

# The number of columns $(1, k-2, 1)$ in the intersection array of a distance-regular graph

大阪教育大学 平木 彰

Osaka Kyoiku University. Akira Hiraki

## 1 Definitions

Let  $\Gamma = (V\Gamma, E\Gamma)$  be a connected graph with usual distance  $\partial_\Gamma$ .

Let  $d = \max\{\partial_\Gamma(x, y) \mid x, y \in V\Gamma\}$  denote the diameter of  $\Gamma$ . For vertex  $u$ , let  $\Gamma_j(u) := \{x \in V\Gamma \mid \partial_\Gamma(u, x) = j\}$  the set of vertices which are at distance  $j$  from  $u$ .

For  $x, y \in V\Gamma$  with  $\partial_\Gamma(x, y) = i$ , let

$$\begin{aligned} C(x, y) &:= \Gamma_{i-1}(x) \cap \Gamma_1(y), \\ A(x, y) &:= \Gamma_i(x) \cap \Gamma_1(y) \\ \text{and} \quad B(x, y) &:= \Gamma_{i+1}(x) \cap \Gamma_1(y). \end{aligned}$$

A graph  $\Gamma$  is said to be *distance-regular* if

$$c_i = |C(x, y)|, \quad a_i = |A(x, y)| \quad \text{and} \quad b_i = |B(x, y)|$$

depend only on  $i = \partial_\Gamma(x, y)$  rather than individual vertices.

Then  $c_i, a_i, b_i$  are called the *intersection numbers* of  $\Gamma$  and the array

$$\iota(\Gamma) = \begin{pmatrix} * & c_1 & c_2 & \cdots & c_j & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_j & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_j & \cdots & b_{d-1} & * \end{pmatrix}$$

is called the *intersection array* of  $\Gamma$ . Let

$$\ell(c, a, b) := |\{i \mid (c_i, a_i, b_i) = (c, a, b)\}|.$$

For example the dodecahedron is a distance-regular graph with

$$\iota(\Gamma) = \begin{pmatrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{pmatrix}$$

and

$$\ell(1, 0, 2) = 1, \quad \ell(1, 1, 1) = 2, \quad \ell(2, 0, 1) = 1.$$

## 2 Bannai-Ito conjecture

The following are well-known basic properties of the intersection numbers.

**Lemma 1** *A distance-regular graph  $\Gamma$  is a regular graph of valency  $k = b_0$ . And :*

- (1)  $k = b_0 \geq b_1 \geq \cdots \geq b_{d-1} \geq 1$ .
- (2)  $1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k$ .
- (3)  $c_i + a_i + b_i = k$  for all  $i$ .

The reader is referred to [1] and [4] for general theory of distance-regular graphs.

**Conjecture. ( Bannai-Ito )** For fixed integer  $k$  with  $k \geq 3$  there are only finitely many distance-regular graphs of valency  $k$ .

This conjecture is equivalent to construction of a diameter bound for distance-regular graphs in terms of valency  $k$ .

Fix an integer  $k$  with  $k \geq 3$ . Let  $\Gamma$  be a distance-regular graph with valency  $k$ . From the above lemma there are only finitely many kind of column vectors  ${}^t(c, a, b)$  with  $c + a + b = k$ . To solve the conjecture we have to bound  $\ell(c, a, b)$  by a function of  $k$  or absolute constant.

Bannai and Ito proved a lot of interesting results concerning this problem by using eigenvalue technique. For example:

**Result 2** [2, Bannai-Ito (1987)]

$$\ell(c, a, c) \leq 10k2^k.$$

For the special case of the Bannai-Ito's result, we have the following results.

**Result 3** [7, Higashitani-Suzuki (1992)]

If  $k \geq 5$ , then

$$\ell(1, k-2, 1) \leq 46\sqrt{k-3}.$$

**Result 4** [9, Hiraki (1996)]

$$\ell(1, k-2, 1) \leq 20.$$

For the small valency we have:

**Result 5** [3, Biggs-Boshier and Shawe-Taylor (1986)] [5, Brouwer-Koolen (1999)]

$$\ell(1, 1, 1) \leq 3 \quad \text{and} \quad \ell(1, 2, 1) \leq 1.$$

We recall the sketch of the proof of the above results.

