

On some subclasses of univalent functions

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ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and w is a fixed point in U . In this paper we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in U .

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

We recall here the definitions of the well - known classes of starlike, convex, close to convex and α - convex functions:

$$\begin{aligned} S^* &= \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U \right\}, \\ S^c &= \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in U \right\} \\ CC &= \left\{ f \in A : \exists g \in S^*, \operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U \right\}, \\ M_\alpha &= \left\{ f \in A : \frac{f(z) \cdot f'(z)}{z} \neq 0, \operatorname{Re} J(\alpha, f; z) > 0, \quad z \in U \right\} \end{aligned}$$

where $J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right)$.

Let w be a fixed point in U and $A(w) = \{f \in \mathcal{H}(U) : f(w) = f'(w) - 1 = 0\}$.

In [3] S. Kanas and F. Ronning introduced the following classes:

$$\begin{aligned} S(w) &= \{f \in A(w) : f \text{ is univalent in } U\} \\ ST(w) &= S^*(w) = \left\{ f \in S(w) : \operatorname{Re} \frac{(z-w)f'(z)}{f(z)} > 0, \quad z \in U \right\} \\ CV(w) &= S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re} \frac{(z-w)f''(z)}{f'(z)} > 0, \quad z \in U \right\}. \end{aligned}$$

The class $S^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $S^c(w)$ is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones for uniformly starlike and convex functions introduced by A. W. Goodman in [1] and [2], except that in this case the point w is fixed.

It is obvious that exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \text{ if and only if } f(z) = (z - w)g'(z) \in S^*(w).$$

Let denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$ that are regular in U and satisfy $p(w) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in U$.

2 Preliminary results

It is easy to see that a function $f(z) \in A(w)$ have the series expansions:

$$f(z) = (z - w) + a_2(z - w)^2 + \dots$$

In [7] J. K. Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$:

Teorema 2.1 *If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$, then*

$$(1) \quad |B_n| \leq \frac{2}{(1+d)(1-d)^n}, \quad \text{where } d = |w| \text{ and } n \geq 1.$$

Using the above result, S. Kanas and F. Ronning obtain in [3]:

Teorema 2.2 *Let $f \in S^*(w)$ and $f(z) = (z - w) + a_2(z - w)^2 + \dots$ Then*

$$(2) \quad \begin{aligned} |a_2| &\leq \frac{2}{1-d^2}, \quad |a_3| \leq \frac{3+d}{(1-d^2)^2}, \\ |a_4| &\leq \frac{2}{3} \cdot \frac{(2+d)(3+d)}{(1-d^2)^3}, \quad |a_5| \leq \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4} \end{aligned}$$

where $d = |w|$.

Remark 2.1 *It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.*

The next theorem is the result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [4], [5], [6]).

Teorema 2.3 *Let h convex in U and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and p satisfied the Briot - Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } p(z) \prec h(z).$$

3 Main results

Let consider the integral operator $L_a : A(w) \rightarrow A(w)$ defined by

$$(3) \quad f(z) = L_a F(z) = \frac{1+a}{(z-w)^a} \cdot \int_w^z F(t) \cdot (t-w)^{a-1} dt, \quad a \in \mathbb{R}, \quad a \geq 0.$$

We denote by $D(w) = \left\{ z \in U : Re \left[\frac{w}{z} \right] < 1 \text{ and } Re \left[\frac{z(1+z)}{(z-w)(1-z)} \right] > 0 \right\}$, with $D(0) = \bar{U}$, and $s(w) = \{f : \bar{D}(w) \rightarrow \mathbb{C}\} \cap S(w)$, where w is a fixed point in $\bar{U}$.

