

# Remarks on the $L^p$ -boundedness of wave operators for fourth-order Schrödinger operators on the line

By

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## Abstract

We discuss a recent progress [MWY] on the  $L^p$ -boundedness of wave operators for fourth-order Schrödinger operators on the line.

## § 1. Introduction and main result

### § 1.1. Introduction

This is a survey of [MWY] on the  $L^p$ -boundedness of wave operators for the following fourth-order Schrödinger operator on  $\mathbb{R}$ :

$$H = H_0 + V(x), \quad x \in \mathbb{R},$$

where  $H_0 = \frac{d^4}{dx^4}$  and  $V \in L^\infty(\mathbb{R}; \mathbb{R})$  satisfies, with some  $\mu > 1$ ,

$$(1.1) \quad |V(x)| \lesssim \langle x \rangle^{-\mu}, \quad x \in \mathbb{R}.$$

By the Kato–Rellich theorem,  $H$  is self-adjoint on  $L^2(\mathbb{R})$  with domain  $D(H) = D(H_0) = H^4(\mathbb{R})$ . The main object of this paper is the *wave operator* defined by

$$W_\pm f = W_\pm(H, H_0)f := \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} f, \quad f \in L^2(\mathbb{R}).$$

It is well-known that if  $\mu > 1$ , then  $W_\pm$  are well-defined as partial isometries on  $L^2(\mathbb{R})$ . In particular, they are bounded on  $L^2(\mathbb{R})$ . Moreover, they are asymptotically complete in the sense that  $\text{Ran}(W_\pm) = \mathcal{H}_{ac}(H)$ , or equivalently, the following inverse wave operators also exist:

$$W_\pm(H_0, H)f = \lim_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH} P_{ac}(H)f, \quad f \in L^2(\mathbb{R}),$$

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where  $\mathcal{H}_{\text{ac}}(H)$  and  $P_{\text{ac}}(H)$  denote the absolutely continuous spectral subspace of  $H$  and the projection onto  $\mathcal{H}_{\text{ac}}(H)$  (see [Ku]). Note that  $W_{\pm}(H_0, H) = W_{\pm}^*$ . Under the same condition, it is also known that  $\sigma_{\text{sc}}(H) = \emptyset$ , so  $\mathcal{H}_{\text{ac}}(H)$  is an orthogonal complement of the pure point spectral subspace  $\mathcal{H}_{\text{pp}}(H)$  of  $H$ . Then we are interested in various mapping properties of  $W_{\pm}$ , especially the  $L^p$ -boundedness for  $p \neq 2$ . To explain how important this topic is, we recall that the intertwining property  $HW_{\pm} = W_{\pm}H_0$  and the asymptotic completeness imply

$$f(H)P_{\text{ac}}(H) = W_{\pm}f(H_0)W_{\pm}^*$$

for any Borel measurable function  $f$  on  $\mathbb{R}$ , where  $f(H)$  is defined by the spectral theorem. By virtue of this formula, the  $L^p$ -boundedness of  $W_{\pm}$  can be applied to deduce  $L^p$ - $L^q$  estimates for  $f(H)P_{\text{ac}}(H)$  to that of  $f(H_0)$ , namely one has

$$\|f(H)P_{\text{ac}}(H)\|_{L^p \rightarrow L^q} \leq \|W_{\pm}\|_{L^q \rightarrow L^q} \|f(H_0)\|_{L^p \rightarrow L^q} \|W_{\pm}^*\|_{L^p \rightarrow L^p},$$

whenever the right hand side is finite. More generally, for any function spaces  $X, Y$  such that  $X \cap L^2$  and  $Y \cap L^2$  are dense in  $L^2$ , we have

$$\|f(H)P_{\text{ac}}(H)\|_{X \rightarrow Y} \leq \|W_{\pm}\|_{Y \rightarrow Y} \|f(H_0)\|_{X \rightarrow Y} \|W_{\pm}^*\|_{X \rightarrow X}.$$