Let  $\Gamma$  be a distance-regular graph with :

$$\left\{ \begin{array}{cccccccccccc} * & c_1 & \cdots & c_1 & c_{r+1} & \cdots & c_s & c & \cdots & c & c' & \cdots \\ 0 & a_1 & \cdots & a_1 & a_{r+1} & \cdots & a_s & a & \cdots & a & a' & \cdots \\ k & b_1 & \cdots & b_1 & b_{r+1} & \cdots & b_s & b & \cdots & b & b' & \cdots \end{array} \right\}$$

$\underbrace{\hspace{1.5cm}}_{r} \qquad \qquad \qquad \underbrace{\hspace{1.5cm}}_t$

where  $r := \ell(c_1, a_1, b_1)$ ,  $t = \ell(c, a, b)$  and  $s = \min\{j \mid (c_{j+1}, a_{j+1}, b_{j+1}) = (c, a, b)\}$ .

Let  $A := A(\Gamma)$  be adjacency matrix of  $\Gamma$ . Let

$$k = \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_d$$

be the eigenvalues of  $A$ , and  $m(\theta_j)$  denote the multiplicity of  $\theta_j$  in  $A$ .

It is known that the  $m(\theta_j)$  is bounded from below by a function  $f_1(r, k)$  of  $r$  and  $k$ . ( See [13] and [2]. ) On the other hand by Biggs formula for  $m(\theta_1)$  it can be bounded from above by a function  $f_2(k, s, t)$  of  $k, s$  and  $t$ . Next we bound  $s$  by a function  $f_3(k, r)$  of  $k$  and  $r$ . Combining these we have

$$f_1(k, r) \leq m(\theta_1) \leq f_2(k, s, t) \leq f_4(k, r, t).$$

For our cases the above inequality gives us an upper bound for  $t$  by a function of  $k$  or absolute constant.

The bound for  $s$  that they used are as follows.

|     |                    | $k = 5$  | $k = 9$  | $k = 17$  | $k \sim 2^n$ |
|-----|--------------------|----------|----------|-----------|--------------|
| B-I | $2^{k-3} r$        | $4r$     | $64r$    | $2^{14}r$ | $2^{2^n}r$   |
| H-S | $(k-3)r$           | $2r$     | $6r$     | $14r$     | $2^n r$      |
| H   | $(r+1)\log_2(k-1)$ | $2(r+1)$ | $3(r+1)$ | $4(r+1)$  | $n(r+1)$     |

In [10] we have the following result.

**Lemma 6** *Let  $\Gamma$  be a distance-regular graph of valency  $k \geq 3$  with  $r := \ell(1, a_1, b_1) \geq 2$ . Then*

$$\max\{i \mid c_i = 1\} \leq 2r + 2.$$

□

Suppose  $t := \ell(1, k-2, 1) \geq 2$ . Then we have  $s + t = \max\{i \mid c_i = 1\} \leq 2r + 2$  from the above lemma. Apply this new bound for  $s$  to Bannai-Ito's method. We have the following improvement of the bound for  $t$

**Theorem 7** Let  $\Gamma$  be a distance-regular graph of valency  $k \geq 5$  and  $t := \ell(1, k-2, 1)$ . Then the following hold.

- (1) If  $k = 5$ , then  $t \leq 14$ .
- (2) If  $k \geq 6$ , then  $t \leq 7$ .
- (3) If  $k \geq 9$ , then  $t \leq 6$ .
- (4) If  $k \geq 12$ , then  $t \leq 4$ .
- (5) If  $k \geq 20$ , then  $t \leq 3$ .
- (6) If  $k \geq 30$ , then  $t \leq 2$ .
- (7) If  $k \geq 58$ , then  $t \leq 1$ .

