

ON SUPERMANIFOLDS ASSOCIATED WITH THE COTANGENT BUNDLE

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INTRODUCTION

We study here the problem of classification of complex analytic supermanifolds. Clearly, with any holomorphic vector bundle E over a complex manifold M one can associate the so-called split supermanifold $(M, \wedge \mathcal{E})$, where $\mathcal{E}$ is the sheaf of holomorphic sections of E . On the other hand, each supermanifold $(M, \mathcal{O})$ can be deformed into a split one which is called the retract of $(M, \mathcal{O})$. Thus, our problem is reduced to the problem of classification of holomorphic vector bundles and to the problem of classification of complex analytic supermanifolds with a given retract. We give here a survey of results concerning the second problem. We consider the case when $E = T(M)^*$ is the cotangent bundle of M , though some important facts exposed in Sections 1 and 3 are valid in the general case. Thus, we deal mainly with the problem of classification of complex supermanifolds with retract (M, Ω) , where Ω is the sheaf of holomorphic forms on a complex manifold M .

Section 1 contains necessary definitions and some preliminary facts, including the theorem of Green reducing our classification problem to a problem of non-abelian cohomology theory. In Section 2 we give a direct construction of supermanifolds with retract (M, Ω) starting from a d -closed $(1, 1)$ -form or from a holomorphic line bundle on M (see [11]). In particular, we see that for any compact Kähler manifold M with $\dim M > 1$ there exist non-split supermanifolds of this sort. In Section 3 we construct a non-abelian cochain complex in the sense of [8, 12], whose 1-cohomology set gives a solution of our problem. This complex is actually of a type considered by Nijenhuis and Richardson [7] in connection with the deformation theory of algebras, i.e., it is related to a differential graded Lie superalgebra. The corresponding differential Lie superalgebra was introduced in [10]; its elements are derivations of the sheaf of smooth differential forms on M . For a compact manifold M , our complex gives rise to a finite-dimensional affine algebraic variety which can serve as a moduli variety for our classification problem; it is analogous to the Kuranishi family of complex structures on a compact manifold (see [5]). The detailed exposition of this theory see in [13, 15]. Section 4 contains applications to the case when M is a flag manifold.

1991 *Mathematics Subject Classification*. Primary 18G50, 32C11, 32C35, 58A50.

Key words and phrases. Complex supermanifold, tangent sheaf, non-abelian cochain complex, cohomology set, flag manifold, Π -symmetric supergrassmannian.

Work supported in part by the Russian Foundation for Fundamental Research (Grant 98-01-00329).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

1. COMPLEX SUPERMANIFOLDS

We consider here complex analytic supermanifolds, i.e., $\mathbb{Z}_2$ -graded ringed spaces $(M, \mathcal{O})$ locally isomorphic to $(\tilde{U}, \bigwedge_{\mathcal{F}_n}(\xi_1, \dots, \xi_m))$, where $\tilde{U}$ is an open subset of $\mathbb{C}^n$ and $\mathcal{F}_n$ the sheaf of holomorphic functions in $\mathbb{C}^n$ and the exterior algebra sheaf $\mathcal{F}_{n|m} = \bigwedge_{\mathcal{F}_n}(\xi_1, \dots, \xi_m)$ is $\mathbb{Z}_2$ -graded in the usual way. Such a local isomorphism gives us a *chart* on an open subset $U \subset M$. The coordinates $z_1, \dots, z_n$ of $\mathbb{C}^n$ are called *even local coordinates* on U , while $\xi_1, \dots, \xi_m$ are called *odd ones*. If U and V are two open subsets of M admitting two charts with local coordinates x_i ($i = 1, \dots, n$), ξ_j ($j = 1, \dots, m$) and y_i ($i = 1, \dots, n$), η_j ($j = 1, \dots, m$), then in $U \cap V$ we can write

$$(1) \quad \begin{aligned} y_i &= \varphi_i(x_1, \dots, x_n, \xi_1, \dots, \xi_m), \quad i = 1, \dots, n; \\ \eta_j &= \psi_j(x_1, \dots, x_n, \xi_1, \dots, \xi_m), \quad j = 1, \dots, m, \end{aligned}$$

where φ_i, ψ_j are, respectively, even and odd sections of $\mathcal{F}_{n|m}$ called the *transition functions*. We write $\dim(M, \mathcal{O}) = n|m$.

