

Solvability of complex Ginzburg-Landau equation in a general domain

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1 Introduction

In this paper we shall study the following complex Ginzburg-Landau equation in a general domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$:

$$(CGL) \begin{cases} \partial_t u - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u &= f & \text{in } \Omega \times (0, \infty), \\ u &= 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+ := (0, \infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $q \geq 2$ are constants; $i = \sqrt{-1}$ is the imaginary unit; $u_0 : \Omega \rightarrow \mathbb{C}$ is an initial function; $f : \Omega \times (0, \infty) \rightarrow \mathbb{C}$ is an external force; $u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{C}$ is a complex valued unknown function. In extreme cases, equation (CGL) includes two well-known equations: heat equation (when $\alpha = \beta = 0$) and Schrödinger equation (when $\lambda = \kappa = 0$). Thus we see that the equation (CGL) is “intermediate” between nonlinear heat and Schrödinger equations. From $\lambda > 0$, we can regard (CGL) as a parabolic type equation, and from $\kappa > 0$, we can find that (CGL) has a negative feedback mechanism in the nonlinear term. By these insights, we can expect “smoothing effect” and “global solvability”, respectively.

2 Notations and Preliminaries

In what follows, we identify $\mathbb{C}$ with $\mathbb{R}^2$: $u = u_1 + iu_2 \in \mathbb{C} \mapsto U = (u_1, u_2)^T \in \mathbb{R}^2$.

$$\begin{aligned} \mathbb{L}^2(\Omega) &:= L^2(\Omega) \times L^2(\Omega), & (U, V)_{\mathbb{L}^2} &:= (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}, \\ \mathbb{L}^q(\Omega) &:= L^q(\Omega) \times L^q(\Omega), & |U|_{\mathbb{L}^q}^q &:= |u_1|_{L^q}^q + |u_2|_{L^q}^q, \\ \mathbb{H}_0^1(\Omega) &:= H_0^1(\Omega) \times H_0^1(\Omega), & (U, V)_{\mathbb{H}_0^1} &:= (u_1, v_1)_{H_0^1} + (u_2, v_2)_{H_0^1}. \end{aligned}$$

We introduce the following matrix I , which is a linear operator in $\mathbb{R}^2$ into itself:

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We use the nabla symbol $\nabla = (D_1, \dots, D_N) : \mathbb{H}_0^1 \rightarrow (L^2)^N \times (L^2)^N$ as $\nabla U = (\nabla u_1, \nabla u_2)^T$. Then, the following properties are fundamental:

(i) Skew-symmetric property of the matrix I :

$$(IU \cdot V)_{\mathbb{R}^2} = -(U \cdot IV)_{\mathbb{R}^2}; \quad (IU \cdot U)_{\mathbb{R}^2} = 0 \quad \text{for each } U, V \in \mathbb{R}^2. \quad (2.1)$$

(ii) Commutative property of the matrix I and the differential operator D_i :

$$ID_i = D_i I : \mathbb{H}_0^1 \rightarrow \mathbb{L}^2 \quad (i = 1, \dots, N). \quad (2.2)$$

(iii) Consequences from orthogonality of a vector V and IV :

$$(U \cdot V)_{\mathbb{R}^2}^2 + (U \cdot IV)_{\mathbb{R}^2}^2 = |U|_{\mathbb{R}^2}^2 |V|_{\mathbb{R}^2}^2 \quad \text{for each } U, V \in \mathbb{R}^2; \quad (2.3)$$

$$(U, V)_{\mathbb{L}^2}^2 + (U, IV)_{\mathbb{L}^2}^2 \leq |U|_{\mathbb{L}^2}^2 |V|_{\mathbb{L}^2}^2 \quad \text{for each } U, V \in \mathbb{L}^2(\Omega). \quad (2.4)$$

Now we define two functionals $\varphi, \psi : \mathbb{L}^2(\Omega) \rightarrow (-\infty, +\infty]$ by

$$\varphi(U) := \frac{1}{2} \int_{\Omega} |\nabla U(x)|_{\mathbb{R}^2}^2 dx \quad (\text{if } U \in \mathbb{H}_0^1(\Omega)), \quad +\infty \quad (\text{otherwise}), \quad (2.5)$$

$$\psi(U) := \frac{1}{q} \int_{\Omega} |U(x)|_{\mathbb{R}^2}^q dx \quad (\text{if } U \in \mathbb{L}^q(\Omega) \cap \mathbb{L}^2(\Omega)), \quad +\infty \quad (\text{otherwise}). \quad (2.6)$$

Then subdifferential of these functionals are, respectively, single valued and

$$\partial\varphi(U)(\cdot) = -\Delta U(\cdot) \quad (\text{where } D(-\Delta) := \{U \in \mathbb{H}_0^1(\Omega) \mid \Delta U \in \mathbb{L}^2(\Omega)\}), \quad (2.7)$$

$$\partial\psi(U)(\cdot) = |U(\cdot)|_{\mathbb{R}^2}^{q-2} U(\cdot) \quad (\text{where } D(|\cdot|_{\mathbb{R}^2}^{q-2}) := \mathbb{L}^{2(q-1)}(\Omega) \cap \mathbb{L}^2(\Omega)). \quad (2.8)$$

Proposition 2.1 (Brezis, H. [2] Theorem 9.). *Let B be maximal monotone and $\phi : \mathbb{H} \rightarrow \mathbb{R}_{\infty}$ be proper, convex and lower semi-continuous. Suppose*

$$\varphi((1 + \mu B)^{-1}u) \leq \varphi(u), \quad \forall \mu > 0, \quad \forall u \in D(\varphi). \quad (2.9)$$

Then $\partial\phi + B$ is maximal monotone.