The detailed proof of the theorem will be found in [11]. □

### 3 Sketch of the proof of Lemma 6.

Let  $\Gamma$  be a distance-regular graph. A subgraph  $\Delta$  is called *strongly closed* if

$$C(x, y) \cup A(x, y) \subseteq \Delta \quad \text{for any } x, y \in \Delta.$$

Then  $c_i(\Delta) = c_i(\Gamma)$  and  $a_i(\Delta) = a_i(\Gamma)$ . If  $\Delta$  is regular of valency  $k_\Delta$ , then  $b_i(\Delta) = k_\Delta - c_i(\Delta) - a_i(\Delta)$ . Hence a strongly closed subgraph of a distance-regular graph is also distance-regular if it is regular.

There are examples of a non-regular strongly closed subgraph of a distance-regular graph. However Suzuki proved the following result.

**Result 8** [12, Suzuki (1995)]

Let  $\Gamma$  be a distance-regular graph with  $r := \ell(c_1, a_1, b_1)$ . A strongly closed subgraph of diameter  $q$  with  $r+1 \leq q$  is one of the following graph :

- (i) A distance-regular graph. In particular,  $b_{q-1} > b_q$ .
- (ii) A distance-biregular graph.
- (iii) The 3-subdivision graph of a complete graph, or of a Moore graph. In particular,  $r \in \{3, 6\}$ .

On the other hand we have the following existence condition for strongly closed subgraphs of a distance-regular graph.

**Result 9** [10, Hiraki (2001)]

Let  $\Gamma$  be a distance-regular graph of diameter  $d$  and  $r := \ell(c_1, a_1, b_1)$ . Let  $q$  be an integer with  $r+1 \leq q \leq d-1$ . If  $b_{q-1} > b_q$  and  $c_{q+r} = 1$ , then there exists a sequence of strongly closed subgraphs

$$\Delta_{r+1} \subseteq \Delta_{r+2} \subseteq \cdots \subseteq \Delta_q,$$

where the diameter of  $\Delta_i$  is  $i$ . Moreover  $\Delta_q$  is distance-regular.

**Proof of Lemma 6.** Suppose  $c_{2r+3} = 1$  to derive a contradiction.

Since  $b_r > b_{r+1}$  and  $c_{2r+1} = 1$ , there exists a strongly closed subgraph  $\Delta$  of diameter  $r+1$  with

$$\iota(\Delta) = \begin{pmatrix} * & 1 & \cdots & 1 & 1 \\ 0 & a_1 & \cdots & a_1 & a_{r+1} \\ k' & b'_1 & \cdots & b'_1 & * \end{pmatrix}.$$

Since  $r \geq 2$ , a graph  $\Delta$  has to be an ordinary polygon. ( See [4, Theorem 6.8.1]. ) Hence we have  $(a_1, a_{r+1}) = (0, 1)$  and thus  $\ell(1, 1, k-2) \in \{1, 2\}$  from [8].

Suppose  $\ell(1, 1, k-2) = 1$ . Then  $b_{r+1} > b_{r+2}$  and  $c_{2r+2} = 1$ . There exists a strongly closed subgraph  $\Delta'$  of diameter  $r+2$  with

$$\iota(\Delta') = \begin{pmatrix} * & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & a \\ a+1 & a & \cdots & a & a-1 & * \end{pmatrix}.$$

But no such  $\Delta'$  exists from [6]. We have a contradiction.

Suppose  $\ell(1, 1, k-2) = 2$ . Then  $b_{r+1} = b_{r+2} > b_{r+3}$  and  $c_{2r+3} = 1$ . Thus there exists a sequence  $\Delta_{r+1} \subseteq \Delta_{r+2} \subseteq \Delta_{r+3}$  of strongly closed subgraphs. Since  $b_{r+1} = b_{r+2}$  and  $a_{r+1} = a_{r+2} = 1$ , a strongly closed subgraph  $\Delta_{r+2}$  of diameter  $r+2$  have to be the 3-subdivision graph of a complete graph, or of a Moore graph from Result 8. In particular, we have  $r \in \{3, 6\}$ . By counting the number of the 3-subdivision graphs of a complete graph, or of a Moore graph in  $\Delta_{r+3}$ , we have  $a+2$  is a divisor of 216. So there are only finitely many possible parameters for  $(a, r)$ . Each of them are ruled out by integrality condition of  $m(\theta_1)$ .

The lemma is proved. □

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