

THE UNIVERSAL DEFINING EQUATIONS OF ABELIAN SURFACES WITH LEVEL 3 STRUCTURE

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1. PRELIMINARIES

Let $A = \mathbb{C}^2/\tau\mathbb{Z}^2 + \mathbb{Z}^2$, $\tau \in \mathbb{H}_2$ be a principally polarized abelian surface, and we put $\Lambda = \Lambda_1 \oplus \Lambda_2 = \tau\mathbb{Z}^2 + \mathbb{Z}^2$. Let H be a hermitian form on $\mathbb{C}^2$ given by $(\text{Im } \tau)^{-1}$, and $E = \text{Im } H$, that is, $E(v, w) = \text{Im}(v(\text{Im } \tau)^{-1}\bar{w}) = {}^t v_1 w_2 - {}^t v_2 w_1$ for $v = \tau v_1 + v_2$ and $w = \tau w_1 + w_2$. In particular (1): $E(\Lambda, \Lambda) \subset \mathbb{Z}$. We define $\alpha : \Lambda \rightarrow \mathbb{C}^\times$ by $\alpha(\lambda) = (-1)^{{}^t \lambda_1 \lambda_2}$ for $\lambda = \tau \lambda_1 + \lambda_2$. Then (2): $\alpha(\lambda + \mu) = \alpha(\lambda)\alpha(\mu) \exp \pi i E(\lambda, \mu)$.

We put $L_0 = L(H, \alpha)$, that is the quotient of trivial line bundle $\mathbb{C} \times \mathbb{C}^2$ by the action of Λ

$$((a, v), \lambda) \mapsto (e_\lambda(v)a, v + \lambda) \quad a \in \mathbb{C}, v \in \mathbb{C}^2, \lambda \in \Lambda$$

$$e_\lambda(v) = \alpha(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)).$$

Let $L = L_0^k = L(kH, \alpha^k)$, $K(L) = \{x \in A \mid T_x^* L \cong L\} = A_k = \{x \in A \mid kx = 0\}$. We decompose $K(L) = K(L)_1 \oplus K(L)_2$. Then the Riemann-Roch theorem says

$$\dim H^0(A, L) = \#K(L)_1 = \#K(L)_2.$$

We can construct the standard basis of $H^0(A, L)$. Let B be a symmetric form on $\mathbb{C}^2$ given by $B(v, w) = {}^t v(\text{Im } \tau)^{-1} w$. We consider $M = L(H, \alpha)$ for any pair (H, α) which satisfies (1) and (2). For $x \in K(M)_1$, we define

$$\vartheta_x^M(v) = \exp\left(\frac{\pi}{2} B(v, v) - \frac{\pi}{2} (H - B)(x + 2v, x)\right) \sum_{\lambda \in \Lambda_1} \exp\left(\pi (H - B)(x + v, \lambda) - \frac{\pi}{2} (H - B)(\lambda, \lambda)\right),$$

then $\{\vartheta_x^M\}_{x \in K(M)_1}$ form a basis of $H^0(A, M)$.

Now we put $k = 3$, $L = L_0^3$. Then the canonical map

$$\varphi : \bigoplus_{n=0}^{\infty} \text{Sym}^n H^0(A, L) \longrightarrow \bigoplus_{k=0}^{\infty} H^0(A, L^n)$$

is surjective by the theorem of Koizumi([Ko1, Corollary 4.7]), and $\ker \varphi$ is generated by the elements of degree 2 and 3 by the theorem of Sekiguchi([S, Main Theorem]).

2. QUADRATIC EQUATIONS

We consider the map $\varphi_2 : \text{Sym}^2 H^0(A, L) \rightarrow H^0(A, L^2)$. Then $\dim \ker \varphi_2 = 45 - 36 = 9$, thus we have 9 linearly independent equations.

Lemma 1 (Addition formula). *We denote $Z_2 = K(L^2)_1 \cap A_2$. For any $x_1, x_2 \in K(L)_1$,*

$$\varphi_2(\vartheta_{x_1}^L \vartheta_{x_2}^L) = \sum_{z \in Z_2} \vartheta_{y_2+z}^{L^2}(0) \cdot \vartheta_{y+z}^{L^2}$$

with $y_1, y_2 \in K(L^2)_1$ such that $y + y_2 = x_1$ and $y - y_2 = x_2$.

