

The sets of non-escaping points of generalized Chebyshev mappings

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1 Introduction

Let G_c be the polynomial self-mapping of $\mathbf{C}^2$ defined by

$$G_c(x, y) = (x^2 - cy, y^2 - cx).$$

It admits an invariant line $\{x = y\}$ on which it acts as the quadratic polynomial

$$f_c(z) = z^2 - cz.$$

If c is real, the map G_c admits an invariant plane $\{x = \bar{y}\}$, on which it acts as

$$F_c(z) = z^2 - c\bar{z}.$$

The purpose of this paper is to understand the dynamics of F_c as a self-map of $\mathbf{C}$. The mapping G_c can be extended as a holomorphic self-map of $\mathbf{CP}^2$

$$g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2].$$

Ueda [1999] shows that any holomorphic map on $\mathbf{CP}^2$ of degree 2 is equivalent to one of the following maps :

- (1) $U_1([x : y : z]) = [x^2 : y^2 : z^2],$
- (2) $U_2([x : y : z]) = [x^2 + yz : y^2 : z^2],$
- (3) $U_3([x : y : z]) = [x^2 + yz : y^2 + xz : z^2],$
- (4) $U_4([x : y : z]) = [x^2 + \lambda xy + y^2 : z^2 + xy : yz].$

Note that g_c is equivalent to U_3 .

The map

$$F_c(z) = z^2 - c\bar{z}$$

has a connection with a physical model when $c = 2$. It is Chebyshev map

$$F_2(z) = z^2 - 2\bar{z}.$$

A. Lopes [1990,1992] considered the dynamics of F_2 as a special kind of Potts model and showed that triple point phase transition (three equilibrium states) exists. He conjectured that if $c > 2$, a Cantor set with expanding dynamics exists. It is known that for expanding systems equilibrium states are unique. We explain triple point phase transition. We consider the pressure

$$P(t) = \sup_{\nu \in M(f)} \{h(\nu) - \frac{t}{2} \int \log |\det(Df(z))| d\nu(z)\}.$$

$M(f)$ denotes the set of invariant probabilities and $h(\nu)$ is the entropy of ν . For each t , if the measure $\mu(t)$ is the solution of the variation problem, $\mu(t)$ is called the equilibrium measure. Multiple equilibrium measures of $F_2(z) = z^2 - 2\bar{z}$ are stated as follows :

$$(1) \text{ if } -\frac{4}{3} < t \text{ then } \mu(t) = \mu(= \frac{3}{\pi^2} \frac{1}{\sqrt{-(z\bar{z})^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27}} dx dy),$$

$$(2) \text{ if } t = -\frac{4}{3} \text{ then there exist triple point phase transition } \mu(t) :$$

$$\mu \text{ (not magnetic), } \frac{1}{2}\delta_{p_2} + \frac{1}{2}\delta_{p_3} \text{ (magnetic), } \delta_{p_1} \text{ (anti-ferromagnetic),}$$

$$(3) \text{ if } t < -\frac{4}{3} \text{ then there exist two equilibrium states } \mu(t) :$$

$$\frac{1}{2}\delta_{p_2} + \frac{1}{2}\delta_{p_3}, \delta_{p_1}.$$

We give an affirmative answer to Lopes's conjecture. More generally, we show an analogue of the result which are well known for quadatic polynomials. In the paper we assume that c is real.

2 Dynamics of $G_c(x, y)$ and $F_c(z)$

We show the following two theorems. Let $K(g) = \{z \in \mathbb{C} \mid g^n(z) : n = 0, 1, 2, \dots, \text{ is bounded}\}$.

Theorem 1. $K(F_c)$ is connected with the simply connected complement in $\mathbb{CP}^1$ if and only if $-4 \leq c \leq 2$.

Theorem 2. If $c > 2$, then

- (1) $K(F_c)$ is a Cantor set;
- (2) the two-dimensional Lebesgue measure of $K(F_c)$ is 0;
- (3) F_c restricted to $K(F_c)$ is topological conjugate to the shift on 4 symbols;
- (4) the measure of maximal entropy of $G_c(x, y)$ is supported in the real plane $\{x = \bar{y}\}$.

We see the analogue as follows .

Let $f_c(z) = z^2 + c$ and $F_c(z) = z^2 - c\bar{z}$.

(a) $K(f_c)$ is connected with the simply connected complement if and only if $-2 \leq c \leq \frac{1}{4}$.

(A) $K(F_c)$ is connected with the simply connected complement if and only if $-4 \leq c \leq 2$.

Note that $f_c(x)$ on $[-2, \frac{1}{4}]$ and $F_c(x)$ on $[-4, 2]$ are topological conjugate.

(b) If $c < -2$ then,

- (1) $K(f_c)$ is a Cantor set ;
- (2) the one dimensional Lebsgue measure of $K(f_c)$ is 0;
- (3) $\{K(f_c), f_c\}$ and $\{\Sigma_2, \sigma\}$ are equivalent;
- (4) Julia set of f_c is included in the set $[-q, q]$.

(B) If $c > 2$ then,

- (1) $K(F_c)$ is a Cantor set ;
- (2) the two dimensional Lebesgue measure of $K(F_c)$ is 0;
- (3) $\{K(F_c), F_c\}$ and $\{\Sigma_4, \sigma\}$ are equivalent;
- (4) the smallest Julia set of G_c is included in the set $\{x = \bar{y}\}$.