Here is a classical example of a complex supermanifold. Let M be a complex manifold of dimension n . By definition, this is a ringed space $(M, \mathcal{F})$, where $\mathcal{F}$ is the sheaf of holomorphic functions on M . Extending this sheaf to the sheaf $\Omega = \bigoplus_{p=0}^n \Omega^p$ of holomorphic exterior forms on M , we get the graded ringed space (M, Ω) . This is a supermanifold of dimension $n|n$. In fact, let U be an open subset of M , where a chart with local coordinates $x_1, \dots, x_n$ is defined. Clearly, the sheaf $\Omega|_U$ can be identified with $\bigwedge_{\mathcal{F}_n}(dx_1, \dots, dx_n)$. Denoting $\xi_j = dx_j$, we see that x_i, ξ_j are local coordinates for (M, Ω) . If V is another open subset with local coordinates y_i and $\eta_j = dy_j$, then the transition functions in $U \cap V$ have the form

$$(2) \quad \begin{aligned} y_i &= \varphi_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \\ \eta_j &= \sum_{k=1}^n \frac{\partial y_j}{\partial x_k} \xi_k, \quad j = 1, \dots, n, \end{aligned}$$

where φ_i are the usual transition functions for M .

The transition functions (2) are very simple: y_i do not depend on ξ_j , while η_j contain only terms of degree 1 in ξ_j . We express this fact by saying that (M, Ω) is a *split* complex supermanifold. Quite similarly, we may associate a split complex supermanifold with any holomorphic vector bundle E over a complex manifold M ; our example corresponds to the case $E = T(M)^*$ (the cotangent bundle).

Consider now the following problem: how can we add to (2) additional terms of degrees 2, 4 etc. for y_i and of degrees 1, 3 etc. for η_j , in order to get a supermanifold structure on M , whose structure sheaf $\mathcal{O}$ is not isomorphic to Ω ? The supermanifolds obtained in this way are called *non-split supermanifolds with retract* (M, Ω) , and we would like to classify them up to isomorphism. A similar problem can be posed for an arbitrary holomorphic vector bundle E .

For the complex grassmannians $M = \text{Gr}_{n,k}$ (and more generally, for complex manifolds of flags), examples of supermanifolds with retract (M, Ω) were given by Manin. These are the so-called *Π -symmetric supergrassmannians* $\Pi \text{Gr}_{n|n,k|k}$ defined in [6]. It is proved in [9] that $\Pi \text{Gr}_{n|n,k|k}$ is non-split whenever $n > 2$.

The supermanifolds with a given retract can be classified in terms of the 1-cohomology with values in an automorphism sheaf of the structure sheaf of the

retract. In our case, consider the sheaf $\mathcal{A}ut_{(2)}\Omega$ of automorphisms a of the $\mathbb{Z}_2$ -graded algebra sheaf Ω such that $a(\psi) - \psi \in \bigoplus_{p \geq 2} \Omega^p$ for any $\psi \in \Omega$. The group $\text{Aut } \mathbf{T}(M)^*$ acts on the automorphism sheaf of Ω by inner automorphisms leaving invariant the subsheaf $\mathcal{A}ut_{(2)}\Omega$ and hence on the 1-cohomology of this subsheaf. If $(M, \mathcal{O})$ is a supermanifold with retract (M, Ω) , then we may assume that its transition functions (1) have the functions (2) as their first terms and thus are obtained from (2) by an automorphism $g_{UV} \in \Gamma(U \cap V, \mathcal{A}ut_{(2)}\Omega)$. The following theorem (in a more general form) was proved by Green [2].

Theorem 1.1. *The automorphisms g_{UV} form a Čech 1-cocycle of an open cover of M with values in the sheaf $\mathcal{A}ut_{(2)}\Omega$. This correspondence gives rise to a bijection between the isomorphism classes of supermanifolds with retract (M, Ω) and the orbits of the group $\text{Aut } \mathbf{T}(M)^*$ on $H^1(M, \mathcal{A}ut_{(2)}\Omega)$ under the action described above. The split supermanifold (M, Ω) corresponds to the unit element $e \in H^1(M, \mathcal{A}ut_{(2)}\Omega)$.*