Denoting $s^*(w) = S^*(w) \cap s(w)$, where w is a fixed point in $\bar{U}$, we obtain:

Teorema 3.1 *Let w be a fixed point in U and $F(z) \in s^*(w)$. Then $f(z) = L_a F(z) \in S^*(w)$, where the integral operator L_a is defined by (3).*

Proof. By differentiating (3) we obtain:

$$(4) \quad (1+a)F(z) = a \cdot f(z) + (z-w) \cdot f'(z).$$

From (4) we have:

$$(5) \quad (1+a)F'(z) = (1+a)f'(z) + (z-w)f''(z).$$

Using (4) and (5) we obtain:

$$(6) \quad \frac{(z-w)F'(z)}{F(z)} = \frac{(1+a) \cdot (z-w) \cdot \frac{f'(z)}{f(z)} + (z-w)^2 \frac{f''(z)}{f(z)}}{a + (z-w) \frac{f'(z)}{f(z)}}.$$

With notation $p(z) = \frac{(z-w)f'(z)}{f(z)}$, where $p(z) \in \mathcal{H}(U)$ and $p(0) = 1$, we have:

$$(z-w)p'(z) = p(z) + (z-w)^2 \cdot \frac{f''(z)}{f(z)} - [p(z)]^2$$

and thus:

$$(7) \quad (z-w)^2 \frac{f''(z)}{f'(z)} = (z-w)p'(z) - p(z)[1-p(z)].$$

Using (6) and (7) we obtain:

$$(8) \quad \frac{(z-w)F'(z)}{F(z)} = p(z) + \frac{(z-w) \cdot p'(z)}{a+p(z)}.$$

Using $F(z) \in s^*(w)$ from (8) we have:

$$p(z) + \frac{z-w}{a+p(z)} \cdot p'(z) \prec \frac{1+z}{1-z} \equiv h(z)$$

or

$$p(z) + \frac{1-\frac{w}{z}}{a+p(z)} \cdot zp'(z) \prec \frac{1+z}{1-z}.$$

From hypothesis we have $\operatorname{Re} \left[\frac{1}{1-\frac{w}{z}} \cdot h(z) + \frac{a}{1-\frac{w}{z}} \right] > 0$ and thus from Theorem 2.3

we obtain $p(z) \prec \frac{1+z}{1-z}$ or $\operatorname{Re} \left(\frac{(z-w)f'(z)}{f(z)} \right) > 0, z \in U$. This means $f(z) \in S^*(w)$.

Definition 3.1 Let $f \in S(w)$ where w is a fixed point in U . We say that f is w -close to convex if exists a function $g \in S^*(w)$ such that $\operatorname{Re} \frac{(z-w)f'(z)}{g(z)} > 0, z \in U$. We denote this class with $CC(z)$.

Remark 3.1 If we consider $f = g, g \in S^*(w)$, we have $S^*(w) \subset CC(w)$.

If we take $w = 0$ we obtain the well known close to convex functions.

Teorema 3.2 Let w be a fixed point in U and $f \in CC(w)$, $f(z) = (z-w) + \sum_{n=2}^{\infty} b_n \cdot (z-w)^n$,

with respect to the function $g \in S^*(w)$, $g(z) = (z-w) + \sum_{n=2}^{\infty} a_n \cdot (z-w)^n$. Then

$$|b_n| \leq \frac{1}{n} \left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot \frac{2}{(1+d)(1-d)^{n-k}} \right]$$

where $d = |w|$, $n \geq 2$ and $a_1 = 1$.

Proof. Let $f \in CC(w)$ with respect to the function $g \in S^*(w)$. Then there exists a function $p \in \mathcal{P}(w)$ such that

$$\frac{(z-w)f'(z)}{g(z)} = p(z)$$

where $p(z) = 1 + \sum_{n=1}^{\infty} B_n(z-w)^n$.

Using the hypothesis through identification of $(z-w)^n$ coefficients we obtain:

$$(9) \quad n \cdot b_n = a_n + \sum_{k=1}^{n-1} a_k \cdot B_{n-k}$$

where $a_1 = 1$ and $n \geq 2$.

