

Some isomorphism theorems of cohomology groups for completely integrable connections

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1 Introduction

About 14 years ago, the author proved an isomorphism theorem between the cohomology group of complex of global meromorphic sections derived from a completely integrable connection and the cohomology group of kernel sheaf with values in the sheaf of functions asymptotically developable to the formal series 0 for the connection ([5]). Recently, several rescerchers, who are interested in the intersection theory for differential equations with singular points, pushed him to prove the C^∞ version (cf. [1], [2]). In this paper, firstly, we give a short review of the isomorphism theorem in asymptotic analysis and some examples with concrete calculation of basis for the cohomology groups. Secondly, we explain the C^∞ version.

2 Isomorphism Theorem in Asymptotic Analysis

Let M be a complex manifold and let H be a divisor on M at most normal crossing singularities. we denote by $\Omega^p(*H)$ the sheaf of germs of meromorphic p -forms which are holomorphic in $M - H$ and have poles on H and denote by $\mathcal{S}$ a locally free sheaf of $\mathcal{O}$ -modules of rank m on M . Put $\mathcal{S}\Omega^p(*H) = \Omega^p(*H) \otimes_{\mathcal{O}} \mathcal{S}$ for $p = 0, \dots, n$. For $p = 0$, instead of $\mathcal{S}\Omega^0(*H)$, we use frequently $\mathcal{S}(*H)$ of which the restriction to U , $\mathcal{S}(*H)|_U$ is isomorphic to

$$(\mathcal{O}(*H))^m|_U = (C^m \otimes_C \mathcal{O}(*H))|_U$$

and the isomorphism is denoted by g_U .

Let ∇ be a connection on $\mathcal{S}\Omega^0(*H)$: ∇ is an additive mapping of $\mathcal{S}\Omega^0(*H)$ into $\mathcal{S}\Omega^1(*H)$ satisfying "Leipnitz' rule"

$$\nabla(f \cdot u) = u \otimes df + f \cdot \nabla(u),$$

for all sections $f \in \Omega^0(*H)(U)$ and $u \in \mathcal{S}\Omega^1(*H)(U)$. We suppose that the connection is integrable, that is, the composite mapping

$$\nabla_2 : \mathcal{S}\Omega^0(*H) \longrightarrow \mathcal{S}\Omega^1(*H) \longrightarrow \mathcal{S}\Omega^2(*H),$$

is a zero mapping.

If we take adequately an open covering $\{U_k\}_k$ on M , then to give the connection ∇ means as follows: for each U_k , the mapping

$$g_{U_k} \circ \nabla \circ g_{U_k}^{-1} : \Omega^0(*H)(U_k)^m \longrightarrow \Omega^1(*H)(U_k)^m,$$

is induced by a mapping

$$\nabla_k : \Omega^0(*H)(U_k)^m \longrightarrow \Omega^1(*H)(U_k)^m,$$

which is represented by $(d + \Omega_k)$ under a generator system

$$\langle e_{k,1}, \dots, e_{k,m} \rangle$$

of $(\mathcal{O}(U_k))^m$, i.e.

$$\nabla_k(\langle e_{k,1}, \dots, e_{k,m} \rangle u) = \langle e_{k,1}, \dots, e_{k,m} \rangle (du + \Omega_k u)$$

where Ω_k is an m -by- m matrix of meromorphic 1-forms on U_k at most with poles in $U_k \cap H$.

Let $x_1, \dots, x_n$ be holomorphic local coordinates on U_k and suppose

$$U_k \cap H = \{(x_1, \dots, x_n) \mid x_1 \cdots x_{n''} = 0\},$$

then Ω_k is of the form

$$\Omega_k = \sum_{i=1}^{n''} x^{-p_i} x_i^{-1} A_i(x) dx_i + \sum_{i=n''+1}^n x^{-p_i} A_i(x) dx_i,$$

where $p_i = (p_{i1}, \dots, p_{in''}, 0, \dots, 0) \in \mathbb{N}^n$ and $A_i(x)$ is an m -by- m matrix of holomorphic functions in U_k for $i = 1, \dots, n$.