For an arbitrary complex supermanifold $(M, \mathcal{O})$, denote by $\mathcal{T} = \text{Der } \mathcal{O}$ the sheaf of derivations of the structure sheaf $\mathcal{O}$. The sheaf $\mathcal{T}$ is called the *tangent sheaf* of M . The tangent sheaf is in a natural way a sheaf of $\mathbb{Z}_2$ -graded left $\mathcal{O}$ -modules. On the other hand, it can be regarded as a sheaf of complex Lie superalgebras under the bracket

$$(3) \quad [u, v] = uv + (-1)^{p(u)p(v)+1}vu.$$

Sections of $\mathcal{T}$ (*holomorphic vector fields* on $(M, \mathcal{O})$) form the Lie superalgebra $\mathfrak{v}(M, \mathcal{O}) = \Gamma(M, \mathcal{T})$; it is finite-dimensional whenever M is compact.

In what follows, we shall use the cohomology groups $H^p(M, \mathcal{T})$ with values in the tangent sheaf; they are finite-dimensional vector spaces whenever M is compact. The bracket (3) induces a bracket in $H^*(M, \mathcal{T}) = \bigoplus_{p \geq 0} H^p(M, \mathcal{T})$ giving a graded Lie superalgebra that contains $H^0(M, \mathcal{T}) = \mathfrak{v}(M, \mathcal{O})$ as a subalgebra.

If the supermanifold $(M, \mathcal{O})$ is split, then $\mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p$ is a $\mathbb{Z}$ -graded sheaf of Lie superalgebras. E.g., for $\mathcal{O} = \Omega$ the grading is given by

$$\mathcal{T}_p = \text{Der}_p \Omega = \{v \in \mathcal{T} \mid v(\Omega^q) \subset \Omega^{q+p} \text{ for all } q \in \mathbb{Z}\}.$$

The structure of the sheaf $\mathcal{T} = \text{Der } \Omega$ is described by the following theorem proved essentially by Frölicher and Nijenhuis [1].

Theorem 1.2. *There is the following exact sequence of locally free analytic sheaves on M :*

$$0 \rightarrow \Omega^{p+1} \otimes \Theta \xrightarrow{i} \mathcal{T}_p \xrightarrow{\alpha} \Omega^p \otimes \Theta \rightarrow 0,$$

Here $\Theta = \text{Der } \mathcal{F}$ is the tangent sheaf of the manifold M , the mapping α is the restriction of a derivation of degree p onto the subsheaf $\mathcal{F}$, and i identifies any sheaf homomorphism $\Omega^1 \rightarrow \Omega^{p+1}$ with a derivation of degree p that is zero on $\mathcal{F}$.

This sequence is split, the splitting mapping $l : \Omega \otimes \Theta \rightarrow \mathcal{T}$ being defined by

$$l(\varphi) = [i(\varphi), d],$$

where d is the exterior derivative regarded as a section of $\mathcal{T}_1$.

Corollary. *There is the following decomposition into the direct sum of sheaves of vector spaces:*

$$\mathcal{T} = i(\Omega \otimes \Theta) \oplus l(\Omega \otimes \Theta).$$

Note that $\Omega \otimes \Theta$ is the so-called sheaf of holomorphic *vector-valued forms*. Also, for $p = 0$ the derivation $l(u)$, $u \in \Theta$, is the classical Lie derivative along the vector field u .

As in the classical Lie theory, there exists a natural relationship between automorphisms and derivations of the sheaf Ω (see [16]). Let us denote

$$\mathcal{T}_{\bar{0}(2p)} = \bigoplus_{k \geq p} \mathcal{T}_{2k}.$$

Then we have the exponential mapping

$$\exp : \mathcal{T}_{\bar{0}(2)} \rightarrow \mathcal{A}ut_{(2)}\Omega.$$

It is expressed by the usual exponential series which is actually a polynomial, since any $v \in \mathcal{T}_{\bar{0}(2)}$ satisfies $v^k = 0$ for any $k > [\frac{m}{2}]$. One proves that $\exp$ is bijective. Thus it is an isomorphism of sheaves of sets (but in general not of groups). We denote $\log = \exp^{-1}$. One proves that

$$(4) \quad \lambda_2 : \mathcal{A}ut_{(2)}\Omega \rightarrow \mathcal{T}_2,$$

where $\lambda_2(a)$ is the 2-component of $\log a \in \mathcal{T}_{\bar{0}(2)}$, is a homomorphism of sheaves of groups.

