

Cospectral graphs of the Grassmann graphs

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(joint work with Edwin van Dam)

Let q be a prime power, and V be a n -dimensional space over the $GF(q)$ the field with q elements. Let $1 \leq e \leq n - 1$ be an integer.

The **Grassmann Graph** $G_q(n, e)$ has as vertices the e -dimensional subspaces and $S \sim T$ iff their intersection is $(e - 1)$ -dimensional.

To construct graphs with the same spectrum as $G_q(n, e)$ we first will look at a partial linear space.

Let n, e be positive integers such that $4 \leq 2e \leq n$.

Let V be a n -dimensional vector space over $GF(q)$ and

let H be a $2e$ -dimensional subspace of V .

We first construct the partial linear space

$$\mathcal{LG}_q(n, e, e + 1).$$

Its points are the e -dimensional subspaces of V .

There are two kinds of lines:

Lines of the first kind: $(e + 1)$ -dimensional subspaces L of V which are not a subspace of H . A line L has as points the e -dimensional subspaces contained in L .

b Lines of the second kind: $(e - 1)$ -dimensional spaces M contained in H . A line M has as points the e -dimensional spaces contained in H which contain M as a subspace.

Now $\mathcal{LG}_q(n, e, e + 1)$ has

$\begin{bmatrix} n \\ e \end{bmatrix}$ points,

$\begin{bmatrix} n \\ e+1 \end{bmatrix}$ lines,

each point is incident with $\begin{bmatrix} n-e \\ 1 \end{bmatrix}$ lines

and each line is incident with $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$ points.

Through any pair of points there is at most one line.

If P and Q are points then they are on a line iff $P \cap Q$ is $(e - 1)$ -dimensional.

Define $P_q(n, e + 1)$ as the line graph of $\mathcal{LG}_q(n, e, e + 1)$, that is its vertices are the lines of $\mathcal{LG}_q(n, e, e + 1)$ and two lines are adjacent iff they have exactly one point in common.

Theorem 1 (i) $P_q(n, e + 1)$ is cospectral with $G_q(n, e + 1)$,
(ii) $P_q(n, e + 1)$ is distance-regular iff $n = 2e + 1$.
(iii) $P_q(2e + 1, e + 1)$ is not isomorphic to the Grassmann graph $G_q(2e + 1, e + 1)$.

(i) Let N be the point-line incidence matrix. Then $NN^T - \begin{bmatrix} n-e \\ 1 \end{bmatrix} I$ is the adjacency matrix of the point graph. As the point graph is clearly $G_q(n, e)$, we know the spectrum of NN^T . Now except for the zero eigenvalue the spectrum of NN^T is the same as the $N^T N$. This implies that $P_q(n, e + 1)$ is cospectral with $G_q(n, e + 1)$ as $NN^T - \begin{bmatrix} e+1 \\ 1 \end{bmatrix} I$ is the adjacency matrix for $P_q(n, e + 1)$.

(ii) If $n < 2e + 1$, then there is $e + 1$ -dimensional space L which intersects H in a $(e - 1)$ -dimensional space M . Now in $P_q(n, e + 1)$ the distance between L and M is 2 and it is easy to see that they have $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} e+1 \\ 1 \end{bmatrix}$ common neighbours where in the Grassmann graph $c_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2$.

If $n = 2e + 1$, then it is possible to check that it is distance-regular. An easy way to see this is true we use a result by Fiol and Garriga which states that if a graph has the same spectrum as a distance-regular graph Γ with diameter d is distance-regular iff for all vertices x we have $k_d(x) = k_d(\Gamma)$. And this is easily checked.

(iii) Let $n = 2e + 1$. Let K be an $(e + 2)$ -dimensional space which intersects H in $e + 1$ dimensions. Now the $(e + 1)$ -dimensional subspaces of K which are not contained in H form a maximal clique of size $\begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$ in $P_q(2e + 1, e + 1)$, whereas the Grassmann graph $G_q(2e + 1, e + 1)$ has maximal cliques of sizes $\begin{bmatrix} e+2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$.

This shows the theorem.

(i) By looking at maximal cliques in $P_q(2e + 1, e + 1)$, it is easy to see that it is not vertex-transitive. The group $P\Gamma L(2e + 1)_{2e}$ is an automorphism group of the graph. It was shown by M. Tagami that this is the full automorphism group.

(ii) For large q and e we were able to show that the local graph of a line of type 1 is not cospectral to the local graph of a line of type 2. We suspect that this is always the case. This implies, for example, that the Terwilliger Algebra depends on the base vertex for $P_q(2e + 1, e + 1)$.