For the proof, see [LB, (1.3), Chapter 7], or also see [M1, p339].

Now we use the following notation: $K(L^2)_1 = \{^t(a, b) \mid a, b \in \{0, 3, 6, 9, 12, 15\}\}$, $K(L)_1 = \{^t(a, b) \mid a, b \in \{0, 6, 12\}\}$ and $Z_2 = \{^t(a, b) \mid a, b \in \{0, 9\}\}$. Since $K(L^2)_1 = K(L)_1 \oplus Z_2$, we can take $K(L)_1$ as the representative system of $K(L^2)_1/Z_2$.

For $y \in K(L)_1$, let $W_y \subset H^0(A, L^2)$ be the space spanned by $\{\vartheta_{y+z}^{L^2}\}_{z \in Z_2}$ and $V_y \subset \text{Sym}^2 H^0(A, L)$ be the space spanned by $\{\vartheta_{y+u}^L \vartheta_{y-u}^L\}_{u \in K(L)_1}$. Then φ_2 maps V_y onto W_y , and we write $\varphi_2^y : V_y \rightarrow W_y$ the restriction of φ_2 . We decompose

$$\text{Sym}^2 H^0(A, L) = \bigoplus_{y \in K(L)_1} V_y, \quad H^0(A, L^2) = \bigoplus_{y \in K(L)_1} W_y.$$

For simplicity, we write $X_{a,b} = \vartheta_y^L$, $Y_{a,b} = \vartheta_x^{L^2}$ and $q(y) = \vartheta_y^{L^2}(0)$, with y (or x) $= \begin{pmatrix} a \\ b \end{pmatrix}$. Then φ_2^y is given by

$$\varphi_2^y(X_{y+u} X_{y-u}) = \sum_{z \in Z_2} q(u+z) Y_{y+z}.$$

For example the case of $y = ^t(0, 0) = 0$, we have

$$\varphi_2^0 \begin{pmatrix} X_{0,0}^2 \\ X_{6,0} X_{12,0} \\ X_{0,6} X_{0,12} \\ X_{6,6} X_{12,12} \\ X_{6,12} X_{12,6} \end{pmatrix} = \begin{pmatrix} q\begin{pmatrix} 0 \\ 0 \end{pmatrix} & q\begin{pmatrix} 0 \\ 6 \end{pmatrix} & q\begin{pmatrix} 0 \\ 9 \end{pmatrix} & q\begin{pmatrix} 0 \\ 15 \end{pmatrix} \\ q\begin{pmatrix} 6 \\ 0 \end{pmatrix} & q\begin{pmatrix} 6 \\ 6 \end{pmatrix} & q\begin{pmatrix} 6 \\ 9 \end{pmatrix} & q\begin{pmatrix} 6 \\ 15 \end{pmatrix} \\ q\begin{pmatrix} 9 \\ 0 \end{pmatrix} & q\begin{pmatrix} 9 \\ 6 \end{pmatrix} & q\begin{pmatrix} 9 \\ 9 \end{pmatrix} & q\begin{pmatrix} 9 \\ 15 \end{pmatrix} \\ q\begin{pmatrix} 15 \\ 0 \end{pmatrix} & q\begin{pmatrix} 15 \\ 6 \end{pmatrix} & q\begin{pmatrix} 15 \\ 9 \end{pmatrix} & q\begin{pmatrix} 15 \\ 15 \end{pmatrix} \\ q\begin{pmatrix} 6 \\ 12 \end{pmatrix} & q\begin{pmatrix} 3 \\ 3 \end{pmatrix} & q\begin{pmatrix} 15 \\ 12 \end{pmatrix} & q\begin{pmatrix} 15 \\ 3 \end{pmatrix} \end{pmatrix} \begin{pmatrix} Y_{0,0} \\ Y_{0,9} \\ Y_{9,0} \\ Y_{9,9} \end{pmatrix}.$$