2. A CONSTRUCTION OF NON-SPLIT SUPERMANIFOLDS

Here we give a direct construction of non-split supermanifolds with retract (M, Ω) (see [11]). Let $\mathcal{Z}\Omega^1$ denote the subsheaf of Ω^1 consisting of closed forms and $\beta : \mathcal{Z}\Omega^1 \rightarrow \Omega^1$ the inclusion mapping. Consider the mapping $\mu : \mathcal{Z}\Omega^1 \rightarrow \mathcal{A}ut_{(2)}\Omega$ given by

$$\mu(\psi) = \exp(\psi d) = \text{id} + \psi d, \quad \psi \in \mathcal{Z}\Omega^1.$$

One verifies easily that this is a homomorphism of sheaves of groups. It follows that we have the cohomology homomorphism (i.e. a mapping, taking 0 to the unit element)

$$\mu^* : H^1(M, \mathcal{Z}\Omega^1) \rightarrow H^1(M, \mathcal{A}ut_{(2)}\Omega).$$

Using Theorem 1.2 and the homomorphism λ_2 given by (4), we get

Proposition 2.1. *Suppose that $\dim M > 1$ and that $\zeta, \zeta' \in H^1(M, \mathcal{Z}\Omega^1)$. If $\mu^*(\zeta) = \mu^*(\zeta')$, then $\beta^*(\zeta) = \beta^*(\zeta')$.*

Let $\mathfrak{U} = (U, V, \dots)$ be an open cover of M and let $\psi = (\psi_{UV})$ be a cocycle from $\mathcal{Z}^1(\mathfrak{U}, \mathcal{Z}\Omega^1)$. Then the above construction assigns to ψ the supermanifold given by the cocycle $g = (g_{UV}) \in \mathcal{Z}^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\Omega)$, where

$$(5) \quad g_{UV} = \text{id} + \psi_{UV} d.$$

Due to Theorem 1.1, we see from Proposition 2.1 that this supermanifold is non-split if and only if the cohomology class of ψ in $H^1(M, \Omega^1)$ is non-zero.

Now we pass to an important case, where a "closed cocycle" ψ appears. Let ω be a $(1,1)$ -form on M satisfying $d\omega = 0$. Then, clearly, $\bar{\partial}\omega = 0$, and hence ω determines a Dolbeault cohomology class $[\omega] \in H^{1,1}(M, \mathbb{C})$. If we denote by $D : H^{1,1}(M, \mathbb{C}) \rightarrow H^1(M, \Omega^1)$ the Dolbeault isomorphism, then it turns out that $D([\omega])$ can be represented by a closed Čech cocycle. Denote by $\Phi^{p,q}$ the sheaf of smooth complex-valued (p,q) -forms on M . Then we have the exact sequence of sheaves:

$$0 \rightarrow \mathcal{Z}\Omega^1 \rightarrow \Phi_{\partial}^{1,0} \xrightarrow{\bar{\partial}} \mathcal{Z}\Phi^{1,1} \rightarrow 0,$$

where $\Phi_{\partial}^{1,0} \subset \Phi^{1,0}$ is the subsheaf of ∂ -closed $(1,0)$ -forms and $\mathcal{Z}\Phi^{1,1} \subset \Phi^{1,1}$ the subsheaf of d -closed $(1,1)$ -forms. Consider the corresponding connecting homomorphism

$$\delta^* : \Gamma(M, \mathcal{Z}\Phi^{1,1}) \rightarrow H^1(M, \mathcal{Z}\Omega^1).$$

Then $\beta^*\delta^*\omega$ is the Dolbeault class of ω . As a result, we get the mapping

$$\mu^* \circ \delta^* : \Gamma(M, \mathcal{Z}\Phi^{1,1}) \rightarrow H^1(M, \text{Aut}_{(2)}\Omega).$$

Thus, any $(1,1)$ -form ω on M such that $d\omega = 0$ determines a supermanifold with retract (M, Ω) . To obtain an expression of the corresponding cocycle g , we consider an open cover $\mathcal{U} = (U, V, \dots)$ of M such that $\omega = \bar{\partial}\psi_U$ in any U , where $\psi_U \in \Phi_{\partial}^{1,0}(U)$. Then $\delta^*\omega$ is represented by the cocycle $\psi = (\psi_{UV}) \in Z^1(\mathcal{U}, \mathcal{Z}\Omega^1)$, where $\psi_{UV} = \psi_V - \psi_U$ in $U \cap V \neq \emptyset$. Finally, the cocycle g is given by (5).