Since  $f(H_0) = \mathcal{F}^{-1}f(|\xi|^4)\mathcal{F}$  is a Fourier multiplier, there are many techniques such as the integration by parts, the stationary phase method, the Fourier restriction theorem and so on, to establish various mapping properties for  $f(H_0)$ , while this is not always the case for  $f(H)$ . Because of such a useful feature, the  $L^p$ -boundedness of the wave operators has been extensively studied for the Schrödinger operator  $-\Delta + V(x)$  on  $\mathbb{R}^d$  in any dimensions and there are a few results for higher-order Schrödinger operators on  $\mathbb{R}^d$  and other models. I take this opportunity to record an almost complete list at present of existing literature as follows:

- $-\Delta + V(x)$  on  $\mathbb{R}^d$  with regular time-independent potentials  $V(x)$ : [Ya1, Ya2, Ya3, Ya4, We1, AY, JY1, DF, Ya5, FY, JY2, Be, Ya6, GG1, GG2, EGG1, Ya7, BS1, BS2, We2, Ya9, Ya10];
- $-\Delta + \delta$  on  $\mathbb{R}^d$  with point-interactions  $\delta$ : [DMW, DMSY, CMY, Ya8, We2];
- $-\Delta + V(t, x)$  on  $\mathbb{R}^d$  with time-dependent potentials  $V(t, x)$ : [Je, SW1, SW2];
- $(-\Delta)^m + V(x)$  on  $\mathbb{R}^d$  with integer  $m \geq 2$ : [GG3, EG1, EGG2, EG2];
- Other models: [Cu1, CT] on a matrix Schrödinger operator arising from nonlinear problems and [Cu2] on the discrete Schrödinger operator on  $\mathbb{Z}$ .

Let us focus on the higher-order Schrödinger operator  $L_m = (-\Delta)^m + V$  on  $\mathbb{R}^d$ , explaining known literature more precisely. Zero is said to be a resonance of  $L_m$  if there exists  $f \in L^2_{-s} \setminus L^2$  with some  $s = s(m, d) > 0$  such that  $L_m f = 0$  in the distribution sense, where  $L^2_s = \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \langle x \rangle^s f \in L^2(\mathbb{R}^d)\}$  is the weighted  $L^2$ -space. We say that  $L_m$  is regular if  $L_m$  has neither zero eigenvalue nor zero resonance (see below the precise definition in the present case). For  $L_2 = \Delta^2 + V$  on  $\mathbb{R}^3$ , [GG3] proved  $L^p$ -boundedness of wave operators for all  $1 < p < \infty$  under the conditions that  $L_2$  is regular and has no embedded eigenvalues in  $(0, \infty)$  and  $|V(x)| \lesssim \langle x \rangle^{-\mu}$  with  $\mu > 9$ . Later, this result has been extended by [EG1, EG2] to the general  $L_m$  with integer  $m \geq 2$  and  $d > 2m$  and sufficiently regular and decaying potentials. These results are basically generalizations of Yajima's seminal work [Ya1] to the higher-order case. In particular,  $L_m$  is assumed to be regular in all these results and, to the author's best knowledge, there seems to be no results on the  $L^p$ -boundedness of the wave operators for the higher-order Schrödinger operators if there is a zero resonance or in low dimensions  $d = 1, 2$ .

As a simplest model of higher-order Schrödinger operators with zero resonances, we have considered in [MWY] the one dimensional fourth-order case and obtained the  $L^p$ -boundedness of wave operators for all  $1 < p < \infty$  whatever the zero energy is a regular point or a resonance. Moreover, we have also proved some weak-type boundedness in the endpoint cases  $p = 1, \infty$  and that the wave operators are neither bounded on  $L^1$  nor on  $L^\infty$  in the regular case.

