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Growth property of singular solutions of linear partial differential equations in the complex domain in $\mathbf{C}^{d+1}$

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ABSTRACT. Let $P(z, \partial)$ be a linear partial differential operator with coefficients holomorphic in a neighbourhood Ω of $z = 0$ in $\mathbf{C}^{d+1}$. Consider equation $P(z, \partial)u(z) = f(z)$, where $u(z)$ and $f(z)$ admit singularities on the surface $\{z_0 = 0\}$. We assume that $|f(z)| \leq A|z_0|^c$ in a region $\Omega(\theta)$ which is sectorial with respect to z_0 . The main result of this paper is the following:

There is an exponent γ^* such that for some class of operators if $\forall \varepsilon > 0 \exists C_\varepsilon$ such that $|u(z)| \leq C_\varepsilon \exp(\varepsilon|z_0|^{-\gamma^*})$ in $\Omega(\theta)$, then $|u(z)| \leq C|z_0|^{c'}$ for some constants c' and C .

First we give the notations briefly. The coordinates of $\mathbf{C}^{d+1}$ are denoted by $z = (z_0, z_1, \dots, z_d) = (z_0, z') \in \mathbf{C} \times \mathbf{C}^d$. $|z| = \max\{|z_i|; 0 \leq i \leq d\}$ and $|z'| = \max\{|z_i|; 1 \leq i \leq d\}$. Its dual variables are $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_d)$. $\mathbf{N}$ is the set of all nonnegative integers $\mathbf{N} = \{0, 1, 2, \dots\}$. The differentiation is denoted by $\partial_i = \partial/\partial z_i$, and $\partial = (\partial_0, \partial_1, \dots, \partial_d) = (\partial_0, \partial')$. For a multi-index $\alpha = (\alpha_0, \alpha') \in \mathbf{N} \times \mathbf{N}^d$, $|\alpha| = \alpha_0 + |\alpha'| = \sum_{i=0}^d \alpha_i$. Define $\partial^\alpha = \prod_{i=0}^d \partial_i^{\alpha_i}$. We denote $\partial^{\alpha'} = \prod_{i=1}^d \partial_i^{\alpha_i}$ by $\partial^{\alpha'}$.

We define spaces of holomorphic functions in some regions to state the results. Let $\Omega = \Omega_0 \times \Omega'$ be a polydisk with $\Omega_0 = \{z_0 \in \mathbf{C}^1; |z_0| < R\}$ and $\Omega' = \{z' \in \mathbf{C}^d; |z'| < R\}$ for some positive constant R . Put $\Omega_0(\theta) = \{z_0 \in \Omega_0 - \{0\}; |\arg z_0| < \theta\}$ and $\Omega(\theta) = \Omega_0(\theta) \times \Omega'$. $\mathcal{O}(\Omega)$ ($\mathcal{O}(\Omega')$, $\mathcal{O}(\Omega(\theta))$) is the set of all holomorphic functions on Ω (resp. Ω' , $\Omega(\theta)$). $\mathcal{O}(\Omega(\theta))$ contains multi-valued functions, if $\theta > \pi$.

We introduce $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ and $\text{Asy}_{\{\kappa\}}(\Omega(\theta))$, which are subspaces of $\mathcal{O}(\Omega(\theta))$ and fundamental function spaces in this paper.

Definition 1. $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ ($0 < \kappa < +\infty$) is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that for any $\varepsilon > 0$ and any θ' with $0 < \theta' < \theta$

$$(1) \quad |u(z)| \leq C \exp(\varepsilon|z_0|^{-\kappa}) \quad z \in \Omega(\theta')$$

holds for a constant $C = C(\varepsilon, \theta')$. We put $\mathcal{O}_{(+\infty)}(\Omega(\theta)) = \mathcal{O}(\Omega(\theta))$ for $\kappa = +\infty$.

Definition 2. $\mathcal{O}_{\text{reg},c}(\Omega(\theta))$ is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that any θ' with $0 < \theta' < \theta$

$$(2) \quad |u(z)| \leq C|z_0|^c \quad z \in \Omega(\theta')$$

holds for a constant $C = C(\theta')$.

We say that $u(z) \in \mathcal{O}(\Omega(\theta))$ is slowly increasing in $\Omega(\theta)$, if $u(z) \in \bigcup_{|c| < +\infty} \mathcal{O}_{\text{reg},c}(\Omega(\theta))$.

Now let $P(z, \partial)$ be an m -th order linear partial differential equation with coefficients in $\mathcal{O}(\Omega)$

$$(3) \quad P(z, \partial) = \sum_{|\alpha| \leq m} a_\alpha(z) \partial^\alpha = \sum_{|\alpha| \leq m} z_0^{j_\alpha} b_\alpha(z) \partial^\alpha,$$

where $j_\alpha \in \mathbf{N}$ is the valuation of $a_\alpha(z)$ with respect to z_0 , $a_\alpha(z) = z_0^{j_\alpha} b_\alpha(z)$. Let us define some quantities for $P(z, \partial)$:

$$(4) \quad \begin{cases} e_* := \min\{j_\alpha - \alpha_0; \alpha \in \mathbf{N}^{d+1}\}, \Delta = \{\alpha \in \mathbf{N}^{d+1}; j_\alpha - \alpha_0 = e_*\} \\ k^* := \max\{|\alpha|; \alpha \in \Delta\}. \end{cases}$$

Put

$$(5) \quad \mathfrak{P}(z, \partial) = \sum_{\alpha \in \Delta} z_0^{j_\alpha} b_{\alpha,0}(z') \partial^\alpha.$$

Let us introduce an index which plays an important role in this paper.