Using Proposition 2.1, we deduce the following result.

Theorem 2.1. *If M is a compact Kähler manifold, then we have a linear mapping $\tilde{\delta} : H^{1,1}(M, \mathbb{C}) \rightarrow H^1(M, \mathcal{Z}\Omega^1)$ such that $\beta^* \circ \tilde{\delta} = D$. The mapping $\mu^* \circ \tilde{\delta} : H^{1,1}(M, \mathbb{C}) \rightarrow H^1(M, \text{Aut}_{(2)}\Omega)$ is injective, whenever $n > 1$, and takes 0 to e .*

Applying the above construction, we can associate a supermanifold $(M, \mathcal{O})$ with retract (M, Ω) with any holomorphic line bundle L over M . The closed $(1,1)$ -form ω will be here the curvature form of a Hermitian metric on L . More precisely, we have the mapping

$$\mu^* \circ \mathcal{D}^* : \text{Pic}(M) = H^1(M, \mathcal{F}^\times) \rightarrow H^1(M, \text{Aut}_{(2)}\Omega)$$

corresponding to the homomorphism of sheaves of groups

$$\mu \circ \mathcal{D} : \mathcal{F}^\times \rightarrow \text{Aut}_{(2)}\Omega,$$

where $\mathcal{D}$ is the logarithmic differential, i.e.,

$$\mathcal{D}f = f^{-1}df = d \log f, \quad f \in \mathcal{F}^\times.$$

Let $L \in \text{Pic}(M)$ be given by a cocycle $h = (h_{UV}) \in Z^1(\mathcal{U}, \mathcal{F}^\times)$. Then $(M, \mathcal{O})$ is determined by the following cocycle $g = (g_{UV}) \in Z^1(M, \text{Aut}_{(2)}\Omega)$:

$$g_{UV} = \text{id} + (h_{UV}^{-1}dh_{UV})d.$$

For example, the canonical line bundle $K_M = \bigwedge^n T(M)^*$ gives rise to a supermanifold called the *canonical supermanifold* over M . It corresponds to the canonical form defined by Koszul [4] and is not necessarily non-split.

3. A GENERAL CLASSIFICATION THEOREM

We retain the notation of the preceding sections. Here we are going to express the cohomology set $H^1(M, \text{Aut}_{(2)}\Omega)$ (see Theorem 1.1) in terms of differential forms on M . To do this, we use a non-linear complex similar to the non-linear de Rham and Dolbeault complexes studied, e.g., in [3, 8, 15]. Actually, a general complex of this sort was considered in [7], but it was used there only in the finite-dimensional situation. We consider here the split supermanifold (M, Ω) , but the cotangent bundle can be easily replaced by an arbitrary holomorphic vector bundle over M in all general theorems formulated below.

The first step is the construction of a fine resolution of the sheaf $\mathcal{T} = \text{Der } \Omega$. Theorem 1.2 implies that $\mathcal{T}$ is a locally free analytic sheaf on M , and hence we can form the standard Dolbeault—Serre resolution of $\mathcal{T}$. More precisely, we set

$$\begin{aligned}\mathcal{R}_{p,q} &= \Phi^{0,q} \otimes \mathcal{T}_p, \\ \mathcal{R} &= \bigoplus_{p \geq -1, q \geq 0} \mathcal{R}_{p,q}, \\ \bar{\partial}(\varphi \otimes u) &= (\bar{\partial}\varphi) \otimes u, \quad \varphi \in \mathcal{R}_{0,q}, u \in \mathcal{T}_p.\end{aligned}$$

Then the sequence

$$(6) \quad 0 \rightarrow \mathcal{T} \xrightarrow{i} \mathcal{R}_{*,0} \xrightarrow{\bar{\partial}} \mathcal{R}_{*,1} \xrightarrow{\bar{\partial}} \dots$$

is the desired resolution. However, it is convenient to write this resolution in a more complicated form, using derivations of the sheaf Φ of smooth forms. Our purpose is to obtain a resolution endowed with a bracket operation that extends the operation (3) given in $\mathcal{T}$.

