1 \cdot z^2 + (z^{-(2\gamma+a)}) \cdot z(2\gamma + a) = 0. \quad (22)$$

Therefore the function (2) satisfies the equation (1).

When $\alpha = \delta$ and $p \neq 0$, we have

$$\varphi_{2+\delta} \cdot z^2 + \varphi_{1+\delta} \cdot z(2\delta + a) = 0 \quad (23)$$

instead of (15). Therefore in the same way we obtain

$$\varphi(z) = (z^{-(2\delta+a)})_{-(1+\delta)} = \varphi_{[2](a,b)}. \quad (24)$$

When $p = 0$, the case is $\gamma = \delta = \frac{1}{2}(1 - a)$, we have

$$\varphi(z) = (z^{-1})_{\frac{1}{2}(a-3)} = \varphi_{[3](a,b)}. \quad (25)$$

Notice that in our N-fractional calculus operator (NFCO)-method, the original homogeneous linear second order ordinary differential equation (1) is reduced to a variable separable form one.

3 Familiar forms of Solutions

In this section we show the translated forms (familiar forms) of the solutions obtained in §2.

Theorem 2. The solutions shown in Theorem 1 are written like as the following familiar forms;

(i)

$$\begin{aligned}\varphi_{[1](a,b)} &= -e^{i\pi\gamma} \frac{\Gamma(\gamma + a - 1)}{\Gamma(2\gamma + a)} z^{-\gamma-a+1} \\ &\left(\left| \frac{\Gamma(\gamma + a - 1)}{\Gamma(2\gamma + a)} \right| < \infty, \quad \gamma = \frac{1}{2}(1 - a - \sqrt{p}), \quad p \neq 0 \right).\end{aligned}\quad (1)$$

(ii)

$$\begin{aligned}\varphi_{[2](\nu,b)} &= -e^{i\pi\delta} \frac{\Gamma(\delta + a - 1)}{\Gamma(2\delta + a)} z^{-\delta-a+1} \\ &\left(\left| \frac{\Gamma(\delta + a - 1)}{\Gamma(2\delta + a)} \right| < \infty, \quad \gamma = \frac{1}{2}(1 - a + \sqrt{p}), \quad p \neq 0 \right)\end{aligned}\quad (2)$$

(iii)

$$\varphi_{[3](a,b)} = \begin{cases} e^{i\pi\frac{1}{2}(3-a)} \Gamma(\frac{a}{2} - \frac{1}{2}) z^{\frac{1}{2}-\frac{a}{2}} & \text{for } \frac{1}{2}(a-3) \notin Z^-, p=0 \\ (\log z)_{-\frac{1}{2}(a-3)+1} & \text{for } \frac{1}{2}(a-3) \in Z^-, p=0 \end{cases} \quad (3)$$

where $p = (a-1)^2 - 4b$.

4 A Special case

When $a = 1$ and $b = -\nu^2$, we have the following corollary from Theorem 1.
Corollary 1. Let $\varphi \in F = \{\varphi : 0 \neq |\varphi_\nu| < \infty, \nu \in R\}$ then the second order homogeneous Euler's equation

$$L(\varphi; z; 1, -\nu^2) = 0 \quad (z \neq 0) \quad (1)$$

has particular solutions of the forms in fractional differintegrated form as follows;

(i)

$$\varphi = (z^{-(2\nu+1)})_{-(1+\nu)} = \varphi_{[1](1,-\nu^2)} \quad (2)$$

(ii)

$$\varphi = (z^{2\nu-1})_{\nu-1} = \varphi_{[2](1,-\nu^2)} \quad (3)$$

and

(iii)

$$\varphi = (z^{-1})_{-1} = \varphi_{[3](1,-\nu^2)} \quad (\text{when } \nu = 0) \quad (4)$$

5 Some Illustrative Example

[I] Let $a = 1$ and $b = -1$. The equation is

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot z - \varphi = 0 \quad (z \neq 0) \quad (1)$$

and the solutions are

$$\varphi = \varphi_{[1](1,-1)} = (z^{-3})_{-2} = \frac{1}{\Gamma(3)} z^{-1} \quad (2)$$

and

$$\varphi = \varphi_{[2](1,-1)} = (z)_0 = z. \quad (3)$$

The function given by (2) and (3) satisfy the equation (1) clearly.