Lemma 2.1. *If $\phi = \varphi$ and $B = \partial\psi$ given by (2.5) and (2.8), then the inequality (2.9) holds.*

Proof. Let $U \in \mathbb{C}_0^1(\Omega)$ and $V := (1 + \mu\partial\psi)^{-1}U$. For a.e. $x \in \Omega$, $V(x) + \mu|V(x)|_{\mathbb{R}^2}^{q-2}V(x) = U(x)$. Thus defining $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2 ; V \mapsto V + \mu|V|_{\mathbb{R}^2}^{q-2}V$, we have $G(V(x)) = U(x)$. Note that G is of class C^1 and bijective from $\mathbb{R}^2$ into itself, and its Jacobian determinant is given by

$$\det DG(V) = (1 + \mu|V|_{\mathbb{R}^2}^{q-2})\{1 + \mu(q-1)|V|_{\mathbb{R}^2}^{q-2}\} \neq 0 \quad \text{for each } V \in \mathbb{R}^2.$$

Applying the inverse function theorem, we have $G^{-1} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. Hence $V(x) = G^{-1}(U(x))$. This shows $(1 + \mu\partial\psi)^{-1}\mathbb{C}_0^1(\Omega) \subset \mathbb{C}_0^1(\Omega)$. Let $U \in \mathbb{H}_0^1(\Omega)$, $V := (1 + \mu\partial\psi)^{-1}U$ and $U_n \in \mathbb{C}_0^1(\Omega)$ satisfying $U_n \rightarrow U$ in $\mathbb{H}^1(\Omega)$. Let $V_n := (1 + \mu\partial\psi)^{-1}U_n \in \mathbb{C}_0^1(\Omega)$. Since

$$|V_n - V|_{\mathbb{L}^2} = |(1 + \mu\partial\psi)^{-1}U_n - (1 + \mu\partial\psi)^{-1}U|_{\mathbb{L}^2} \leq |U_n - U|_{\mathbb{L}^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

we have $V_n \rightarrow V$ in $\mathbb{L}^2(\Omega)$. Also differentiating $G(V_n(x)) = U_n(x)$ gives

$$(1 + \mu|V_n(x)|_{\mathbb{R}^2}^{q-2})\nabla V_n(x) + \mu(q-2)|V_n(x)|_{\mathbb{R}^2}^{q-4}(V_n(x) \cdot \nabla V_n(x))_{\mathbb{R}^2}V_n(x) = \nabla U_n(x). \quad (2.10)$$

Multiplying (2.10) by $\nabla V_n(x)$, we have $|\nabla V_n(x)|_{\mathbb{R}^2}^2 \leq (\nabla U_n(x) \cdot \nabla V_n(x))_{\mathbb{R}^2}$. Therefore we have $|\nabla V_n|_{\mathbb{L}^2} \leq |\nabla U_n|_{\mathbb{L}^2} \rightarrow |\nabla U|_{\mathbb{L}^2}$. Thus the boundedness of $\{\nabla V_n\}$ gives $V \in \mathbb{H}_0^1(\Omega)$, and we have $(1 + \mu\partial\psi)^{-1}D(\varphi) \subset D(\varphi)$. In addition, by weak lower semi-continuity of the norm, we have $|\nabla V|_{\mathbb{L}^2} \leq |\nabla U|_{\mathbb{L}^2}$. $\square$

Now since the trivial inclusion $\lambda\partial\varphi + \kappa\partial\psi \subset \partial(\lambda\varphi + \kappa\psi)$ holds, we have shown

$$\lambda\partial\varphi + \kappa\partial\psi = \partial(\lambda\varphi + \kappa\psi) \quad \text{for all } \lambda, \kappa > 0. \quad (2.11)$$

Here, we can reduce (CGL) to the following evolution equation:

$$(E) \begin{cases} \frac{d}{dt}U(t) + \partial(\lambda\varphi + \kappa\psi)(U(t)) + \alpha I\partial\varphi(U(t)) + \beta I\partial\psi(U(t)) - \gamma U(t) = F(t), & t \in (0, \infty), \\ U(0) = U_0. \end{cases}$$

We introduce the following region:

$$\text{CGL}(r) := \left\{ (x, y) \in \mathbb{R}^2 \mid xy \geq 0 \text{ or } \frac{|xy| - 1}{|x| + |y|} < r \right\}. \quad (2.12)$$

Also, we use the constant $c_q \in [0, \infty)$ which denotes a strength of the nonlinearity:

$$c_q := \frac{q - 2}{2\sqrt{q - 1}} \quad (2.13)$$

3 Main Results

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a general domain with smooth boundary, $F \in L^2(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$ and $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \text{CGL}(c_q^{-1})$. If the initial value $U_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$, then there exists a solution $U \in C([0, \infty); \mathbb{L}^2(\Omega))$ of the equation (E) satisfying*

- (i) $U \in W^{1,2}(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$;
- (ii) $U(t) \in D(\partial\varphi) \cap D(\partial\psi)$ for a.e. $t \in (0, \infty)$ and satisfies (E) for a.e. $t \in (0, \infty)$;
- (iii) $\partial\varphi(U(\cdot)), \partial\psi(U(\cdot)) \in L^2(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a general domain with smooth boundary, $F \in L^2(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$ and $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \text{CGL}(c_q^{-1})$. If the initial value $U_0 \in \mathbb{L}^2(\Omega)$, then there exists a solution $U \in C([0, \infty); \mathbb{L}^2(\Omega))$ of the equation (E) satisfying*

- (i) $U \in W_{\text{loc}}^{1,2}((0, \infty); \mathbb{L}^2(\Omega))$;
- (ii) $U(t) \in D(\partial\varphi) \cap D(\partial\psi)$ for a.e. $t \in (0, \infty)$ and satisfies (E) for a.e. $t \in (0, \infty)$;
- (iii) $\varphi(U(\cdot)), \psi(U(\cdot)) \in L^1(0, T)$ and $t\varphi(U(t)), t\psi(U(t)) \in L^\infty(0, T)$ for all $T > 0$;
- (iv) $\sqrt{t}\frac{d}{dt}U(t), \sqrt{t}\partial\varphi(U(t)), \sqrt{t}\partial\psi(U(t)) \in L^2(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$.

4 Key Inequalities

Lemma 4.1. *The following inequalities hold for all $U \in D(\partial\varphi) \cap D(\partial\psi)$:*

$$|(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}| \leq c_q(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}, \quad (4.1)$$

$$|(\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}| \leq c_q(\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2} \leq c_q(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}, \quad (4.2)$$

where $\partial\psi_\mu(U) = \partial\psi((1 + \mu\partial\psi)^{-1}U)$ is Yosida approximation of $\partial\psi(U)$.