Consider the sheaf of graded Lie algebras $\text{Der } \Phi$ and denote

$$\bar{D} = \text{ad } \bar{\partial}.$$

Clearly, $\bar{D}$ is a derivation of bidegree $(0, 1)$ of $\text{Der } \Phi$, and

$$\bar{D}^2 = \frac{1}{2}[\bar{D}, \bar{D}] = \frac{1}{2} \text{ad}[\bar{\partial}, \bar{\partial}] = 0.$$

Set

$$\mathcal{S} = \{u \in \text{Der } \Phi \mid u(\bar{f}) = u(df) = 0 \text{ for any } f \in \mathcal{F}\}.$$

One sees readily that $\mathcal{S}$ is a subsheaf of bigraded subalgebras of $\text{Der } \Phi$ that is invariant under $\bar{D}$. Moreover, $\mathcal{T}$ is identified with the kernel of the mapping $\bar{D} : \mathcal{S}_{*,0} \rightarrow \mathcal{S}_{*,1}$. Thus, we get the sequence

$$(7) \quad 0 \rightarrow \mathcal{T} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\bar{D}} \mathcal{S}_{*,1} \xrightarrow{\bar{D}} \dots$$

By [10], this is a fine resolution of $\mathcal{T}$ isomorphic to (6). Moreover, i is a homomorphism of graded Lie algebra sheaves, and hence the natural bracket in $\mathcal{S}$ may be used to calculate the bracket in $H^*(M, \mathcal{T})$ induced by the Lie bracket defined in $\mathcal{T}$. We also need the sheaf of groups

$$\mathcal{P}\text{Aut } \Phi = \{a \in \text{Aut } \Phi \mid a(\bar{\psi}) = \bar{\psi} \text{ for all } \psi \in \Omega\}.$$

and its subsheaf

$$\mathcal{P}Aut_{(2)}\Phi = \{a \in Aut\Phi \mid a(\psi) - \psi \in \bigoplus_{p \geq 2} \Phi^p, \psi \in \Phi\}.$$

The sheaf of groups $\mathcal{P}Aut_{(2)}\Phi$ acts on S by the automorphisms $\text{Int } a(u) = aua^{-1}$.

Consider now the triple (K^0, K^1, K^2) , where

$$K^0 = \Gamma(M, \mathcal{P}Aut_{(2)}\Phi), \quad K^p = \bigoplus_{k \geq 2} \Gamma(M, \mathcal{S}_{2k,p}), \quad p = 1, 2,$$

and define the mappings $\delta_0 : K^0 \rightarrow K^1$ and $\delta_1 : K^1 \rightarrow K^2$ by

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1}, \\ \delta_1(u) &= \bar{D}u - \frac{1}{2}[u, u] = -\frac{1}{2}[u - \bar{\partial}, u - \bar{\partial}]. \end{aligned}$$

Clearly, $\delta_1(0) = 0$.

Proposition 3.1.

(1) *The mapping δ_0 is a crossed homomorphism, i.e.,*

$$\delta_0(ab) = \delta_0(a) + a\delta_0(b)a^{-1}, \quad a, b \in K^0.$$

(2) *The corresponding affine action of K^0 on K^1 is given by*

$$\rho(a)(u) \stackrel{\text{def}}{=} \delta_0(a) + aua^{-1} = a(u - \bar{\partial})a^{-1} + \bar{\partial}.$$

(3) *The mapping δ_1 satisfies*

$$\delta_1(\rho(a)(u)) = a\delta_1(u)a^{-1}.$$

This proposition shows that the triple $K = (K^0, K^1, K^2)$ with coboundary mappings δ_p and actions Int of K^0 on K^p , $p = 1, 2$, is a non-abelian cochain complex in the sense of [8, 15]. In particular, we can define its 1-cohomology set

$$H^1(K) = \text{Ker } \delta_1 / \rho$$

with the distinguished point 0. Using the machinery of non-abelian complexes, we get the following result (see [13]).

Theorem 3.1. *We have an isomorphism of sets with distinguished points*

$$\nu : H^1(K) \rightarrow H^1(M, Aut_{(2)}\Omega).$$

The mapping ν can be expressed quite explicitly. Take $z \in K^1$ such that $\delta_1(z) = 0$. There exists an open cover $\mathcal{U} = (U, V, \dots)$ of M such that $z = \delta_0(a_U)$, where $a_U \in \Gamma(U, \mathcal{P}Aut_{(2)}\Phi)$ for any U . Define $b \in Z^1(\mathcal{U}, \mathcal{P}Aut_{(2)}\Phi)$ by $b_{UV} = a_U^{-1}a_V$. One sees that b_{UV} preserve the subsheaf $\Omega|_{U \cap V}$, and hence we may regard b as a cocycle from $Z^1(\mathcal{U}, Aut_{(2)}\Omega)$. Then ν maps the cohomology class of z onto that of b .