From (9) we have

$$|b_n| \leq \frac{1}{n} \left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot |B_{n-k}| \right], \quad a_1 = 1, \quad n \geq 2.$$

Applying the above and the estimates (1) we obtain the result.

Remark 3.2 If we use the estimates (2) we obtain the same estimates for the coefficients b_n , $n = 2, 3, 4, 5$.

Definition 3.2 Let $\alpha \in \mathbb{R}$ and w be a fixed point in U . For $f \in S(w)$ we denote by $J(\alpha, f, w; z) = (1 - \alpha) \frac{(z-w)f'(z)}{f(z)} + \alpha \left[1 + \frac{(z-w)f''(z)}{f'(z)} \right]$. We say that f is $w - \alpha$ -convex function if $\frac{f(z) \cdot f'(z)}{z-w} \neq 0$ and $\operatorname{Re} J(\alpha, f, w; z) > 0$, $z \in U$. We denote this class with $M_\alpha(w)$.

Remark 3.3 It is easy to observe that $M_\alpha(0)$ is the well known class of α -convex functions.

Teorema 3.3 Let w be a fixed point in U , $\alpha \in \mathbb{R}$, $\alpha \geq 0$ and $m_\alpha(w) = M_\alpha(w) \cap s(w)$.

1. If $f \in m_\alpha(w)$ then $f \in S^*(w)$. This means $m_\alpha(w) \subset S^*(w)$.
2. If $\alpha, \beta \in \mathbb{R}$, with $0 \leq \frac{\beta}{\alpha} < 1$, then $m_\alpha(w) \subset m_\beta(w)$.

Proof. From $f \in m_\alpha(w)$ we have $\operatorname{Re} J(\alpha, f, w; z) > 0$, $z \in U$. Using the notation $p(z) = \frac{(z-w)f'(z)}{f(z)}$, with $p \in \mathcal{H}(U)$ and $p(0) = 1$, we obtain:

$$\operatorname{Re} J(\alpha, f, w; z) = \operatorname{Re} \left[p(z) + \alpha \cdot \frac{(z-w)p'(z)}{p(z)} \right] > 0, \quad z \in U \quad \text{or}$$

$$p(z) + \frac{\alpha \left(1 - \frac{w}{z} \right)}{p(z)} \cdot zp'(z) \prec \frac{1+z}{1-z} \equiv h(z).$$

For $\alpha = 0$ we have $p(z) \prec \frac{1+z}{1-z}$.

Using the hypothesis we have for $\alpha > 0$, $\operatorname{Re} \left[\frac{1}{\alpha \left(1 - \frac{w}{z}\right)} \cdot h(z) \right] > 0$ and from Theo-

rem 2.3 we obtain $p(z) \prec \frac{1+z}{1-z}$.

This means that $\operatorname{Re} \frac{(z-w)f'(z)}{f(z)} > 0$, $z \in U$ and $\alpha \geq 0$ or $f \in S^*(w)$.

If we denote by $A = \operatorname{Re} p(z)$ and by $B = \operatorname{Re} \frac{(z-w)p'(z)}{p(z)}$ we have $A > 0$ and $A + B \cdot \alpha > 0$, where $\alpha \geq 0$.

Using the geometric interpretation of the equation $y(x) = A + B \cdot x$, $x \in [0, \alpha]$ we obtain

$$y(\beta) = A + B \cdot \beta > 0 \text{ for every } \beta \in [0, \alpha].$$

This means $\operatorname{Re} \left[p(z) + \beta \cdot \frac{(z-w)p'(z)}{p(z)} \right] > 0$, $z \in U$ or $f \in m_\beta(w)$.

Remark 3.4 From the above theorem we have:

$$m_1(w) \subseteq s^c(w) \subseteq m_\alpha(w) \subseteq s^*(w)$$

where $0 \leq \alpha \leq 1$ and $s^c(w) = S^c(w) \cap s(w)$.

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