Proof. Using the definition of Yosida approximation, and letting $V := (1 + \mu\partial\psi)^{-1}U$, we can reduce (4.2) to (4.1). Thus it is enough to show (4.1).

Calculating the right-hand side of (4.1) by integration by parts, we have

$$(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2} = \int_{\Omega} \left\{ (q-2)|U|_{\mathbb{R}^2}^{q-4} |(U \cdot \nabla U)_{\mathbb{R}^2}|^2 + |U|_{\mathbb{R}^2}^{q-2} |\nabla U|_{\mathbb{R}^2}^2 \right\}. \quad (4.3)$$

Also, by integration by parts with (2.1) and (2.2), the left-hand side of (4.1) becomes

$$\begin{aligned} (\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2} &= (\nabla U, (q-2)|U|_{\mathbb{R}^2}^{q-4} (U \cdot \nabla U)_{\mathbb{R}^2} IU + |U|_{\mathbb{R}^2}^{q-2} I\nabla U)_{\mathbb{L}^2} \\ &= (q-2) \int_{\Omega} |U|_{\mathbb{R}^2}^{q-4} (U \cdot \nabla U)_{\mathbb{R}^2} \cdot (IU \cdot \nabla U)_{\mathbb{R}^2}. \end{aligned} \quad (4.4)$$

Thus by Young's inequality, (2.3) and (4.3), we obtain the desired (4.1) as follows.

$$\begin{aligned} |(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}| &\leq (q-2) \int_{\Omega} |U|_{\mathbb{R}^2}^{q-4} |(U \cdot \nabla U)_{\mathbb{R}^2} \cdot (IU \cdot \nabla U)_{\mathbb{R}^2}| \\ &\leq (q-2) \int_{\Omega} |U|_{\mathbb{R}^2}^{q-4} \frac{1}{2\sqrt{q-1}} \{ (q-1) |(U \cdot \nabla U)_{\mathbb{R}^2}|^2 + (IU \cdot \nabla U)_{\mathbb{R}^2}^2 \} \\ &= c_q \int_{\Omega} |U|_{\mathbb{R}^2}^{q-4} \{ (q-2) |(U \cdot \nabla U)_{\mathbb{R}^2}|^2 + |U|_{\mathbb{R}^2}^2 |\nabla U|_{\mathbb{R}^2}^2 \} \\ &= c_q (\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}. \end{aligned} \quad \square$$

5 Solvability of Approximate Equation

We treat the following equation:

$$\text{(AE)} \begin{cases} \frac{d}{dt}U(t) + \partial(\lambda\varphi + \kappa\psi)(U(t)) + \alpha I\partial\varphi(U(t)) + B(U(t)) &= F(t), \quad t \in (0, \infty), \\ U(0) &= U_0, \end{cases}$$

where $B : \mathbb{L}^2(\Omega) \rightarrow \mathbb{L}^2(\Omega)$ is Lipschitz with Lipschitz constant L_B .

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a general domain, $F \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$, $\lambda, \kappa > 0$, $\alpha \in \mathbb{R}$ and $B : \mathbb{L}^2(\Omega) \rightarrow \mathbb{L}^2(\Omega)$ be Lipschitz. If $U_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$, then there exists a unique solution $U \in C([0, \infty); \mathbb{L}^2(\Omega))$ of (AE) satisfying*

- (i) $U \in W^{1,2}(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$;
- (ii) $U(t) \in D(\partial\varphi) \cap D(\partial\psi)$ for a.e. $t \in (0, \infty)$ and satisfies (AE) for a.e. $t \in (0, \infty)$;
- (iii) $\partial\varphi(U(\cdot)), \partial\psi(U(\cdot)) \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$.

In order to prove Proposition 5.1, we approximate monotone perturbation term $\alpha I\partial\varphi(U)$ by $\alpha I\partial\varphi_{\nu}(U)$, where $\partial\varphi_{\nu}$ is Yosida approximation of $\partial\varphi$: $\partial\varphi_{\nu}(U) = \partial\varphi((1 + \nu\partial\varphi)^{-1}U)$.

$$\text{(AE)}_{\nu} \begin{cases} \frac{d}{dt}U(t) + \partial(\lambda\varphi + \kappa\psi)(U(t)) + \alpha I\partial\varphi_{\nu}(U(t)) + B(U(t)) &= F(t), \quad t \in (0, \infty), \\ U(0) &= U_0. \end{cases}$$

Since $\alpha I\partial\varphi_{\nu}(\cdot) + B(\cdot)$ is Lipschitz in $\mathbb{L}^2(\Omega)$, approximate equation (AE)_{ν} has a unique solution $U = U_{\nu} \in C([0, \infty); \mathbb{L}^2(\Omega))$ by the general theory of subdifferential operator (e.g. [2], [11]). Note that this approximate solution U_{ν} has the same regularities as those of the desired solution of Proposition 5.1. Then by the standard argument in the maximal monotone operator

theory, we can show $\{U_{\nu}\}_{\nu \downarrow 0}$ is Cauchy in $C([0, T]; \mathbb{L}^2(\Omega))$, as well as $\{\frac{d}{dt}U_{\nu_n}\}$, $\{\partial\varphi(U_{\nu_n})\}$ and $\{\partial\psi(U_{\nu_n})\}$ are bounded in $L^2(0, T; \mathbb{L}^2(\Omega))$. Hence by the demiclosedness of $\frac{d}{dt}$, $\partial\varphi$ and $\partial\psi$,

$$\begin{aligned} U_{\nu_n} &\rightarrow U \quad \text{in } C([0, T]; \mathbb{L}^2(\Omega)), \\ \frac{dU_{\nu'_n}}{dt} &\rightharpoonup \frac{dU}{dt} \quad \text{in } L^2(0, T; \mathbb{L}^2(\Omega)), \\ \partial\varphi(U_{\nu'_n}) &\rightharpoonup \partial\varphi(U) \quad \text{in } L^2(0, T; \mathbb{L}^2(\Omega)), \\ \partial\psi(U_{\nu'_n}) &\rightharpoonup \partial\psi(U) \quad \text{in } L^2(0, T; \mathbb{L}^2(\Omega)), \end{aligned}$$