Definition 3. (*Minimal irregularity*)

$$(6) \quad \begin{cases} \gamma^* := \min\left\{\frac{j_\alpha - \alpha_0 - e_*}{|\alpha| - k^*}; \alpha \in \mathbf{N}^{d+1}, |\alpha| > k^*\right\}, & \text{if } k^* < m, \\ \gamma^* := \infty, & \text{if } k^* = m. \end{cases}$$

Let us introduce conditions on $P(z, \partial)$.

Condition 0. If $\alpha = (\alpha_0, \alpha') \in \Delta$, then $\alpha' = (0, 0, \dots, 0)$.

The following condition is more strict than Condition 0.

Condition 1. $P(z, \partial)$ satisfies Condition 0 and $b_{(k^*, 0, 0, \dots, 0)}(0) \neq 0$.

Suppose that $P(z, \partial)$ satisfies Condition 0. Then $\mathfrak{P}(z, \partial)$ is an ordinary differential operator,

$$(7) \quad \mathfrak{P}(z, \partial) = \sum_{\alpha \in \Delta} z_0^{e_*} b_{\alpha,0}(z') z_0^{\alpha_0} \partial_0^{\alpha_0},$$

and $\{z_0 = 0\}$ is regular singular. Define the indicial polynomial $\chi_P(z', \lambda)$ of $\mathfrak{P}(z, \partial)$,

$$(8) \quad \chi_P(z', \lambda) := \sum_{\alpha \in \Delta} b_{\alpha,0}(z') \lambda(\lambda - 1) \cdots (\lambda - \alpha_0 + 1).$$

Further suppose that $P(z, \partial)$ satisfies Condition 1. Then $\chi_P(z', \lambda)$ is a polynomial of λ with degree k^* in $\{z; |z| \leq R\}$. Hence there exist real constants a_0, a_1 and b_0 such that all the roots of $\chi_P(z', \lambda) = 0$ for $|z| \leq R$ are contained in $\{\lambda; a_0 \leq \Re \lambda \leq a_1, |\Im \lambda| \leq b_0\}$.

Now let us consider

$$(Eq) \quad P(z, \partial)u(z) = f(z).$$

We have results concerning the growth properties of solutions of (Eq).

Theorem 4. Suppose that $P(z, \partial)$ satisfies Condition 1. Let $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of (Eq). Suppose that $f(z) \in \mathcal{O}_{\text{reg},c}(\Omega(\theta))$. Then there is a polydisk U centered at $z = 0$ such that $u(z) \in \mathcal{O}_{\text{reg},c'}(U(\theta))$ for any $c' < \min\{c - e_*, a_0\}$.

Theorem 5. Suppose that $P(z, \partial)$ satisfies Condition 0. Let $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of (Eq). Suppose that $f(z) \in \mathcal{O}_{\text{reg},c}(\Omega(\theta))$. Then there is a polydisk U centered at $z = 0$ and a constant c'' such that $u(z) \in \mathcal{O}_{\text{reg},c''}(U(\theta))$.

We show Theorem 4 by constructing a parametrix and Theorem 5 follows from Theorem 4. The proof theorems and the details of this paper will be appeared in the forthcoming paper.

We give some examples satisfying Condition 1:

(a). Operators of normal type with respect to ∂_0 ,

$$\partial_0^{k^*} + \sum_{\alpha_0 < k^*} a_\alpha(z) \partial^\alpha.$$

(b). Operators of Fuchsian type.

(c). Other concrete examples are

$$I_d + z_0^2 \partial_0 + z_0 \partial_1^2, \quad z_0 \partial_0^2 + a(z) \partial_0 + \partial_1^3.$$

The present paper follows Ōuchi [4]. The class of operators considered in [4] was more strict than that of this paper. The main Theorem in [4] was the following:

If $u(z)$ grows at most some exponential order near $z_0 = 0$, that is, for any $\varepsilon > 0$ $|u(z)| \leq C_\varepsilon \exp(\varepsilon |z_0|^{-\gamma^*})$ near $z_0 = 0$, and if $f(z)$ behaves asymptotically $f(z) \sim \sum_{n=0}^{+\infty} f_n(z') z_0^n$ as $z_0 \rightarrow 0$ in a sectorial region $\Omega(\theta)$, where $|f_n(z')| \leq AB^n \Gamma(n/\gamma^* + 1)$, then $u(z)$ has also the asymptotic expansion like $f(z)$ as z_0 tends to 0.

It was an extension of the main result of [1] and [2]. But in the present paper we treat a wider class of operators which contains that of [4]. So even if $f(z)$ has a Gevrey type asymptotic expansion, $u(z)$ does not always have. Hence, Theorem 4 in this paper is somewhat different. Roughly speaking,

if $u(z)$ grows at most some exponential order near $\{z_0 = 0\}$, and if $f(z)$ has the slowly increasing singularities on $\{z_0 = 0\}$, then the growth order of singularities of $u(z)$ are also slowly increasing.

We can show the results in [4], by using Theorem 4.

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