[II] Let $a = 5$ and $b = 1$. The equation is

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot 5z + \varphi = 0 \quad (z \neq 0) \quad (4)$$

and the solutions are

$$\begin{aligned} \varphi = \varphi_{[1](5,1)} &= (z^{-1-2\sqrt{3}})_{1-\sqrt{3}} \\ &= K z^{-2-\sqrt{3}} \quad (K = e^{i\pi(\sqrt{3}-1)} \frac{1}{\Gamma(2+\sqrt{3})} \Gamma(1+2\sqrt{3})) \end{aligned} \quad (5)$$

and

$$\begin{aligned} \varphi = \varphi_{[2](5,1)} &= (z^{-1+2\sqrt{3}})_{1+\sqrt{3}} \\ &= K' z^{-2+\sqrt{3}} \quad (K' = e^{-i\pi(1+\sqrt{3})} \frac{1}{\Gamma(2-\sqrt{3})} \Gamma(1-2\sqrt{3})) \end{aligned} \quad (6)$$

The function given by (5) and (6) satisfy the equation (4) clearly.

Indeed in the case of (5), we have

$$\varphi_1 = K(-2 - \sqrt{3})z^{-3-\sqrt{3}} \quad (7)$$

and

$$\varphi_2 = K(2 + \sqrt{3})(3 + \sqrt{3})z^{-4-\sqrt{3}}. \quad (8)$$

Therefore we obtain

$$LHS \text{ of } (4) = K z^{-2-\sqrt{3}} \{(2 + \sqrt{3})(3 + \sqrt{3}) + 5(-2 - \sqrt{3}) + 1\} = 0. \quad (9)$$

6 Commentary

(I) Usually the Euler's equation is solved by the change of independent variable

$$z = e^t. \quad (1)$$

Since

$$\varphi_1 = \frac{d\varphi}{dz} = \frac{d\varphi}{dt} \cdot \frac{dt}{dz} = \frac{d\varphi}{dt} e^{-t}, \quad (2)$$

$$\varphi_2 = \frac{d^2\varphi}{dz^2} = \frac{d}{dz} \left(\frac{d\varphi}{dz} \right) = \left(\frac{d^2\varphi}{dt^2} - \frac{d\varphi}{dt} \right) e^{-2t}, \quad (3)$$

therefore, for example, for §5, (1) we have

$$\frac{d^2\varphi}{dt^2} - \varphi = 0. \quad (4)$$

Particular solutions to this equation are given by

$$\varphi = e^t = z \quad (5)$$

and

$$\varphi = e^{-t} = z^{-1}. \quad (6)$$

(II) Equation §2 (1)

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot az + \varphi \cdot b = 0 \quad (7)$$

can be solved with setting

$$\varphi = z^\lambda \phi \quad (\phi = \phi(z)) \quad (8)$$

too.

(III) Nonhomogeneous equations shall be discussed in a next paper.

References

- [1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.

- [2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.
- [3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.
- [4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.
- [5] K. Nishimoto ; Ring and Field Produced from The Set of N- Fractional Calculus Operator, J. Frac Calc. Vol. 24, Nov. (2003), 29 - 36.
- [6] K. Nishimoto ; An application of fractional calculus to the nonhomogeneous Gauss equations, J. Coll. Engng, Nihon Univ., B,-28 (1987), 1 - 8.
- [7] K. Nishimoto and S. L. Kalla ; Application of Fractional Calculus to Ordinary Differential Equation of Fuchs Type, Rev. Tec. Ing. Univ. Zulia, Vol. 12, No 1 (1989).
- [8] K. Nishimoto ; Application of Fractional Calculus to Gauss Type Partial Differential Equations, J. Coll. Engng, Nihon Univ., B,-30 (1989), 81 - 87.
- [9] Shih- Tong Tu, S.-J. Jaw and Shy -Der Lin ; An application of fractional calculus to Chebychev 's equation, Chung Yuan J., Vol. XIX (1990), 1 - 4.
- [10] K. Nishimoto, H. M. Srivastava and Shih -Tong Tu ; Application of Fractional Calculus in Solving Certain Classes of Fuchsian Differential Equations, J. Coll. Engng, Nihon Univ., B,-32 (1991), 119 - 126.
- [11] K. Nishimoto ; A Generalization of Gauss' Equation by Fractional Calculus Method, J. Coll. Engng, Nihon Univ., B,-32 (1991), 79 - 87.
- [12] Shy -Der Lin, Shih -Tong Tu and K. Nishimoto ; A generalization of Legendre's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 35 - 43.