### § 1.2. Main result

To state the results, we first recall the precise definition of zero resonances. Let  $W_\sigma = \bigcap_{s > \sigma} L^2_{-s}$ . We say that

- zero is a *first kind resonance* if there exists  $f \in W_{3/2} \setminus \{0\}$  but no any  $f \in W_{1/2} \setminus \{0\}$  such that  $Hf = 0$  in the distribution sense;
- zero is a *second kind resonance* if there exists  $f \in W_{1/2} \setminus \{0\}$  but no any  $f \in L^2 \setminus \{0\}$  such that  $Hf = 0$  in the distribution sense;
- zero is a *regular point* if  $H$  has neither zero eigenvalue nor zero resonances.

The main result of [MWY] is as follows:

**Theorem 1.1** (Weighted  $L^p$ -boundedness). *Suppose that  $H$  has no embedded eigenvalue in  $(0, \infty)$  and (1.1) holds for some  $\mu$  satisfying*

$$(1.2) \quad \mu > \begin{cases} 15 & \text{if zero is a regular point of } H, \\ 21 & \text{if zero is a first kind resonance of } H, \\ 29 & \text{if zero is a second kind resonance of } H. \end{cases}$$

Then, for any  $1 < p < \infty$  and any even  $A_p$ -weight  $w_p$ , we have

$$(1.3) \quad W_{\pm}, W_{\pm}^* \in \mathbb{B}(L^p(w_p))$$

Moreover, we have the following weak boundedness in the endpoint cases  $p = 1, \infty$ :

$$(1.4) \quad \begin{aligned} W_{\pm} &\in \mathbb{B}(L^1(w_1), L^{1,\infty}(w_1)) \cap \mathbb{B}(\mathcal{H}^1(\mathbb{R}), L^1(\mathbb{R})), \\ W_{\pm}^* &\in \mathbb{B}(L^1(w_1), L^{1,\infty}(w_1)) \cap \mathbb{B}(L^\infty(\mathbb{R}), \text{BMO}(\mathbb{R})) \end{aligned}$$

for any even  $A_1$ -weight  $w_1$ .

Here  $\mathbb{B}(X, Y)$  denotes the space of bounded operators from  $X$  to  $Y$  and  $\mathbb{B}(X) = \mathbb{B}(X, X)$ .  $A_p$  is the Muckenhoupt class,  $L^p(w)$ ,  $L^{1,\infty}(w)$ ,  $\mathcal{H}^1(\mathbb{R})$  and  $\text{BMO}(\mathbb{R})$  are the weighted  $L^p$ , weighted weak  $L^1$ , Hardy and Bounded Mean Oscillation spaces on  $\mathbb{R}$ , respectively. The norm  $\|\cdot\|_{L^p(w)}$  is given by

$$\begin{aligned} \|f\|_{L^p(w)} &= \left( \int |f(x)|^p w(x) dx \right)^{1/p}, \\ \|f\|_{L^{1,\infty}(w)} &= \sup_{\lambda > 0} \lambda w(\{x \mid |f(x)| > \lambda\}). \end{aligned}$$

Typical examples of  $A_p$ -weights we have in mind are the polynomial weights  $|x|^a$  and  $\langle x \rangle^a$  for  $-1 < a < p - 1$  if  $p > 1$  and  $-1 < a \leq 0$  if  $p = 1$ .

**Remark 1.2.** (1) Not only the weighted  $L^p$ -boundedness (1.3), we also obtain the following bounds with an explicit dependence on the weights:

$$\begin{aligned} \|W_{\pm} f\|_{L^p(w_p)} &\lesssim [w_p]_{A_p}^{\max\{1, 1/(p-1)\}} \|f\|_{L^p(w_p)}, \quad 1 < p < \infty, \\ \|W_{\pm} f\|_{L^{1,\infty}(w_1)} &\lesssim [w_1]_{A_1} (1 + \log[w]_{A_1}) \|f\|_{L^1(w_1)}, \end{aligned}$$

where  $[w]_{A_p}$  is the  $A_p$ -characteristic constant of  $w$  (see [MWY, Appendix A]). The same bounds also hold for  $W_{\pm}^*$ .