We write M the 5×4 representation matrix above, and let M_k be the matrix removing the k -th row vector from M . Then for $h_k = (-1)^{k+1} \det M_k$, we see

$$(h_1, \dots, h_5) \cdot ^t(X_{0,0}^2, \dots, X_{6,12} X_{12,6}) = 0.$$

Theorem 1. *We have 9 quadratic equations:*

- (q1) $h_1 X_{0,0}^2 + h_2 X_{6,0} X_{12,0} + h_3 X_{0,6} X_{0,12} + h_4 X_{6,6} X_{12,12} + h_5 X_{6,12} X_{12,6} = 0,$
- (q2) $h_1 X_{6,0}^2 + h_2 X_{12,0} X_{0,0} + h_3 X_{6,6} X_{6,12} + h_4 X_{12,6} X_{0,12} + h_5 X_{12,12} X_{0,6} = 0,$
- (q3) $h_1 X_{12,0}^2 + h_2 X_{0,0} X_{6,0} + h_3 X_{12,6} X_{12,12} + h_4 X_{0,6} X_{6,12} + h_5 X_{0,12} X_{6,6} = 0,$
- (q4) $h_1 X_{0,6}^2 + h_2 X_{6,6} X_{12,6} + h_3 X_{0,12} X_{0,0} + h_4 X_{6,12} X_{12,0} + h_5 X_{6,0} X_{12,12} = 0,$
- (q5) $h_1 X_{6,6}^2 + h_2 X_{12,6} X_{0,6} + h_3 X_{6,12} X_{6,0} + h_4 X_{12,12} X_{0,0} + h_5 X_{12,0} X_{0,12} = 0,$
- (q6) $h_1 X_{12,6}^2 + h_2 X_{0,6} X_{6,6} + h_3 X_{12,12} X_{12,0} + h_4 X_{0,12} X_{6,0} + h_5 X_{0,0} X_{6,12} = 0,$
- (q7) $h_1 X_{0,12}^2 + h_2 X_{6,12} X_{12,12} + h_3 X_{0,0} X_{0,6} + h_4 X_{6,0} X_{12,6} + h_5 X_{6,6} X_{12,0} = 0,$
- (q8) $h_1 X_{6,12}^2 + h_2 X_{12,12} X_{0,12} + h_3 X_{6,0} X_{6,6} + h_4 X_{12,0} X_{0,6} + h_5 X_{12,6} X_{0,0} = 0,$
- (q9) $h_1 X_{12,12}^2 + h_2 X_{0,12} X_{6,12} + h_3 X_{12,0} X_{12,6} + h_4 X_{0,0} X_{6,6} + h_5 X_{0,6} X_{6,0} = 0.$

Here $h_k = (-1)^{k+1} \det M_k$ ($1 \leq k \leq 5$).

We can regard q or h as functions in τ . Then

$$q \begin{pmatrix} 3a \\ 3b \end{pmatrix} = \sum_{m \in \mathbb{Z}^2} \exp 6\pi i \tau [m - \frac{1}{12} \begin{pmatrix} 2a \\ 2b \end{pmatrix}],$$

and each h_i is quartic polynomial in these q 's, and belongs to $M_2(\Gamma^2(12))$.

Theorem 2. For each $1 \leq k \leq 5$, h_k is contained in the space $M_2(\Gamma^2(3), \varepsilon)$, with a character ε of $\Gamma^2(3)/\Gamma^2(12)$ such that $\varepsilon^2 = 1$.

For the proof of this theorem, we use the fact that the group $G = \Gamma^2(3)/\Gamma^2(12) \cong \Gamma^2/\Gamma^2(4)$, and G is generated by the elements

$$\begin{pmatrix} 1_2 & 3S \\ 0 & 1_2 \end{pmatrix}, \quad \begin{pmatrix} 1_2 & 0 \\ 3S & 1_2 \end{pmatrix}, \quad {}^tS = S.$$

The ring structure of the graded ring of Siegel modular forms of degree 2, level 3 is already known by Freitag and Salvati Manni [FS]. They showed

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma^2(3)) = \mathbb{C}[t_1, \dots, t_5, f_1, \dots, f_5]$$

with $t_1, \dots, t_5 \in M_1(\Gamma^2(3))$ and $f_1, \dots, f_5 \in M_3(\Gamma^2(3))$. They have 5 relations in weight 5, and 15 relations in weight 6.