- [13] N. S. Sohi, L. P. Singh and K. Nishimoto ; A generalization of Jacobi's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 45 - 51.
- [14] K. Nishimoto ; Solutions of Gauss equation in fractional calculus, J. Frac. Calc. Vol. 3, May(1993), 29 - 37.
- [15] K. Nishimoto ; Solutions of homogeneous Gauss equations, which have a logarithmic function, in fractional calculus, J. Frac. Calc. Vol. 5, May (1994), 11 - 25.
- [16] K. Nishimoto ; Application of N-transformation and N-fractional calculus method to nonhomogeneous Bessel equations (I), J. Frac. Calc. Vol. 8, Nov. (1995), 25 - 30.
- [17] K. Nishimoto ; Operator N^ν method to nonhomogeneous Gauss and Bessel equations , J. Frac. Calc. Vol. 9, May (1996), 1 - 15.
- [18] K. Nishimoto and Susana S. de Romero ; N-fractional calculus operator N^ν method to nonhomogeneous and homogeneous Whittaker equations (I), J. Frac. Calc. Vol. 9, May (1996), 17 - 22.
- [19] K. Nishimoto and Judith A. de Duran ; N-fractional calculus operator N^ν method to nonhomogeneous Fukuvara equations (I), J. Frac. Calc. Vol. 9, May (1996), 23 - 31.
- [20] K. Nishimoto ; N-fractional calculus operator N^ν method to nonhomogeneous Gauss equation, J. Frac. Calc. Vol. 10, Nov. (1996), 33 - 39.
- [21] K. Nishimoto ; Kummer's twenty-four functions and N-fractional calculus, Nonlinear Analysis, Theory, Method & Applications, Vol. 30, No.2 (1997), 1271 - 1282.
- [22] Shih -Tong Tu, Ding -Kuo Chyan and Wen -Chieh Luo ; Some solutions to the nonhomogeneous Jacobi equations Via fractional calculus operator N^ν methods, J. Frac. Calc. Vol. 12, Nov.(1997), 51 - 60.
- [23] Shih -Tong Tu, Ding -Kuo Chyan and Erh -Tsung Chin ; Solutions of Gegenbauer and Chebyshev equations via operator N^ν methods, J. Frac. Calc. Vol. 12, Nov.(1997), 61 - 69.

- [24] K. Nishimoto ; N-method to Hermite equations, J. Frac. Calc. Vol. 13, May (1998), 21 - 27.
- [25] K. Nishimoto ; N-method to Weber equations, J. Frac. Calc. Vol. 14, Nov. (1998), 9 - 21.
- [26] K. Nishimoto ; N-method to generalized Laguerre equations, J. Frac. Calc. Vol. 14, Nov. (1998), 1 - 8.
- [27] Shy -Der Lin, Jaw -Chian Shyu, K. Nishimoto and H. M. Srivastava ; Explicit Solutions of Some General Families of Ordinary and Partial Differential Equations Associated with the Bessel Equation by Means of Fractional Calculus, J. Frac. Calc. Vol. 25, May (2004), 33 - 45.
- [28] K. Nishimoto ; Solutions to Some Extended Hermite's Equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 29, May (2006), 45-56.
- [29] Tsuyako Miyakoda ; Solutions to An Extended Hermite's Equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 30, Nov. (2006), 23 - 32.
- [30] K. Nishimoto ; N-Fractional Calculus Operator Method to Chebyshev's Equations , J. Frac. Calc. Vol. 33, May (2008), 71- 90.
- [31] K. Nishimoto ; N-fractional Calculus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May (2005), 83 - 88.
- [32] K. Nishimoto ; N-fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006), 35 - 44.
- [33] David Dummit and Richard M. Foote ; Abstract Algebra, Prentice Hall (1991).
- [34] K. B. Oldham and J. Spanier ; The Fractional Calculus, Academic Press (1971).
- [35] A.C. McBride ; Fractional Calculus and Integral Transforms of Generalized Functions, Research Notes, Vol. 31, (1979), Pitman.
- [36] S.G. Samko, A.A. Kilbas and O. I. Marichev ; Fractional Integrals and Derivatives, and Some Their Applications (1987), Nauka, USSR.

- [37] K. S. Miller and B. Ross ; An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, (1993).
- [38] V. Kiryakova ; Generalized fractional calculus and applications, Pitman Research Notes, No.301, (1994), Longman.
- [39] A. Carpinteri and F. Mainard (Ed.); Fractals and Fractional Calculus in Continuum Mechanics, (1997), Springer, Wien, New York.
- [40] Igor Podlubny ; Fractional Differential Equations (1999), Academic Press.
- [41] R. Hilfer (Ed.) ; Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapore, New Jersey, London, Hong Kong.

Tsuyako Miyakoda
Osaka Institute of Technology,
Nakamiya 565 - 0871, Osaka, JAPAN
Fax : + 81 - 66 - 879 - 7871

Katsuyuki Nishimoto
Institute for Applied Mathematics,
Descartes Press Co.
2- 13- 10 Kaguike, Koriyama, 963 -
8833, JAPAN
Fax : + 81 - 24 - 922 - 7596