for some sub sequence $\{\nu'_n\}_{n \in \mathbb{N}} \subset \{\nu_n\}_{n \in \mathbb{N}}$. Then by the definition of Yosida approximation,

$$\begin{aligned} |U_{\nu_n} - J_{\nu_n}U_{\nu_n}|_{L^2(0, T; \mathbb{L}^2)}^2 &= \int_0^T |U_{\nu_n}(s) - J_{\nu_n}U_{\nu_n}(s)|_{\mathbb{L}^2}^2 ds \\ &= \nu_n^2 \int_0^T |\partial\varphi_{\nu_n}(U_{\nu_n}(s))|_{\mathbb{L}^2}^2 ds \leq C_2 \nu_n^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This means $J_{\nu_n}U_{\nu_n} \rightarrow U$ in $L^2(0, T; \mathbb{L}^2(\Omega))$. Now since $\partial\varphi_{\nu}(U_{\nu}) = \partial\varphi(J_{\nu}U_{\nu})$, we have

$$\frac{dU}{dt} + \lambda\partial\varphi(U) + \kappa\partial\psi(U) + \alpha I\partial\varphi(U) + B(U) = F \quad \text{in } L^2(0, T; \mathbb{L}^2(\Omega)),$$

in the limit of the approximate equation $(\text{AE})_{\nu'_n}$. That is, U is a desired solution of (AE) .

6 Proof of Theorem 1

For the first step to prove Theorem 1, we approximate the equation (E) by

$$(\text{E})_{\mu} \begin{cases} \frac{d}{dt}U(t) + \partial(\lambda\varphi + \kappa\psi)(U(t)) + \alpha I\partial\varphi(U(t)) + \beta I\partial\psi_{\mu}(U(t)) - \gamma U(t) = F(t), & t \in (0, \infty), \\ U(0) = U_0, \end{cases}$$

where $\partial\psi_{\mu}(U) := \partial\psi((1 + \mu\partial\psi)^{-1}U)$ is Yosida approximation of $\partial\psi(U)$. This approximate equation $(\text{E})_{\mu}$ is exactly the same form as that of (AE) , whence by Proposition 5.1, $(\text{E})_{\mu}$ has a solution $U = U_{\mu} \in C([0, \infty); \mathbb{L}^2(\Omega))$. Note that U_{μ} has the regularities stated in Proposition 5.1. In order to prove Theorem 1, we first derive some a priori estimates.

Lemma 6.1. *Let U be a solution of $(\text{E})_{\mu}$. Fix $T > 0$. Then there exists a positive constant C_1 depending only on γ , T , $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|_{\mathbb{L}^2}^2$ satisfying*

$$\sup_{t \in [0, T]} |U(t)|_{\mathbb{L}^2}^2 + \int_0^T \varphi(U(s)) ds + \int_0^T \psi(U(s)) ds \leq C_1. \quad (6.1)$$

Proof. Multiplying $(\text{E})_{\mu}$ by $U(t)$, we have, for a.e. $t \in (0, \infty)$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |U(t)|_{\mathbb{L}^2}^2 + 2\lambda\varphi(U(t)) + q\kappa\psi(U(t)) \\ + \alpha(I\partial\varphi(U(t)), U(t))_{\mathbb{L}^2} + \beta(I\partial\psi_{\mu}(U(t)), U(t))_{\mathbb{L}^2} \\ - \gamma|U(t)|_{\mathbb{L}^2}^2 = (F(t), U(t))_{\mathbb{L}^2}. \end{aligned} \quad (6.2)$$

Note that by integration by parts, (2.1) and (2.2), we have

$$\begin{aligned} (I\partial\varphi(U), U)_{\mathbb{L}^2} &= 0, \\ (I\partial\psi_{\mu}(U), U)_{\mathbb{L}^2} &= (I\partial\psi(V), V + \mu\partial\psi(V))_{\mathbb{L}^2} = 0, \end{aligned}$$

where $V := (1 + \mu\partial\psi)^{-1}U$. Hence by (6.2) with Young's inequality, we have

$$\frac{1}{2} \frac{d}{dt} |U(t)|_{\mathbb{L}^2}^2 + 2\lambda\varphi(U(t)) + q\kappa\psi(U(t)) \leq (\gamma_+ + \frac{1}{2})|U(t)|_{\mathbb{L}^2}^2 + \frac{1}{2}|F(t)|_{\mathbb{L}^2}^2$$

where $\gamma_+ := \max\{\gamma, 0\}$. Thus the Gronwall's inequality yields

$$|U(t)|_{\mathbb{L}^2}^2 + 2 \int_0^t \{2\lambda\varphi(U(s)) + q\kappa\psi(U(s))\} ds \leq e^{(2\gamma_++1)t} \left\{ |U_0|_{\mathbb{L}^2}^2 + \int_0^T |F|_{\mathbb{L}^2}^2 \right\}$$

for all $t \in [0, T]$. Therefore we obtain the desired estimate (6.1). $\square$

Lemma 6.2. *Let U be a solution of $(E)_\mu$, and let $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \text{CGL}(c_q^{-1})$. Fix $T > 0$. Then there exist a positive constant C_2 depending only on $\lambda, \kappa, \alpha, \beta, \gamma, T, \varphi(U_0), \psi(U_0), |U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|_{\mathbb{L}^2}^2$ satisfying*

$$\sup_{t \in [0, T]} \varphi(U(t)) + \int_0^T \left| \frac{dU}{ds} \right|_{\mathbb{L}^2}^2 ds + \int_0^T |\partial\varphi(U(s))|_{\mathbb{L}^2}^2 ds + \int_0^T |\partial\psi(U(s))|_{\mathbb{L}^2}^2 ds \leq C_2. \quad (6.3)$$