From this fact, we can rewrite the above functions $h_1, \dots, h_5$ by using $t_1, \dots, t_5$ and $f_1, \dots, f_5$ as follows.

$$\begin{aligned} h_1^2 &= \frac{1}{216}(f_1 t_1 - t_1^4 + 4t_1(t_2^3 + t_3^3 + t_4^3 + t_5^3) - 24t_2 t_3 t_4 t_5), \\ h_1 h_2 &= \frac{1}{108}(f_2 t_2 + 3t_1^2 t_2^2 - 12t_1 t_3 t_4 t_5), \\ h_1 h_3 &= \frac{1}{108}(f_3 t_3 + 3t_1^2 t_3^2 - 12t_1 t_2 t_4 t_5), \\ h_1 h_4 &= \frac{1}{108}(f_4 t_4 + 3t_1^2 t_4^2 - 12t_1 t_2 t_3 t_5), \\ h_1 h_5 &= \frac{1}{108}(f_5 t_5 + 3t_1^2 t_5^2 - 12t_1 t_2 t_3 t_4). \end{aligned}$$

3. CUBIC EQUATIONS

Next we consider the map $\varphi_3 : \text{Sym}^3 H^0(A, L) \rightarrow H^0(A, L^2)$. We need $\dim \ker \varphi_3 = 165 - 81 = 84$ relations. However, in this case, all the generators of $\ker \varphi_3$ is given by the theorem of Birkenhake and Lange.

Let $Z_6 = A_6 \cap K(L^2)_1$. For $\rho \in \widehat{Z}_6 = \text{Hom}(Z_6, \mathbb{C}^\times)$, $y_1 \in K(L^6)_1$ and $y_2 \in K(L^2)_1$ we define

$$\theta_{(y_1, y_2), \rho}(v) = \sum_{a \in Z_6} \rho(a) \vartheta_{y_1 - a}^{L^6}(v) \vartheta_{y_2 - 3a}^{L^2}(v).$$

Theorem 3 (Cubic theta relations [BL, Theorem 3.3]). *Let L be an ample line bundle on A and assume $L = L_0^3$ for a line bundle L_0 . Then all the cubic theta relations are given by the following form:*

$$\begin{aligned} &\theta_{(y_1, y_2), \rho}(0) \sum_{b \in Z_6} \rho(b) \vartheta_{y'_1 + y'_2 + y_3 + 2b}^L \vartheta_{y'_1 - y'_2 + y_3 + 2b}^L \vartheta_{-2y'_1 + y_3 + 2b}^L \\ &= \theta_{(y'_1, y'_2), \rho}(0) \sum_{b \in Z_6} \rho(b) \vartheta_{y_1 + y_2 + y_3 + 2b}^L \vartheta_{y_1 - y_2 + y_3 + 2b}^L \vartheta_{-2y_1 + y_3 + 2b}^L. \end{aligned}$$

Here $\rho \in \widehat{Z}_6$, $y_1, y'_1 \in K(L^6)_1$, $y_2, y'_2 \in K(L^2)_1$ and $y_3 \in K(L^3)_1$ such that

$$\begin{cases} y_1 + y_2 + y_3, & y_1 - y_2 + y_3, & -2y_1 + y_3, \\ y'_1 + y'_2 + y_3, & y'_1 - y'_2 + y_3, & -2y'_1 + y_3 \end{cases}$$

belong to $K(L)_1$.

Using this theorem, we can write down all the 84 generators of $\ker \varphi_3$. Let $W_3 = \{0, 3, 6\}$, and $\widehat{Z}_6^+$ be the set of all the character ρ of Z_6 such that $\rho^3 \equiv 1$, that is, all the character of $W_3^2 \bmod 9 \cong (\mathbb{Z}/3\mathbb{Z})^2$. We define the character $\rho_1, \dots, \rho_4 \in \widehat{Z}_6^+$ by

$$\begin{cases} \rho_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 1, \\ \rho_1 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \omega. \end{cases} \quad \begin{cases} \rho_2 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \omega, \\ \rho_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 1. \end{cases} \quad \begin{cases} \rho_3 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \omega^2, \\ \rho_3 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \omega. \end{cases} \quad \begin{cases} \rho_4 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \omega, \\ \rho_4 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \omega. \end{cases}$$

For $\rho \in \widehat{Z}_6^+$, we define

$$\theta^\rho \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \sum_{a,b \in \mathbb{Z}/6\mathbb{Z}} \rho \begin{pmatrix} 3a \\ 3b \end{pmatrix} \Theta \begin{bmatrix} 6a - 2x & 6b - 2z \\ 18a - 2y & 18b - 2w \end{bmatrix},$$

with

$$\Theta \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau) := \sum_{N \in M_2(\mathbb{Z})} \exp \pi i \left(\begin{pmatrix} 18 & 0 \\ 0 & 6 \end{pmatrix} \left[N + \frac{1}{36} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \tau \right).$$