The connection ∇ is integrable if and only if, for k , $d\Omega_k + \Omega_k \wedge \Omega_k = 0$. For any k, k' , denote by $g_{kk'}$ the isomorphism

$$g_{kk'} : (\mathcal{O}(U_k \cap U_{k'}))^m \longrightarrow (\mathcal{O}(U_k \cap U_{k'}))^m,$$

induced by the isomorphism

$$g_{U_k} g_{U_{k'}}^{-1} : (\mathcal{O}|_{U_k \cap U_{k'}})^m \longrightarrow (\mathcal{O}|_{U_k \cap U_{k'}})^m.$$

Then, by using the generator systems, $g_{kk'}$ is represented by $G_{kk'}$ a matrix of elements in $\mathcal{O}(U_k \cap U_{k'})$, i.e.

$$g_{kk'} \langle e_{k,1}, \dots, e_{k,m} \rangle = \langle e_{k,1}, \dots, e_{k,m} \rangle G_{kk'},$$

and

$$\Omega_k = G_{kk'}^{-1} dG_{kk'} + G_{kk'}^{-1} \Omega_k G_{kk'},$$

in $U_k \cap U_{k'}$.

Denote by M^- the real blow-up of M along H and denote by pr the natural projection from M^- to M . Let $\mathcal{A}^-$ be the sheaf of germs of functions strongly asymptotically developable, and let $\mathcal{A}'^-$ and $\mathcal{A}_0^-$ be the sheaves of germs of functions strongly asymptotically developable to $\mathcal{O}_{M|H}^\wedge$ and to 0, respectively, over the real blow-up M^- . Define the locally free $\mathcal{A}^-$ (resp. $\mathcal{A}'^-$)-sheaf $\mathcal{S}^-\Omega^p(*H)$ (resp. $\mathcal{S}'^-\Omega^p(*H)$) over the real blow-up M^- by $\mathcal{S}^-\Omega^p(*H) = \mathcal{A}^- \otimes_{pr^*\mathcal{O}} pr^*\mathcal{S}\Omega^p(*H)$ (resp. $\mathcal{S}'^-\Omega^p(*H) = \mathcal{A}'^- \otimes_{pr^*\mathcal{O}} pr^*\mathcal{S}^-\Omega^p(*H)$), and the locally free $\mathcal{A}_0^-$ -sheaf $\mathcal{S}_0^-\Omega^p$ by $\mathcal{S}_0^-\Omega^p = \mathcal{A}_0^- \otimes_{pr^*\mathcal{O}} pr^*\mathcal{S}\Omega^p(*H)$ for $p = 0, \dots, n$. Then, by a natural way, we obtain integrable connections

$$\nabla^- : \mathcal{S}^-(*H) \longrightarrow \mathcal{S}^-\Omega^1(*H),$$

$$\nabla'^- : \mathcal{S}'^-(*H) \longrightarrow \mathcal{S}'^-\Omega^1(*H),$$

and

$$\nabla_0^- : \mathcal{S}_0^- \longrightarrow \mathcal{S}_0^-\Omega^1(*H).$$

For simplicity, we use also ∇ instead of ∇^- , ∇'^- and ∇_0^- . By the integrability, we can consider the complexes of sheaves

$$\mathcal{S}^-(*H) \xrightarrow{\nabla} \mathcal{S}^-\Omega^1(*H) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \mathcal{S}^-\Omega^n(*H) \xrightarrow{\nabla} 0$$

$$\mathcal{S}'^-(*H) \xrightarrow{\nabla} \mathcal{S}'^-\Omega^1(*H) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \mathcal{S}'^-\Omega^n(*H) \xrightarrow{\nabla} 0$$

$$\mathcal{S}_0^- \xrightarrow{\nabla} \mathcal{S}_0^-\Omega^1(*H) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \mathcal{S}_0^-\Omega^n(*H) \xrightarrow{\nabla} 0.$$

Suppose here that ∇ satisfies the following condition: for any point $p \in H$, under the local representation of ∇ ,

(H.1) $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n]$,

or

(H.2) $p_{ii} > 0$ and $A_i(0)$ is invertible for all $i \in [1, n'']$ or $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n'']$.

Then, we can assert

Theorem 1. If the assumption (H.1) is satisfied for any point in H , then the above three sequences are exact. If (H.1) or (H.2) is satisfied for any point in H , then the above sequences are exact except the second.

Moreover, we consider the complex $(\Gamma(M^-, \mathcal{S}^-\Omega^\bullet(*H)), \nabla)$ of global sections:

$$\mathcal{S}^-(*H)(M^-)^m \xrightarrow{\nabla} \mathcal{S}^-\Omega^1(*H)(M^-)^m \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \mathcal{S}^-\Omega^1(*H)(M^-)^m \xrightarrow{\nabla} 0.$$

Then, we can prove

Theorem 2. If $H^1(M, \mathcal{S}) = 0$ and if (H.1) or (H.2) is satisfied for any point in H , then the following isomorphism is valid:

$$H^1(\Gamma(M^-, \mathcal{S}^-\Omega^\bullet(*H)), \nabla) \cong H^1(M^-, \text{Ker } \nabla_0^-),$$

where $\mathcal{Ker}\nabla_0$ denote the sheaf of solutions of ∇_0^- .