Proof. Let $V(t) := (1 + \mu\partial\psi)^{-1}U(t)$. Since

$$\begin{aligned} (\partial\psi(U), \partial\psi_\mu(U))_{\mathbb{L}^2} &= \int_{\Omega} |U|_{\mathbb{R}^2}^{q-2} |V|_{\mathbb{R}^2}^{q-2} (U \cdot V)_{\mathbb{R}^2} \geq \int_{\Omega} |V|_{\mathbb{R}^2}^{2(q-1)} = |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2; \\ (U, \partial\psi_\mu(U)) &= q\psi(V) + \mu|\partial\psi(V)|_{\mathbb{L}^2}^2 = q\psi_\mu(U) - \left(\frac{q}{2} - 1\right)\mu|\partial\psi(V)|_{\mathbb{L}^2}^2 \leq q\psi(U), \end{aligned}$$

multiplying $(E)_\mu$ by $\partial\varphi(U(t))$ and $\partial\psi_\mu(U(t))$ yields

$$\frac{d}{dt} \varphi(U(t)) + \lambda|\partial\varphi(U(t))|_{\mathbb{L}^2}^2 + \kappa G(t) + \beta B_\mu(t) = 2\gamma\varphi(U(t)) + (F, \partial\varphi(U(t)))_{\mathbb{L}^2}, \quad (6.4)$$

$$\frac{d}{dt} \psi_\mu(U(t)) + \kappa|\partial\psi_\mu(U(t))|_{\mathbb{L}^2}^2 + \lambda G_\mu(t) - \alpha B_\mu(t) \leq q\gamma_+\psi(U(t)) + (F, \partial\psi_\mu(U(t)))_{\mathbb{L}^2}, \quad (6.5)$$

where $\gamma_+ := \max\{\gamma, 0\}$ and

$$\begin{cases} G := (\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}, \\ G_\mu := (\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2}, \\ B_\mu := (\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}. \end{cases}$$

We add $(6.4) \times \delta^2$ and (6.5) for some $\delta > 0$ to get

$$\begin{aligned} \frac{d}{dt} \{ \delta^2 \varphi(U) + \psi_\mu(U) \} + \delta^2 \lambda |\partial\varphi(U)|_{\mathbb{L}^2}^2 + \kappa |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \\ + \delta^2 \kappa G + \lambda G_\mu + (\delta^2 \beta - \alpha) B_\mu \\ \leq \gamma_+ \{ 2\delta^2 \varphi(U) + q\psi(U) \} + (F, \delta^2 \partial\varphi(U) + \partial\psi_\mu(U))_{\mathbb{L}^2}. \end{aligned} \quad (6.6)$$

Let $\epsilon \in (0, \min\{\lambda, \kappa\})$ be a small parameter. By the inequality of arithmetic and geometric means, and the fundamental property (2.4), we have

$$\begin{aligned} \delta^2 \lambda |\partial\varphi(U)|_{\mathbb{L}^2}^2 + \kappa |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \\ = \epsilon \{ \delta^2 |\partial\varphi(U)|_{\mathbb{L}^2}^2 + |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \} + (\lambda - \epsilon) \delta^2 |\partial\varphi(U)|_{\mathbb{L}^2}^2 + (\kappa - \epsilon) |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \\ \geq \epsilon \{ \delta^2 |\partial\varphi(U)|_{\mathbb{L}^2}^2 + |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \} + 2\sqrt{(\lambda - \epsilon)(\kappa - \epsilon) \delta^2 |\partial\varphi(U)|_{\mathbb{L}^2}^2 |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2} \\ \geq \epsilon \{ \delta^2 |\partial\varphi(U)|_{\mathbb{L}^2}^2 + |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \} + 2\sqrt{(\lambda - \epsilon)(\kappa - \epsilon) \delta^2 (G_\mu^2 + B_\mu^2)}. \end{aligned} \quad (6.7)$$

Note that by the key inequality Lemma 4.2

$$G \geq G_\mu \geq c_q^{-1} |B_\mu|. \quad (6.8)$$

Therefore combining (6.6), (6.7) and (6.8) yields

$$\begin{aligned} \frac{d}{dt} \{ \delta^2 \varphi(U) + \psi_\mu(U) \} + \epsilon \{ \delta^2 |\partial \varphi(U)|_{\mathbb{L}^2}^2 + |\partial \psi_\mu(U)|_{\mathbb{L}^2}^2 \} + J(\delta, \epsilon) |B_\mu| \\ \leq \gamma_+ \{ 2\delta^2 \varphi(U) + q\psi(U) \} + (F, \delta^2 \partial \varphi(U) + \partial \psi_\mu(U))_{\mathbb{L}^2}. \end{aligned} \quad (6.9)$$

where

$$J(\delta, \epsilon) := 2\delta \sqrt{(1 + c_q^{-2})(\lambda - \epsilon)(\kappa - \epsilon) + c_q^{-1}(\delta^2 \kappa + \lambda) - |\delta^2 \beta - \alpha|}.$$

Now we show that $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \text{CGL}(c_q^{-1})$ gives $J(\delta, \epsilon) \geq 0$ for some δ and ϵ . By the continuity of $\epsilon \mapsto J(\delta, \epsilon)$ it suffices to show $J(\delta, 0) > 0$ for some δ . When $\alpha\beta > 0$, it is enough to take $\delta = \sqrt{\alpha/\beta}$. When $\alpha\beta \leq 0$, we have $|\delta^2 \beta - \alpha| = \delta^2 |\beta| + |\alpha|$. Hence

$$J(\delta, 0) = (c_q^{-1} \kappa - |\beta|) \delta^2 + 2\delta \sqrt{(1 + c_q^{-2}) \lambda \kappa + (c_q^{-1} \lambda - |\alpha|)}.$$

Therefore if $|\beta|/\kappa \leq c_q^{-1}$, we have $J(\delta, 0) > 0$ for sufficiently large $\delta > 0$. If $c_q^{-1} < |\beta|/\kappa$, we find that it is enough to see the discriminant is positive:

$$D/4 := (1 + c_q^{-2}) \lambda \kappa - (c_q^{-1} \kappa - |\beta|)(c_q^{-1} \lambda - |\alpha|) > 0. \quad (6.10)$$

Since

$$D/4 > 0 \Leftrightarrow \frac{|\alpha|}{\lambda} \frac{|\beta|}{\kappa} - 1 < c_q^{-1} \left(\frac{|\alpha|}{\lambda} + \frac{|\beta|}{\kappa} \right),$$

the condition $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \text{CGL}(c_q^{-1})$ yields $D > 0$, whence $J(\delta, 0) > 0$ for some δ .