To prove the assertion (1) of Theorem 2, we show the following result for non-conformal maps F_c . If $c > 2$, for any connected component $K(i_1, \dots, i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, \dots, i_n)]$ approaches 0 as $n \rightarrow \infty$. To prove this we introduce a Riemannian metric

$$\frac{1}{\mu} \{(\bar{z}^2 - 3z)dz^2 + (9 - z\bar{z})dzd\bar{z} + (z^2 - 3\bar{z})d\bar{z}^2\},$$

$$\text{where } \mu = -z^2\bar{z}^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27.$$

This metric goes to ∞ on the boundary ∂S . This is a generalization of the invariant measure

$$\frac{1}{\pi\sqrt{x(1-x)}} \quad \text{for } f(x) = 4x(1-x).$$

3 Proofs

We show only the proof of the assertion (4) of Theorem 2 in this paper. Proofs of the other assertions of Theorem 2 and that of Theorem 1 are stated in Uchimura [2001] and so are omitted. In this paper we use the same definitions and notations as are used in Uchimura [2001].

Lemma 1. *The number of periodic points of order n of $g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2]$ with $z \neq 0$ is 4^n .*

Proof. From Corollary 3.2 of [Fornaess and Sibony, 1994], this lemma follows immediately. $\square$

Lemma 2. *If $c > 2$, the number of periodic points of order n of the function $F_c(z) = z^2 - c\bar{z}$ is 4^n .*

Proof. From the proof of Theorem 4.1 of [Uchimura, 2001], we see that there exists a positive integer n such that

$$(F_c)^{-n}(D_c) \subset \frac{c}{2}S.$$

Let N be the smallest integer satisfying the above property. Let

$$\Gamma = (F_c)^{-N}(\text{int}(D_c)).$$

Then it can be proved that Γ is an open connected set. From Proposition 2.2 of [Uchimura, 2001], we know that there exist homeomorphisms φ_k , ($k = 0, 1, 2, 3$), from $\frac{c^2}{4}S$ to S_k with $S_k \subset \frac{c}{2}S$ such that the composition $F_c \circ \varphi_k$ is an identity map. From Proposition 3.1 of [Uchimura, 2001], we have

$$(F_c)^{-1}(\Gamma) \subset \Gamma.$$

Hence
$$\bigcup_{k=0}^3 \varphi_k(\Gamma) \subset \Gamma$$

and so

$$\varphi_k(\Gamma) \subset \Gamma.$$

Applying Fixed Point Theorem to φ_k , we get a fixed point p_k in Γ such that $\varphi_k(p_k) = p_k$. Hence we have 4 fixed points of F_c .

By the similar argument, we can prove this lemma when $n > 1$.

□

Combining Lemma 1 and Lemma 2, we have the following proposition.

Proposition 3. *If $c > 2$, then any periodic point of $G_c(x, y)$ lies in the plane $\{(x, \bar{x}) | x \in \mathbf{C}\}$.*

Let $H = \{(x, \bar{x}) | x \in \mathbf{C}\}$. We denote the Jacobian matrix of the map $G_c(x, y)$ at the point (u, v) by $DG_c(u, v)$. G_c restricted on H is the map $F_c(z)$. The map $F_c(z)$ may be viewed as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$. We denote the Jacobian matrix of the map F_c at (u_1, u_2) by $DF_c(u)$ where $u = u_1 + iu_2$, $u_1, u_2 \in \mathbf{R}$.

Lemma 4. *We consider the map $G_c(x, y)$ when c is real. Let (u, v) be a periodic point. Suppose the periodic point (u, v) lies in H . Then the set of eigenvalues of $DG_c(u, \bar{u})$ are identical with that of $DF_c(u)$.*

Proof. Clearly,

$$DG_c(x, y) = \begin{pmatrix} 2x & -c \\ -c & 2y \end{pmatrix}.$$

Then

$$DG_c(u, \bar{u}) = \begin{pmatrix} 2(u_1 + iu_2) & -c \\ -c & 2(u_1 - iu_2) \end{pmatrix}.$$

On the other hand,

$$DF_c(u) = \begin{pmatrix} 2u_1 - c & -2u_2 \\ 2u_2 & 2u_1 + c \end{pmatrix}.$$

Set

$$U = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

Clearly U is an unitary matrix. Then we can easily prove that

$$U^{-1}DG_c(u, \bar{u})U = DF_c(u). \quad \square$$

In Proposition 3, we show that if $c > 2$, all periodic points of $G_c(x, y)$ lie in H . Next we show they are all repelling.

Proposition 5. *If $c > 2$, then any periodic point of $G_c(x, y)$ is repelling.*

Proof. From Lemma 4, we see that to prove this proposition it suffices to show that any periodic point of $F_c(z)$ is repelling. This follows from the fact that for any connected component $K(i_1, \dots, i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, \dots, i_n)]$ approaches to 0 as $n \rightarrow \infty$.
□

Combining Proposition 5 and Corollary V.2.1. in [Briend, 1997], we can prove the assertion (4) of Theorem 2. □

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