Note that we have the natural isomorphism by the projection pr

$$H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) \cong H^1(\Gamma(M^-, S^-\Omega^\bullet(*H)))$$

and we can rewrite the theorem as

Theorem 2'. If $H^1(M, S) = 0$ and if (H.1) or (H.2) is satisfied for any point in H , then the following isomorphism is valid:

$$H^1(\Gamma(M, S\Omega^\bullet(*H)), \nabla) \cong H^1(M^-, \mathcal{Ker}\nabla_0^-).$$

Example. Consider the case where $M = \mathbb{P}_{\mathbb{C}}^1$, $H = \{\infty\}$ and $\nabla = d + x^{r-1}\wedge$. We can find the basis of $H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla)$:

$$H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) = \mathbb{C} \langle [dx], \dots, [x^{r-2}dx] \rangle.$$

On the other hand, we can find the basis of $H^1(M^-, \mathcal{Ker}\nabla_0^-)$ in the following manner. Let $\{U_k \mid k = 1, \dots, r\}$ be the covering of $M - H$, where

$$U_k = \{x \in \mathbb{C} \mid |x| \geq R, \frac{(4k-5)\pi}{2r} < \arg x < \frac{(4k+1)\pi}{2r}\} \cup \{x \in \mathbb{C} \mid |x| < R\}$$

for $k = 1, \dots, r$. We put $U_{r+1} = U_1$ and for $k = 1, \dots, r$, define 1-cocycles $\{f_{j,j+1}^{(k)}\} (j = 1, \dots, r)$ by

$$f_{j,j+1}^{(k)}(x) = \begin{cases} \exp(-\frac{1}{r}x^r), & (x \in U_j \cap U_{j+1}) (j = k) \\ 0, & (x \in U_j \cap U_{j+1}) (j \neq k) \end{cases}$$

Then, we have

$$\langle \{f_{j,j+1}^{(k)}\}_{j=1, \dots, r}, k = 1, \dots, r \rangle$$

as a basis of $H^1(M^-, \mathcal{Ker}\nabla_0^-)$.

3 Isomorphism Theorem in C^∞ case

We restrict here to treat the case of one variable. We give a C^∞ version of isomorphism theorem of cohomology group. Let M, H, ∇ be as above. Let $\mathcal{P}_0^{(j,h)}$ be the sheaf of germs of $C^\infty(j, h)$ -forms infinitely flat on H over M . Consider the following double complex of sheaves:

$$\begin{array}{ccc} \mathcal{P}_0^{(0,0)} & \xrightarrow{\bar{\partial}} & \mathcal{P}_0^{(0,1)} \\ \nabla \downarrow & & \nabla \downarrow \\ \mathcal{P}_0^{(1,0)} & \xrightarrow{\bar{\partial}} & \mathcal{P}_0^{(1,1)} \end{array}$$

and the complex of global sections

$$\begin{array}{ccc} \mathcal{P}_0^{(0,0)}(M) & \xrightarrow{\bar{\partial}} & \mathcal{P}_0^{(0,1)}(M) \\ \nabla \downarrow & & \nabla \downarrow \\ \mathcal{P}_0^{(1,0)}(M) & \xrightarrow{\bar{\partial}} & \mathcal{P}_0^{(1,1)}(M) \end{array}$$

and the associated simple complex

$$GC^\infty K : \mathcal{P}_0^{(0,0)}(M) \xrightarrow{\nabla_\omega + \bar{\partial}} \mathcal{P}_0^{(0,1)}(M) \oplus \mathcal{P}_0^{(1,0)}(M) \xrightarrow{\nabla_\omega + \bar{\partial}} \mathcal{P}_0^{(1,1)}(M) \longrightarrow 0.$$

Then, we know the following lemma formally due to Malgrange ([6]).

Lemma 3. We have the following isomorphism for $j = 0, 1, 2$:

$$H^j(M^-, \text{Ker} \nabla_0^-) \cong H^j(GC^\infty K).$$

By Theorem 2' and Lemma 3, we can derive the

Theorem 4. We have the following isomorphism:

$$H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) \cong H^1(GC^\infty K).$$

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