Now we take δ and ϵ satisfying $J(\delta, \epsilon) \geq 0$. By Lemma 6.1, integrating (6.9) gives

$$\sup_{t \in [0, T]} \varphi(U(t)) + \int_0^T |\partial \varphi(U(s))|_{\mathbb{L}^2}^2 ds + \int_0^T |\partial \psi_\mu(U(s))|_{\mathbb{L}^2}^2 ds \leq C_2, \quad (6.11)$$

where C_2 depends on the constants stated in Lemma 6.2. We multiply $(E)_\mu$ by $\partial \psi(U)$ to get

$$\begin{aligned} \frac{d}{dt} \psi(U) + \kappa |\partial \psi(U)|_{\mathbb{L}^2}^2 + \lambda (\partial \varphi(U), \partial \psi(U))_{\mathbb{L}^2} \\ = -\alpha (I \partial \varphi(U), \partial \psi(U))_{\mathbb{L}^2} - \beta (I \partial \psi_\mu(U), \partial \psi(U))_{\mathbb{L}^2} + q \gamma \psi(U) + (F, \partial \psi(U))_{\mathbb{L}^2} \\ \leq \frac{\kappa}{4} |\partial \psi(U)|_{\mathbb{L}^2}^2 + \frac{\alpha^2}{\kappa} |\partial \varphi(U)|_{\mathbb{L}^2}^2 + q \gamma_+ \psi(U) + \frac{\kappa}{4} |\partial \psi(U)|_{\mathbb{L}^2}^2 + \frac{1}{\kappa} |F|_{\mathbb{L}^2}^2. \end{aligned} \quad (6.12)$$

Hence by (4.1) and (6.11), integrating (6.12) yields

$$\int_0^T |\partial \psi(U(s))|_{\mathbb{L}^2}^2 ds \leq C_2. \quad (6.13)$$

Finally, combining $(E)_\mu$ with (6.11) and (6.13), we obtain the desired estimate (6.3). $\square$

Now we prove Theorem 1.

Proof of Theorem 1. Let U_μ be a solution of $(E)_\mu$, and fix $T > 0$. By Lemma 6.1 and 6.2, we have a sequence $\mu_n \downarrow 0$ satisfying

$$U_{\mu_n} \rightharpoonup U \quad \text{weakly in } L^2(0, T; \mathbb{H}_0^1(\Omega)), \quad (6.14)$$

$$\frac{dU_{\mu_n}}{dt} \rightharpoonup \frac{dU}{dt} \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega)), \quad (6.15)$$

$$\partial\varphi(U_{\mu_n}) \rightharpoonup G \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega)), \quad (6.16)$$

$$\partial\psi(U_{\mu_n}) \rightharpoonup H \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega)), \quad (6.17)$$

for some function $G, H \in L^2(0, T; \mathbb{L}^2(\Omega))$. Note that we use the weak closedness of $\frac{d}{dt}$ in $L^2(0, T; \mathbb{L}^2(\Omega))$ to (6.15).

First we show $G = \partial\varphi(U)$ in $L^2(0, T; \mathbb{L}^2(\Omega))$. For each $W \in C_0^\infty(\Omega)$ and $w \in C_0^\infty(0, T)$, we have $w(t)W \in L^2(0, T; \mathbb{L}^2(\Omega))$. Hence in the limit of (6.14) and (6.16), we obtain

$$\int_0^T w(s)(G(s), W)_{\mathbb{L}^2} ds = \int_0^T w(s)(U(s), -\Delta W)_{\mathbb{L}^2} ds.$$

Then by the fundamental lemma of calculus of variations, $(G(t), W)_{\mathbb{L}^2} = (U(t), -\Delta W)_{\mathbb{L}^2}$ for a.e. $t \in (0, T)$, so that $-\Delta U(t) = G(t) \in \mathbb{L}^2(\Omega)$. Also by (6.14), $U(t) \in \mathbb{H}_0^1(\Omega)$ a.e. $t \in (0, T)$. Therefore $U(t) \in D(\partial\varphi)$ and $\partial\varphi(U(t)) = -\Delta U(t) = G(t)$ for a.e. $t \in (0, T)$.

Next in order to see $H = \partial\psi(U)$ in $L^2(0, T; \mathbb{L}^2(\Omega))$, we are showing

$$U_{\mu'_n} \rightarrow U \quad \text{in } C(0, T; \mathbb{L}^2(\Omega')) \quad \text{for each bounded } \Omega' \subset \Omega, \quad (6.18)$$

for some subsequence $\{\mu'_n\} \subset \{\mu_n\}$. To confirm this, we use Ascoli's theorem and a diagonal argument. Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be bounded domains in $\mathbb{R}^N$ with smooth boundaries satisfying (i) $\Omega_k \subset \Omega_{k+1} \subset \Omega$ for each $k \in \mathbb{N}$; (ii) for all bounded $\Omega' \subset \Omega$ there exists $k \in \mathbb{N}$ such that $\Omega' \subset \Omega_k$. Fix $k \in \mathbb{N}$. By Lemma 6.1 and 6.2, we have

$$|U_{\mu_n}(t_2) - U_{\mu_n}(t_1)|_{\mathbb{L}^2(\Omega_k)} \leq \left\{ \int_{t_1}^{t_2} \left| \frac{dU_{\mu_n}}{ds} \right|_{\mathbb{L}^2(\Omega)} ds \right\}^{\frac{1}{2}} \left\{ \int_{t_1}^{t_2} ds \right\}^{\frac{1}{2}} \leq \sqrt{C_2} \sqrt{t_2 - t_1}, \quad (6.19)$$

$$|U_{\mu_n}(t)|_{\mathbb{H}^1(\Omega_k)}^2 = |U_{\mu_n}(t)|_{\mathbb{L}^2(\Omega_k)}^2 + |\nabla U_{\mu_n}(t)|_{\mathbb{L}^2(\Omega_k)}^2 \leq C_1 + 2C_2. \quad (6.20)$$