Theorem 4. *The following list contains all of the 84 linearly independent relations of degree 3.*

$$(c1) \quad \sum_{(a,b) \in K(L)_1} X_{a,b}^3 = 3 \frac{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{6,0} X_{12,0} + X_{0,6} X_{6,6} X_{12,6} + X_{0,12} X_{6,12} X_{12,12})$$

$$(c2) \quad = 3 \frac{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{0,6} X_{0,12} + X_{6,0} X_{6,6} X_{6,12} + X_{12,0} X_{12,6} X_{12,12})$$

$$(c3) \quad = 3 \frac{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{6,6} X_{12,12} + X_{0,6} X_{6,12} X_{12,0} + X_{0,12} X_{12,6} X_{6,0})$$

$$(c4) \quad = 3 \frac{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^1 \begin{pmatrix} 0 & 0 \\ 0 & 12 \end{pmatrix}} (X_{0,0} X_{6,12} X_{12,6} + X_{6,6} X_{0,12} X_{12,0} + X_{12,12} X_{0,6} X_{6,0})$$

$$(d1) \quad X_{0,0}^3 + X_{6,0}^3 + X_{12,0}^3 - X_{0,6}^3 - X_{6,6}^3 - X_{12,6}^3 = 3 \frac{\theta^{\rho_1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_1} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{6,0} X_{12,0} - X_{0,6} X_{6,6} X_{12,6}),$$

$$(d2) \quad X_{0,0}^3 + X_{6,0}^3 + X_{12,0}^3 - X_{0,12}^3 - X_{6,12}^3 - X_{12,12}^3 = 3 \frac{\theta^{\rho_1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_1} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{6,0} X_{12,0} - X_{0,12} X_{6,12} X_{12,12}),$$

$$(d3) \quad X_{0,0}^3 + X_{0,6}^3 + X_{0,12}^3 - X_{6,0}^3 - X_{6,6}^3 - X_{6,12}^3 = 3 \frac{\theta^{\rho_2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_2} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{0,6} X_{0,12} - X_{6,0} X_{6,6} X_{6,12}),$$

$$(d4) \quad X_{0,0}^3 + X_{0,6}^3 + X_{0,12}^3 - X_{12,0}^3 - X_{12,6}^3 - X_{12,12}^3 = 3 \frac{\theta^{\rho_2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_2} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{0,6} X_{0,12} - X_{12,0} X_{12,6} X_{12,12}),$$

$$(d5) \quad X_{0,0}^3 + X_{6,6}^3 + X_{12,12}^3 - X_{0,6}^3 - X_{6,12}^3 - X_{12,0}^3 = 3 \frac{\theta^{\rho_3} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_3} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{6,6} X_{12,12} - X_{0,6} X_{6,12} X_{12,0}),$$

$$(d6) \quad X_{0,0}^3 + X_{6,6}^3 + X_{12,12}^3 - X_{0,12}^3 - X_{6,0}^3 - X_{12,6}^3 = 3 \frac{\theta^{\rho_3} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_3} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}} (X_{0,0} X_{6,6} X_{12,12} - X_{0,12} X_{6,0} X_{12,6}),$$

$$(d7) \quad X_{0,0}^3 + X_{6,12}^3 + X_{12,6}^3 - X_{6,0}^3 - X_{12,12}^3 - X_{0,6}^3 = 3 \frac{\theta^{\rho_4} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_4} \begin{pmatrix} 0 & 0 \\ 0 & 12 \end{pmatrix}} (X_{0,0} X_{6,12} X_{12,6} - X_{6,0} X_{12,12} X_{0,6}),$$

$$(d8) \quad X_{0,0}^3 + X_{6,12}^3 + X_{12,6}^3 - X_{12,0}^3 - X_{0,12}^3 - X_{6,6}^3 = 3 \frac{\theta^{\rho_4} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\theta^{\rho_4} \begin{pmatrix} 0 & 0 \\ 0 & 12 \end{pmatrix}} (X_{0,0} X_{6,12} X_{12,6} - X_{12,0} X_{0,12} X_{6,6}),$$