Example. Without going into details, we show, how to express the construction of Section 2 in terms of the complex K .

Let $\omega \in \Gamma(M, \Phi^{1,1})$ be a $(1,1)$ -form satisfying $d\omega = 0$. Consider the derivation $u = \omega\partial$ of Φ . Clearly, $u \in S_{2,1}$. Moreover, $\bar{D}u = [u, u] = 0$, and hence $\delta_1(u) = 0$. By Theorem 3.1, u determines a cohomology class $\tilde{u} \in H^1(M, \text{Aut}_{(2)}\Omega)$. One sees that $\tilde{u} = \mu^*\delta^*(u)$.

In the case when M is compact, Theorem 3.1 allows to use Hodge theory for constructing a moduli variety for our classification problem (see [13]). This variety is actually an algebraic subvariety of $H^1(M, \bigoplus_{k \geq 1} \mathcal{T}_{2k})$. Note the following simple case when this variety coincides with $H^1(M, \mathcal{T}_2)$.

Proposition 3.2. *If $H^1(M, \mathcal{T}_{2q}) = H^2(M, \mathcal{T}_{2q}) = 0$ for all $q \geq 3$, then $\lambda_2^* : H^1(M, \text{Aut}_{(2)}\Omega) \rightarrow H^1(M, \mathcal{T}_2)$ is an isomorphism.*

This can be deduced from Theorem 3.1 (a more direct proof see in [14]).

4. APPLICATIONS TO FLAG MANIFOLDS

In this section, we consider the case when M is a flag manifold of a connected semisimple complex Lie group G . We may identify M with the coset space G/P , where P is a parabolic subgroup of G . The subgroup P is determined by a subset $S \subset \Pi$, where Π is the system of simple roots of G . E.g., P is maximal whenever $|\Pi \setminus S| = 1$. Let Γ denote the subgroup of $\text{Aut } \Pi$ leaving S invariant. It is known that Γ can be interpreted as a group of biholomorphic transformations of M .

Since M is Kähler, the construction of Section 2 gives rise to a non-empty family of non-split supermanifolds having (M, Ω) as their retract. More precisely, Theorem 2.1 implies

Theorem 4.1. *Let $M = G/P$ is a flag manifold of dimension ≥ 2 , where G is simple, and denote $r = |\Pi \setminus S|$. Then there exists a family of distinct non-split supermanifolds parametrized by $\mathbb{CP}^{r-1}/\Gamma$ and having (M, Ω) as their common retract.*

If P is maximal, then this family consists of a unique supermanifold, which is isomorphic to the canonical one.

Now suppose that M is a simply connected irreducible compact Hermitian symmetric space. One proves (see [9]) that the conditions of Proposition 3.2 are satisfied. Moreover, our problem for these manifolds M has the following complete solution.

Theorem 4.2. *Suppose that M is a simply connected irreducible compact Hermitian symmetric space of dimension ≥ 2 .*

If $M = \text{Gr}_{n,s}$, $1 < s < n - 1$, then non-split supermanifolds with retract (M, Ω) are parametrized by $\mathbb{CP}^1/\Gamma$, where

$$\Gamma = \begin{cases} \mathbb{Z}_2 & \text{if } n = 2s \\ \{e\} & \text{otherwise.} \end{cases}$$

Otherwise, there exists (up to isomorphism) precisely one non-split supermanifold with retract (M, Ω) , namely, the canonical one.

It follows that the Π -symmetric supergrassmannian $\Pi \text{Gr}_{n|n,k|k}$ is not rigid, except of the case when $k = 1$ or $n - 1$, i.e., $M = \mathbb{CP}^{n-1}$.

In [9] the Lie superalgebra $\mathfrak{v}((M, \mathcal{O}))$ for all supermanifolds described in Theorem 4.2 is calculated. It is proved, in particular, that $\Pi \operatorname{Gr}_{n|n, k|k}$ is the only homogeneous non-split supermanifold with retract (M, Ω) , where M is a simply connected irreducible compact Hermitian symmetric space.

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