By (6.19), $\{U_{\mu_n}\}$ is uniformly equicontinuous in $C(0, T; \mathbb{L}^2(\Omega_k))$, and by (6.20), $\{U_{\mu_n}(t)\}$ is relatively compact in $\mathbb{L}^2(\Omega)$ for each $t \in (0, T)$. Hence by Ascoli's theorem, we have

$$U_{\mu_n^k} \rightarrow U^k \quad \text{in } C([0, T]; \mathbb{L}^2(\Omega_k)) \quad \text{as } n \rightarrow \infty,$$

for some function $U^k \in C([0, T]; \mathbb{L}^2(\Omega_k))$ and some subsequence $\{\mu_n^k\}_{n \in \mathbb{N}} \subset \{\mu_n\}_{n \in \mathbb{N}}$. Now we take a subsequence successively from $k = 1$ to ∞ : $\{\mu_n^{k+1}\}_{n \in \mathbb{N}} \subset \{\mu_n^k\}_{n \in \mathbb{N}}$ for each $k \in \mathbb{N}$. Then the diagonal sequence $\{\mu_n^n\}_{n \in \mathbb{N}} =: \{\mu'_n\}_{n \in \mathbb{N}}$ satisfies

$$U_{\mu'_n} \rightarrow U^k \quad \text{in } C([0, T]; \mathbb{L}^2(\Omega_k)) \quad \text{as } n \rightarrow \infty \quad \text{for each } k \in \mathbb{N}. \quad (6.21)$$

On the other hand, by (6.14), we have

$$U_{\mu'_n} \rightharpoonup U \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega_k)) \quad \text{as } n \rightarrow \infty \quad \text{for each } k \in \mathbb{N}. \quad (6.22)$$

Thus by the uniqueness of a weak limit, we have $U^k = U$ in $L^2(0, T; \mathbb{L}^2(\Omega_k))$. Finally since $\Omega' \subset \Omega_k$ for some k , we obtain the desired convergence (6.18) from (6.21).

Now we show $H = \partial\psi(U)$ in $L^2(0, T; \mathbb{L}^2(\Omega))$. By the demiclosedness of $U \mapsto |U|_{\mathbb{R}^2}^{q-2}U$ in $L^2(0, T; \mathbb{L}^2(\Omega'))$, we have

$$U(t) \in \mathbb{L}^{2(q-1)}(\Omega') \quad \text{for a.e. } t \in (0, T), \quad (6.23)$$

$$H(t) = |U(t)|_{\mathbb{R}^2}^{q-2}U(t) \quad \text{in } \mathbb{L}^2(\Omega') \quad \text{for a.e. } t \in (0, T). \quad (6.24)$$

Since (6.24) holds for all bounded $\Omega' \subset \Omega$, we have $|U(t)|_{\mathbb{R}^2}^{q-2}U(t) = H(t)$ for a.e. $x \in \Omega$, so that $U(t) \in D(\psi)$ and $H(t) = \partial\psi(U(t))$ for a.e. $t \in (0, T)$.

Finally we are showing that the function U satisfies equation (E). Note that $J_{\mu'_n}U_{\mu'_n} \rightarrow U$ in $L^2(0, T; \mathbb{L}^2(\Omega'))$ by Lemma 6.2 where $J_\mu := (1 + \mu\partial\psi)^{-1}$. By the demiclosedness of $\partial\psi$ in $L^2(0, T; \mathbb{L}^2(\Omega'))$, we find that U satisfies (E) in $L^2(0, T; \mathbb{L}^2(\Omega'))$ for all bounded $\Omega' \subset \Omega$. Hence it also satisfies (E) in $L^2(0, T; \mathbb{L}^2(\Omega))$. $U(0) = U_0$ in $L^2(\Omega)$ can be obtained immediately from (6.18), since $U_{\mu'_n}(0) = U_0$ for each $n \in \mathbb{N}$. $\square$

7 Proof of Theorem 2

Now we are proving Theorem 2. Let $U_{0n} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$ satisfying $U_{0n} \rightarrow U_0$ in $L^2(\Omega)$. By Theorem 1, we have a solution $U_n \in C([0, T]; \mathbb{L}^2(\Omega))$ corresponding to the initial value U_{0n} . First we derive some a priori estimates for the solution of (E) with $U_0 \in \mathbb{H}_0^1 \cap \mathbb{L}^q$.

Lemma 7.1. *Let U be a solution of (E), and fix $T > 0$. Then there exists a positive constant C_1 depending only on γ , T , $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|_{\mathbb{L}^2}^2$ satisfying*

$$\sup_{t \in [0, T]} |U(t)|_{\mathbb{L}^2}^2 + \int_0^T \varphi(U(s)) ds + \int_0^T \psi(U(s)) ds \leq C_1. \quad (7.1)$$

Lemma 7.2. *Let U be a solution of (E) with $U_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$ and $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \text{CGL}(c_q^{-1})$. Fix $T > 0$. Then there exist a positive constant C_2 depending only on $\lambda, \kappa, \alpha, \beta, \gamma, T$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|_{\mathbb{L}^2}^2$ satisfying*

$$\sup_{t \in [0, T]} t\varphi(U(t)) + \int_0^T s \left| \frac{dU}{ds} \right|_{\mathbb{L}^2}^2 ds + \int_0^T s |\partial\varphi(U(s))|_{\mathbb{L}^2}^2 ds + \int_0^T s |\partial\psi(U(s))|_{\mathbb{L}^2}^2 ds \leq C_2. \quad (7.2)$$

Since proofs are almost exactly the same as those of Lemma 6.1 and 6.2, we skip the details.