(2) In contrast with the case of (second-order) Schrödinger operators on  $\mathbb{R}^d$ , it is easy to construct a potential  $V$  in the Schwartz class such that  $\Delta^2 + V$  on  $\mathbb{R}^d$  has an embedded eigenvalues in  $(0, \infty)$ . Indeed, if we take

$$V(x) = \frac{20}{\cosh^2(x)} - \frac{24}{\cosh^4(x)}, \quad f(x) = \frac{1}{\cosh(x)},$$

then  $Hf = f$ . On the other hand, if  $V(x) = O(\langle x \rangle^{-\mu})$  with  $\mu > 15$  then zero is not an eigenvalue of  $H$  ([SWY, Remark 1.2]). Moreover, if  $V \in C^1 \cap L^\infty$  satisfies  $xV'(x) \leq 0$  then  $H$  has no eigenvalues ([FSWY, Theorem 1.11]).

(3) We believe that (1.2) is far from the optimal, while an expected optimal condition

would be  $\langle x \rangle^3 V \in L^1(\mathbb{R})$ . In case of the Schrödinger operator  $-\frac{d^2}{dx^2} + V$  on  $\mathbb{R}$ , an essentially same results (except for the weighted boundedness) as in Theorem 1.1 were proved by [DF, We2] under the condition  $\langle x \rangle V \in L^1(\mathbb{R})$ . Both are heavily relied on the technique of the Jost functions ([DT]). However, it is not known whether a similar technique as that of these papers [DT] can be applied to the fourth-order case even if  $V$  is rapidly decreasing (see [Hi]).

The following is an immediate corollary of Theorem 1.1 and Remark 1.2:

**Corollary 1.3.** *Suppose that  $H$  has no eigenvalues in  $(0, \infty)$  and  $\mu > 29$ . Then (1.3) for all  $1 < p < \infty$  and (1.4) hold.*

We also have the boundedness on the Sobolev space  $W^{s,p}(\mathbb{R})$ .

**Corollary 1.4** ( $W^{s,p}$ -boundedness). *Under the same condition as in Theorem 1.1,*

$$(1.5) \quad W_{\pm}, W_{\pm}^* \in \mathbb{B}(W^{s,p}(\mathbb{R})), \quad 1 < p < \infty,$$

*all for  $0 \leq s \leq 4$ . Moreover, if we assume in addition  $V \in C^N(\mathbb{R})$  and  $V^{(k)} \in L^\infty(\mathbb{R})$  for all  $0 \leq k \leq N$  with some  $N \in \mathbb{N}$ , then (1.5) holds for all  $0 \leq s \leq 4(N+1)$ .*

In Theorem 1.1, we have obtained the  $L^p$ -boundedness of  $W_{\pm}$  for non-endpoint cases and some weak-boundedness for the endpoint case. It is then natural to ask whether  $W_{\pm}$  are bounded on  $L^1(\mathbb{R})$  and  $L^\infty(\mathbb{R})$  or not. The next theorem answers this question negatively in the regular case.

**Theorem 1.5** (Counterexample at the endpoints). *If  $\mu > 15$ , zero is a regular point and  $H$  has no embedded eigenvalue in  $(0, \infty)$ , then  $W_{\pm} \notin \mathbb{B}(L^1(\mathbb{R})) \cup \mathbb{B}(L^\infty(\mathbb{R}))$ . Moreover, if in addition  $V$  is compactly supported, then  $W_{\pm} \notin \mathbb{B}(L^\infty(\mathbb{R}), \text{BMO}(\mathbb{R}))$ .*

**Remark 1.6.**  $H_0$  has a zero resonance of second kind since  $H_0 1 = H_0 x = 0$ ,  $1 \in W_{1/2}$  and  $x \in W_{3/2}$ . Hence, the assumption in Theorem 1.5 rules out the trivial case  $H = H_0$  where  $W_{\pm}$  are the identity operators.

