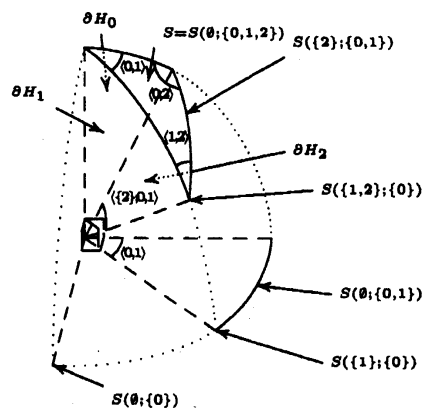


Solid angles of high dimensional spheres

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Spherical simplices. A spherical simplex S is an intersection of n pieces of half spaces $H_k = \{\vec{x} \in \mathbb{R}^n : \vec{p}_k \cdot \vec{x} \geq 0\}$ and the unit sphere $\mathbb{S}^{n-1}$ in the n -dimensional Euclidean space $\mathbb{R}^n$, i.e., $S = \{\vec{x} \in \mathbb{S}^{n-1} : \vec{p}_k \cdot \vec{x} \geq 0 \ (\forall k = 0, \dots, n-1)\}$. See the figure for $n = 3$:



where

$$S(I; J) = \{\vec{x} \in (\sum_{k \in I \cup J} \mathbb{R} \cdot \vec{p}_k) \cap \mathbb{S}^{n-1} : \vec{p}_k \cdot \vec{x} = 0 \ (\forall k \in I), \vec{p}_k \cdot \vec{x} \geq 0 \ (\forall k \in J)\},$$

for disjoint subsets I and non-empty J of $\{0, \dots, n-1\}$ and $\langle I; i, j \rangle$ is the dihedral angle between H_i and H_j subtended by $S(I; J)$.

Volumes of spherical simplices for odd n . The function f_m is the $(m-1)$ -dimensional *Schläfli's normalized volume function*, i.e.,

$$f_m = \frac{\text{Vol}_{m-1}(\cdot)}{\text{Vol}_{m-1}(\mathbb{S}^{m-1})/2^m} = \frac{\text{Vol}_{m-1}(\cdot)}{\pi^{m/2}/(2^{m-1}\Gamma(m/2))}$$

with $f_0(\cdot) = 1$. For $n = 3$, it is well-known that

$$\text{Vol}_2(S) = \sum_{I \subseteq \{0,1,2\}} \sum_{\#I=2} \text{Vol}_1(S(\emptyset; I)) - \pi, \quad \text{which is equivalent to} \quad f_3(S) = \sum_{I \subseteq \{0,1,2\}} \sum_{\#I=2} f_2(S(\emptyset; I)) - 2.$$

Schläfli generalized it to *Schläfli's reduction formula* for odd n :

$$f_n(S) = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k a_k \sum_{I \subseteq \{0, \dots, n-1\}} \sum_{\#I=n-1-2k} f_{n-1-2k}(S(\emptyset; I)),$$

This is an abstract and the details will be published elsewhere. The title in Japanese is “高次元球面の立体角について”.

where a_k is such that $\tan x = \sum_{k=0}^{\infty} a_k \frac{x^{2k+1}}{(2k+1)!}$. It is due to *Schläfli's differential formula*:

$$df_n(S) = \sum_{I \subseteq \{0, \dots, n-1\}}^{\#I=2} f_{n-2}(S(I; I^c)) df_2(S(\emptyset; I)),$$

which is also valid for even n , where $I^c = \{0, \dots, n-1\} \setminus I$. However, it is difficult to represent $f_n(S)$ for even n .

Volumes of orthogonal spherical simplices for $n = 4$. A spherical simplex S is called *orthogonal* if $\langle i, j \rangle = \frac{\pi}{2}$ for $|j - i| \geq 2$. Coxeter gave the formula representing the volume of orthogonal S for $n = 4$:

$$\frac{\pi^2}{2} f_4(S) = \sum_{k=1}^{\infty} \frac{X^k}{k^2} (\cos 2k\alpha - \cos 2k\beta + \cos 2k\gamma - 1) - \alpha^2 + \beta^2 - \gamma^2,$$

where $\alpha = \frac{\pi}{2} - \langle 0, 1 \rangle$, $\beta = \langle 1, 2 \rangle$, $\gamma = \frac{\pi}{2} - \langle 2, 3 \rangle$, and

$$X = \frac{\sqrt{\cos^2 \alpha \cos^2 \gamma - \cos^2 \beta} - \sin \alpha \sin \gamma}{\sqrt{\cos^2 \alpha \cos^2 \gamma - \cos^2 \beta} + \sin \alpha \sin \gamma}.$$

Volumes of spherical simplices for all n . Aomoto gave the formula representing $f_n(S)$:

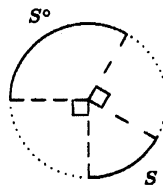
$$2^n \cdot \frac{1}{\sqrt{\det B}} \frac{\pi^{n/2} f_n(S)}{2^{n-1} \Gamma(\frac{n}{2})} / n = \sum_{(\sigma_{i,j})_{0 \leq i \leq j \leq n}} \frac{\prod_{i \leq j} (-2b_{i,j})^{\sigma_{i,j}}}{\prod_{i \leq j} \sigma_{i,j}!} \cdot \frac{\prod_{k=0}^{n-1} \Gamma(\frac{\sigma_{0,k} + \dots + \sigma_{k-1,k} + \sigma_{k,k+1} + \dots + \sigma_{k,n-1} + 1}{2})}{\Gamma(\frac{n}{2} + 1)},$$

where each entry of $(\sigma_{i,j})_{0 \leq i \leq j \leq n}$ runs over all non-negative integers, $B = (b_{i,j})_{j=0, \dots, n-1}^{i=0, \dots, n-1} = K^{-1} \cdot ((-\cos \langle i, j \rangle)_{j=0, \dots, n-1}^{i=0, \dots, n-1})^{-1} \cdot K^{-1}$, $K = (\delta_j^i \cdot \Delta(0, \dots, \hat{i}, \dots, n-1) / \Delta(0, \dots, n-1))_{j=0, \dots, n-1}^{i=0, \dots, n-1}$, $\Delta(i_0, \dots, i_{r-1}) = \det \left((-\cos \langle k, l \rangle)_{l=i_0, \dots, i_{r-1}}^{k=i_0, \dots, i_{r-1}} \right)$, δ_j^i is Kronecker's delta, and the circumflex indicates that the term below it has been omitted.

Polars of spherical simplices. For a spherical simplex S , the polar S° is

$$S^\circ = \{\vec{y} \in \mathbb{S}^{n-1} : \vec{x} \cdot \vec{y} \leq 0, \forall \vec{x} \in S\}.$$

See the figure for $n = 2$.



Main result. For all n , we have simple formulae:

$$\begin{aligned} \sum_{I \subseteq \{0, \dots, n-1\}}^{\#I: \text{ even}} f_{n-\#I}(S(I; I^c)) f_{\#I}((S(\emptyset; I))^\circ) &= 2^{n-1}, \\ \sum_{I \subseteq \{0, \dots, n-1\}}^{\#I: \text{ odd}} f_{n-\#I}(S(I; I^c)) f_{\#I}((S(\emptyset; I))^\circ) &= 2^{n-1}. \end{aligned}$$

Application. From the former, we can get a formula which represents $f_n(S) + f_n(S^\circ)$ by the volumes of lower dimensional spherical simplices for even n , e.g.,

$$f_2(S) + f_2(S^\circ) = 2, \text{ which is trivial for } n = 2,$$

$$f_4(S) + f_4(S^\circ) = 8 - \sum_{I \subseteq \{0,1,2,3\}}^{\#I=2} f_2(S(I; I^c)) f_2((S(\emptyset; I))^\circ),$$

which is already known by Schläfli for $n = 4$,

$$f_6(S) + f_6(S^\circ) = 32 - \sum_{I \subseteq \{0, \dots, 5\}}^{\#I=2} f_4(S(I; I^c)) f_2((S(\emptyset; I))^\circ) - \sum_{I \subseteq \{0, \dots, 5\}}^{\#I=4} f_2(S(I; I^c)) f_4((S(\emptyset; I))^\circ),$$

which seems to be new result for $n = 6$, and so on. For odd n , from the former again, we can also represent $f_n(S)$ distinct from Schläfli's reduction formula, e.g.,

$$f_3(S) = 4 - \sum_{I \subseteq \{0,1,2\}}^{\#I=2} 1 \cdot f_2((S(\emptyset; I))^\circ), \text{ which is trivial for } n = 3,$$

$$f_5(S) = 16 - \sum_{I \subseteq \{0, \dots, 4\}}^{\#I=2} f_3(S(I; I^c)) f_2((S(\emptyset; I))^\circ) - \sum_{I \subseteq \{0, \dots, 4\}}^{\#I=4} 1 \cdot f_4((S(\emptyset; I))^\circ),$$

which seems to be new result for $n = 5$, and so on.

Sketch of proof. We have Schläfli's differential formulae for lower dimensional spherical simplices and polars:

$$\begin{aligned} df_{n-\#I}(S(I; I^c)) &= \sum_{K \subseteq I^c}^{\#K=2} f_{n-\#I-2}(S(I \cup K; (I \cup K)^c)) df_2(S(I; K)), \\ df_{\#J}((S(\emptyset; J))^\circ) &= - \sum_{K \subseteq J}^{\#K=2} f_{\#J-2}((S(\emptyset; J \setminus K))^\circ) df_2(S(J \setminus K; K)). \end{aligned}$$

From differential formulae above, we can calculate the derivations of the left-hand sides of the main results, which are equal to zero. So the left-hand sides are constant, which are determined by the values of the spherical simplex whose all half-spaces are pairwise perpendicular. For such a spherical simplex, all terms of the left-hand side are equal to 1, so, the constant values are 2^{n-1} . $\square$

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