Proof of Theorem 2. Let U_n be a solution of (E) with $U_n(0) = U_{0n} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$, where $U_{0n} \rightarrow U_0$ in $L^2(\Omega)$. By Lemma 7.1 and 7.2, we have $\{m_n\}_{n \in \mathbb{N}} \subset \{n\}_{n \in \mathbb{N}}$ satisfying

$$U_{m_n} \rightharpoonup U \quad \text{weakly in } L_{\text{loc}}^2((0, \infty); \mathbb{H}_0^1(\Omega)), \quad (7.3)$$

$$\sqrt{t} \frac{dU_{m_n}}{dt} \rightharpoonup \sqrt{t} \frac{dU}{dt} \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega)), \quad (7.4)$$

$$\sqrt{t} \partial\varphi(U_{m_n}) \rightharpoonup \sqrt{t} G \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega)), \quad (7.5)$$

$$\sqrt{t} \partial\psi(U_{m_n}) \rightharpoonup \sqrt{t} H \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(\Omega)), \quad (7.6)$$

for some function G, H . Note that we use the weak closedness of $\frac{d}{dt}$ in $L^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$ to (7.4). First by the same argument as those of Theorem 1, we have $G = \partial\varphi(U)$ in $L^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$, so that $G = \partial\varphi(U)$ a.e. $t \in (0, T)$. Next, also by the same argument as those of Theorem 1, we have

$$U_{m'_n} \rightarrow U \quad \text{in } C(\delta, T; \mathbb{L}^2(\Omega')) \quad \text{for each bounded } \Omega' \subset \Omega \text{ and } \delta \in (0, T), \quad (7.7)$$

for some subsequence $\{m'_n\} \subset \{m_n\}$. Therefore this yields $H = \partial\psi(U)$ in $L^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$, so that a.e. $t \in (0, T)$. Now we find that U satisfies equation (E) in the limit ($m'_n \rightarrow \infty$) of the approximate equation of $U_{m'_n}$. Thus in order to finish the proof, it is enough to check

$$U(t) \rightarrow U_0 \quad \text{in } \mathbb{L}^2(\Omega) \quad \text{as } t \downarrow 0. \quad (7.8)$$

First we show $U(t) \rightharpoonup U_0$ weakly in $\mathbb{L}^2(\Omega)$. Multiplying the approximate equation of U_n by each $W \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \frac{d}{dt}(U_n(t), W)_{\mathbb{L}^2} &= \gamma(U_n(t), W)_{\mathbb{L}^2} + (F(t), W)_{\mathbb{L}^2} \\ &\quad - ((\lambda + \alpha I)\partial\varphi(U_n(t)), W)_{\mathbb{L}^2} - ((\kappa + \beta I)\partial\psi(U_n(t)), W)_{\mathbb{L}^2}. \end{aligned} \quad (7.9)$$

Hence integrating (7.9) and taking the absolute value gives

$$\begin{aligned} |(U_n(t) - U_{0n}, W)_{\mathbb{L}^2}| &\leq |\gamma| |W|_{\mathbb{L}^2} \int_0^t |U_n(s)|_{\mathbb{L}^2} ds + |W|_{\mathbb{L}^2} \int_0^t |F(s)|_{\mathbb{L}^2} ds \\ &\quad + (\lambda + |\alpha|) |\nabla W|_{\mathbb{L}^2} \int_0^t |\nabla U_n(s)|_{\mathbb{L}^2} ds \\ &\quad + (\kappa + |\beta|) \int_0^t \int_\Omega |U_n(s)|_{\mathbb{R}^2}^{q-1} |W|_{\mathbb{R}^2} dx ds. \end{aligned}$$

Thus using Hölder's inequality with Lemma 7.1, we have the estimate

$$\begin{aligned} |(U_n(t) - U_{0n}, W)_{\mathbb{L}^2}| &\leq |\gamma| \sqrt{C_1} |W|_{\mathbb{L}^2} t + \left\{ \int_0^t |F(s)|_{\mathbb{L}^2}^2 ds \right\}^{\frac{1}{2}} |W|_{\mathbb{L}^2} t^{\frac{1}{2}} \\ &\quad + (\lambda + |\alpha|) \sqrt{2C_1} |\nabla W|_{\mathbb{L}^2} t^{\frac{1}{2}} + (\kappa + |\beta|) (qC_1)^{\frac{q-1}{q}} |W|_{\mathbb{L}^q} t^{\frac{1}{q}}. \end{aligned} \quad (7.10)$$

Letting $n = m'_n \rightarrow \infty$, we have $|(U(t) - U_0, W)_{\mathbb{L}^2}| \leq Ct^{\frac{1}{q}}$ for sufficiently small $t > 0$, so that $U(t) \rightarrow U_0$ in $\mathcal{D}'(\Omega)$. Since $C^\infty(\Omega) \subset \mathbb{L}^2(\Omega)$ is dense, we have $U(t) \rightharpoonup U_0$ weakly in $\mathbb{L}^2(\Omega)$.

Then we show $|U(t)|_{\mathbb{L}^2}^2 \rightarrow |U_0|_{\mathbb{L}^2}^2$. By the argument of Lemma 7.1, we have

$$|U_n(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ + 1)t} \left\{ |U_{0n}|_{\mathbb{L}^2}^2 + \int_0^t |F(s)|_{\mathbb{L}^2}^2 ds \right\}.$$

Hence letting $n \rightarrow \infty$ gives $|U(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ + 1)t} \{ |U_0|_{\mathbb{L}^2}^2 + \int_0^t |F(s)|_{\mathbb{L}^2}^2 ds \}$. Then letting $t \downarrow 0$, we have $\overline{\lim}_{t \downarrow 0} |U(t)|_{\mathbb{L}^2}^2 \leq |U_0|_{\mathbb{L}^2}^2$. On the other hand, since $U(t) \rightharpoonup U_0$, we have $|U_0|_{\mathbb{L}^2}^2 \leq \underline{\lim}_{t \downarrow 0} |U(t)|_{\mathbb{L}^2}^2$ by the weak lower semicontinuity of the norm. Therefore $|U(t)|_{\mathbb{L}^2}^2 \rightarrow |U_0|_{\mathbb{L}^2}^2$. $\square$

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