$$(e1_\rho) \quad \sum_{a,b \in W_3} \rho \begin{pmatrix} a \\ b \end{pmatrix} X_{2a,6+2b}^2 X_{2a,12+2b} = \frac{\theta^\rho \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}}{\theta^\rho \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}} \left(\sum_{a,b \in W_3} \rho \begin{pmatrix} a \\ b \end{pmatrix} X_{6+2a,6+2b} X_{12+2a,6+2b} X_{2a,12+2b} \right),$$

$$(e2_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,2b}^2 X_{12+2a,2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 4 & 0 \\ 0 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b} X_{6+2a,12+2b} X_{12+2a,2b} \right),$$

$$(e3_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{2a,6+2b}^2 X_{2a,2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 0 & 0 \\ 2 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b} X_{12+2a,6+2b} X_{2a,2b} \right),$$

$$(e4_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,2b}^2 X_{2a,2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 2 & 0 \\ 0 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b} X_{6+2a,12+2b} X_{2a,2b} \right),$$

$$(e5_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b}^2 X_{12+2a,12+2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 4 & 0 \\ 4 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 4 & 0 \\ 4 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,12+2b} X_{6+2a,2b} X_{12+2a,12+2b} \right),$$

$$(e6_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b}^2 X_{12+2a,2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 4 & 0 \\ 2 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 4 & 0 \\ 2 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,12+2b} X_{6+2a,2b} X_{12+2a,2b} \right),$$

$$(e7_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b}^2 X_{2a,12+2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 2 & 0 \\ 4 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 2 & 0 \\ 4 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{12+2a,6+2b} X_{2a,6+2b} X_{2a,12+2b} \right),$$

$$(e8_\rho) \quad \sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,6+2b}^2 X_{2a,2b} = \frac{\theta^\rho \left(\begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix} \right)}{\theta^\rho \left(\begin{smallmatrix} 2 & 0 \\ 2 & 6 \end{smallmatrix} \right)} \left(\sum_{a,b \in W_3} \rho \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix} \right) X_{6+2a,12+2b} X_{6+2a,6+2b} X_{2a,2b} \right).$$

In the last 8 equations $(e1_\rho), \dots, (e8_\rho)$, ρ runs all the characters in $\widehat{Z}_6^+$.

For the coefficients, we have the following:

Theorem 5. For each $\theta^\rho \left(\begin{smallmatrix} 2x & 6y \\ 2z & 6w \end{smallmatrix} \right)$, there is a character χ on $\Gamma^2(3)$ such that $\chi^3 \equiv 1$, and $\theta^\rho \left(\begin{smallmatrix} 2x & 6y \\ 2z & 6w \end{smallmatrix} \right) \in M_1(\Gamma^2(3), \chi)$. These characters are trivial on $\Gamma^2(9)$ and depend only on x, z and ρ . In particular, all the coefficients of the defining equations in Theorem 4 are $\Gamma^2(3)$ -invariant meromorphic functions.

And we can show the following relations.

$$\begin{aligned} \theta^{\rho_1} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \theta^{\rho_1} \left(\begin{smallmatrix} 0 & 6 \\ 0 & 0 \end{smallmatrix} \right)^2 &= t_1 t_2^2 - t_3 t_4 t_5, \\ \theta^{\rho_1} \left(\begin{smallmatrix} 0 & 6 \\ 0 & 0 \end{smallmatrix} \right)^3 &= \frac{1}{24} (t_1^3 + 20 t_2^3 - 4 t_3^3 - 4 t_4^3 - 4 t_5^3 - f_1), \\ \theta^1 \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix} \right) \theta^1 \left(\begin{smallmatrix} 0 & 6 \\ 4 & 0 \end{smallmatrix} \right)^2 &= \frac{1}{9} (t_3 t_4 t_5 + t_1 t_2 t_4 + t_1 t_2 t_5 + t_1 t_4 t_5 + t_2^2 t_3 + t_2 t_3 t_4 + t_3 t_4^2 + t_3 t_5^2), \\ \theta^{\rho_1} \left(\begin{smallmatrix} 0 & 6 \\ 4 & 0 \end{smallmatrix} \right)^3 &= \frac{1}{216} (-t_1^3 + 4(t_2^3 + t_3^3 + t_4^3 + t_5^3) + f_1 + 6 t_1^2 t_3 \\ &\quad + 2 f_3 + 24(t_2 t_4 t_5 + t_2^2 t_4 + t_2 t_4^2 + t_2^2 t_5 + t_2 t_5^2 + t_4 t_5^2 + t_4^2 t_5)). \end{aligned}$$

To prove the theorem, we can show that the group $\Gamma^2(3)/\Gamma^2(36)$ are generated by the following elements:

$$\begin{aligned} &\begin{pmatrix} 1_2 & 3S \\ 0 & 1_2 \end{pmatrix}, \quad \begin{pmatrix} 1_2 & 0 \\ 3S & 1_2 \end{pmatrix}, \quad {}^t S = S, \\ &\begin{pmatrix} U_i & 0 \\ 0 & {}^t U_i^{-1} \end{pmatrix} \quad (1 \leq i \leq 3), \quad U_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}, \\ &\begin{pmatrix} 1_2 & 1_0^0 \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ 3_0^0 & 1_2 \end{pmatrix} \begin{pmatrix} 1_2 & -1_0^0 \\ 0 & 1_2 \end{pmatrix}. \end{aligned}$$

By the theory of the theta series of quadratic forms (cf. [A, Chapter 1, 2]), we can check the modularity for the above generators directory.

4. EXPLICIT FORM OF THE DEFINING EQUATIONS

Finally we consider the problem: find the relations derived from quadratic relations among the cubic relations. Since $\dim(\ker \varphi_2 \otimes H^0(A, L)) = 9 \times 9 = 81 < 84 = \dim \ker \varphi_3$, we need at least 3 cubic relations. In fact we have the following theorem.

Theorem 6 (Main Theorem). *Let $X_{00}, X_{01}, X_{02}, X_{10}, X_{11}, X_{12}, X_{20}, X_{21}$ and X_{22} be the coordinate of $\mathbb{P}^8$. The defining equations of an abelian surface $\mathbb{C}^2/(\tau\mathbb{Z}^2 + \mathbb{Z}^2)$ is given by the following 12 equations.*

$$\begin{aligned} h_1 X_{00}^2 + h_2 X_{10} X_{20} + h_3 X_{01} X_{02} + h_4 X_{11} X_{22} + h_5 X_{12} X_{21} &= 0, \\ h_1 X_{10}^2 + h_2 X_{20} X_{00} + h_3 X_{11} X_{12} + h_4 X_{21} X_{02} + h_5 X_{22} X_{01} &= 0, \\ h_1 X_{20}^2 + h_2 X_{00} X_{10} + h_3 X_{21} X_{22} + h_4 X_{01} X_{12} + h_5 X_{02} X_{11} &= 0, \\ h_1 X_{01}^2 + h_2 X_{11} X_{21} + h_3 X_{02} X_{00} + h_4 X_{12} X_{20} + h_5 X_{10} X_{22} &= 0, \\ h_1 X_{11}^2 + h_2 X_{21} X_{01} + h_3 X_{12} X_{10} + h_4 X_{22} X_{00} + h_5 X_{20} X_{02} &= 0, \\ h_1 X_{21}^2 + h_2 X_{01} X_{11} + h_3 X_{22} X_{20} + h_4 X_{02} X_{10} + h_5 X_{00} X_{12} &= 0, \\ h_1 X_{02}^2 + h_2 X_{12} X_{22} + h_3 X_{00} X_{01} + h_4 X_{10} X_{21} + h_5 X_{11} X_{20} &= 0, \\ h_1 X_{12}^2 + h_2 X_{22} X_{02} + h_3 X_{10} X_{11} + h_4 X_{20} X_{01} + h_5 X_{21} X_{00} &= 0, \\ h_1 X_{22}^2 + h_2 X_{02} X_{12} + h_3 X_{20} X_{21} + h_4 X_{00} X_{11} + h_5 X_{01} X_{10} &= 0, \end{aligned}$$

$$\begin{aligned} X_{00}^3 + X_{01}^3 + X_{02}^3 + X_{10}^3 + X_{11}^3 + X_{12}^3 + X_{20}^3 + X_{21}^3 + X_{22}^3 \\ = 3 \frac{t_1}{t_2} (X_{00} X_{10} X_{20} + X_{01} X_{11} X_{21} + X_{02} X_{12} X_{22}), \\ = 3 \frac{t_1}{t_3} (X_{00} X_{01} X_{02} + X_{10} X_{11} X_{12} + X_{20} X_{21} X_{22}), \\ = 3 \frac{t_1}{t_4} (X_{00} X_{11} X_{22} + X_{01} X_{12} X_{20} + X_{02} X_{21} X_{10}). \end{aligned}$$

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