### § 1.3. Application

We state a few application of Theorem 1.1. Throughout this subsection, we always assume the same assumption as in Theorem 1.1.

**Theorem 1.7** ( $L^p$ - $L^q$  estimates). *Let  $\square_{ABCD}$  be the closed quadrangle by the four vertex points:  $A = (\frac{1}{2}, \frac{1}{2})$ ,  $B = (1, \frac{1}{3})$ ,  $C = (1, 0)$ ,  $D = (\frac{2}{3}, 0)$ , and  $\overline{BC}$  (resp.  $\overline{DC}$ ) the closed line segment linked by two points  $B, C$  (resp.  $D, C$ ). Then, for all  $(\frac{1}{p}, \frac{1}{q}) \in \square_{ABCD} \setminus \{\overline{BC}, \overline{DC}\}$ ,*

$$(1.6) \quad \|e^{-itH} P_{ac}(H) f\|_{L^q(\mathbb{R})} \lesssim |t|^{-\frac{1}{4}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R})}, \quad t \neq 0.$$

**Theorem 1.8** (Spectral multiplier theorem). *If  $f \in L^\infty(\mathbb{R})$  satisfies*

$$\sup_{\delta > 0} \|\eta(\cdot)f(\delta\cdot)\|_{H^s(\mathbb{R})} < \infty, \quad s > 1/2,$$

*with some  $\eta \in C_0^\infty(\mathbb{R} \setminus 0)$ , then  $f(H) \in \mathbb{B}(L^p(\mathbb{R}))$  for all  $1 < p < \infty$ .*

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### § 2. Outline of the proof

In this section, we briefly explain the ideas of the proof of the above theorems for the case when zero is a regular point. It is enough to consider  $W_+$  only, since  $W_\pm f = \overline{W_\mp f}$ . The starting point is the stationary representation formula of  $W_+$ :

$$W_+ = \text{Id} - \frac{2}{\pi i} \int_0^\infty \lambda^3 R_V^+(\lambda^4) V (R_0^+(\lambda^4) - R_0^-(\lambda^4)) d\lambda,$$

where  $R_0^\pm(\lambda^4) = (H_0 - \lambda^4 \mp i0)^{-1}$  and  $R_V^\pm(\lambda^4) = (H - \lambda^4 \mp i0)^{-1}$  are the boundary values of the free and perturbed resolvents, respectively. The integral kernels of  $R_0^\pm(\lambda^4)$  are explicitly given by

$$\begin{aligned} R_0^\pm(\lambda^4, x, y) &= \frac{F_\pm(\lambda|x-y|)}{4\lambda^3} \\ (2.1) \quad &= \frac{F_\pm(\lambda|x|)}{4\lambda^3} - \frac{y}{4\lambda^2} \int_0^1 \text{sgn}(x - \theta y) F'_\pm(\lambda|x - \theta y|) d\theta \end{aligned}$$

where  $F_\pm(s) := \pm i e^{\pm i s} - e^{-s}$ . In particular,  $R_0^\pm(\lambda^4) = O(\lambda^{-3})$ .

We then decompose  $W_+ - \text{Id}$  into the low energy and the high energy parts. The high energy part is easier to treat than the low energy part since the free resolvent decays in  $\lambda \geq 1$ , so we here consider the following low energy part only:

$$(2.2) \quad W_+^L := \int_0^\infty \lambda^3 \chi(\lambda) R_V^+(\lambda^4) V (R_0^+(\lambda^4) - R_0^-(\lambda^4)) d\lambda,$$

where  $\chi \in C_0^\infty(\mathbb{R})$  and  $\chi \equiv 1$  near  $\lambda = 0$ . Setting  $v(x) = \sqrt{|V(x)|}$ ,  $U(x) = \text{sgn } V(x)$  and  $M(\lambda) = U + v R_0^+(\lambda^4) v$ , one has the symmetric second resolvent equation:

$$R_V^+(\lambda^4) V = R_0^+(\lambda^4) v M^{-1}(\lambda) v.$$

Then one of key tools in our argument is the asymptotic expansion of  $M^{-1}(\lambda)$  as  $\lambda \rightarrow +0$  obtained recently by [SWY] which, in the regular case, is of the form

$$(2.3) \quad \begin{aligned} M^{-1}(\lambda) = & Q_2 A_0^0 Q_2 + \lambda Q_1 A_1^0 Q_1 + \lambda^2 (Q_1 A_{21}^0 Q_1 + Q_2 A_{22}^0 + A_{23}^0 Q_2) \\ & + \lambda^3 (Q_1 A_{31}^0 + A_{32}^0 Q_1) + \lambda^3 \tilde{P} + \Gamma_4^0(\lambda), \end{aligned}$$

where  $A_k^0, A_{kj}^0, \tilde{P}, Q_\alpha \in \mathbb{B}(L^2)$ ,  $\|\Gamma_4^0(\lambda)\|_{L^2 \rightarrow L^2} = O(\lambda^4)$  and  $Q_\alpha$  satisfies

$$(2.4) \quad Q_\alpha(x^k v) \equiv 0, \quad \langle x^k v, Q_\alpha f \rangle = 0,$$

for any  $f \in L^2$  and any integer  $0 \leq k \leq \alpha - 1$ . The interest of these properties (2.4) is that, combined with the Taylor expansion formula (2.1), one has

$$(2.5) \quad Q_\alpha v R_0^\pm(\lambda^4) = O(\lambda^{-3+\alpha}), \quad R_0^\pm(\lambda^4) v Q_\alpha = O(\lambda^{-3+\alpha}),$$

which are less singular in  $\lambda \in (0, 1]$  compared with the free resolvent.

### § 2.1. Weighted $L^p$ -boundedness

Substituting (2.3) into (2.2) one can find that  $W_+^L$  is a sum of nine integral operators with integral kernels of the form

$$(2.6) \quad \int_0^\infty \lambda^{\ell-\alpha-\beta} \chi(\lambda) (R_0^+(\lambda^4) v Q_\alpha B Q_\beta v [R_0^+ - R_0^-](\lambda^4)) (x, y) d\lambda,$$

where  $B \in \mathbb{B}(L^2)$  varies from line to line,  $\ell = 6$  or  $7$  and we set  $Q_0 = \text{Id}$ . Note that the integrand is of order  $\lambda^{\ell-6}$  by (2.5). These nine integral operators are classified into the following two classes with respect to the order of  $\lambda$  of the integrands:

- (I)  $Q_\alpha B Q_\beta = Q_2 A_0^0 Q_2, Q_1 A_{21}^0 Q_1, Q_2 A_{22}^0, A_{23}^0 Q_2, Q_1 A_{31}^0, A_{32}^0 Q_1, \lambda^{-4} \Gamma_4^0(\lambda);$
- (II)  $Q_\alpha B Q_\beta = Q_1 A_1^0 Q_1, \tilde{P}.$

The operators in the class (I), whose integral kernels are  $O(\lambda)$ , can be shown to be bounded on  $L^p(\mathbb{R})$  for any  $1 \leq p \leq \infty$ . We shall explain this for  $Q_\alpha B Q_\beta = Q_1 A_{21}^0 Q_1$  as a model case. In such a case, by using (2.1), (2.5) and the identity

$$\begin{aligned} F'_+(\lambda|x|)[F'_+ - F'_-](\lambda|y|) \\ = e^{i\lambda(|x|+|y|)} - e^{i\lambda(|x|-|y|)} + e^{-\lambda(|x|+i|y|)} - e^{-\lambda(|x|-i|y|)}, \end{aligned}$$

we can rewrite (2.6) as a linear combination of following four functions:

$$\begin{aligned} K_{a_1}^\pm(x, y) &= \int_0^\infty e^{i\lambda(|x|\pm|y|)} \lambda \chi(\lambda) a_1^\pm(\lambda, x, y) d\lambda, \\ K_{a_2}^\pm(x, y) &= \int_0^\infty e^{-\lambda(|x|\pm i|y|)} \lambda \chi(\lambda) a_2^\pm(\lambda, x, y) d\lambda, \end{aligned}$$

where  $a_j^\pm$  satisfy, for  $x, y \in \mathbb{R}$ ,  $\lambda \geq 0$  and  $\ell = 0, 1, 2$ ,

$$|\partial_\lambda^\ell a_1^\pm(\lambda, x, y)| + e^{-\lambda|x|} |\partial_\lambda^\ell a_2^\pm(\lambda, x, y)| \lesssim \|\langle x \rangle^{4+2\ell} V\|_{L^1}.$$

Since  $\chi \in C_0^\infty(\mathbb{R})$ ,  $K_{a_j}^\pm \in L^\infty(\mathbb{R}^2)$ . Moreover, since  $(\lambda\chi(\lambda))|_{\lambda=0} = 0$ , applying integration by parts twice to  $K_{a_j}^\pm$  implies

$$|K_{a_j}^\pm(x, y)| \lesssim \langle |x| \pm |y| \rangle^{-2}, \quad x, y \in \mathbb{R}.$$

Since  $\langle |x| \pm |y| \rangle^{-2} \in L_y^\infty L_x^1 \cap L_x^\infty L_y^1$ , classical Schur's lemma yields that the integral operator with kernel  $K_{a_j}^\pm$  is bounded on  $L^p(\mathbb{R})$  for any  $1 \leq p \leq \infty$ .

In the above argument, the crucial point is that we have an additional  $\lambda$  in the integrands for the class (I). For the operators in the class (II) whose integral kernels are  $O(1)$ , this is not the case, so we need more precise informations on the integral kernels to employ the theory of Calderón–Zygmund operators. As a model case, we shall consider the integral (2.6) with  $Q_\alpha B Q_\beta = \tilde{P}$  (in which case  $\ell = 6$ ). In such a case, a similar argument as above yields that (2.6) can be rewritten in the form

$$-\sum_{\pm} \int_0^\infty \left( e^{i\lambda(|x| \pm |y|)} \chi(\lambda) c_1^\pm(\lambda, x, y) + i e^{-\lambda(|x| \pm i|y|)} \chi(\lambda) c_2^\pm(\lambda, x, y) \right) d\lambda$$

with some  $c_j^\pm$  satisfying the same estimates as that for  $a_j^\pm$ . Applying integration by parts twice, we find that this integral is a sum of the leading term

$$k(x, y) = \frac{-1+i}{8} \left\{ \frac{\psi_+}{|x|+|y|} + \frac{\psi_-}{|x|-|y|} + \frac{\psi_-}{|x|+i|y|} + \frac{\psi_-}{|x|-i|y|} \right\},$$

and the error term  $O(\langle |x| - |y| \rangle^{-2})$ . The error term can be dealt as above, where  $\psi_\pm = \psi(|x| \pm |y|)$  are smooth cut-off functions supported in  $\{(x, y) \mid |x| \pm |y| \geq 1\}$ . Although the integral operator  $T_k$  with the kernel  $k$  itself is not a Calderón–Zygmund operator, using the identity

$$k(x, y) = (\chi_+(x) + \chi_-(x)) k(x, y) (\chi_+(y) + \chi_-(y))$$

with  $\chi_\pm$  being the indicator function of  $\mathbb{R}_\pm$ , one can rewrite  $T_k$  as

$$T_k = \left\{ (\chi_+ - \chi_-) T_{k_1} + \chi_+ T_{k_2^+} \chi_+ - \chi_- T_{k_2^+} \chi_- + \chi_+ T_{k_2^-} \chi_+ - \chi_- T_{k_2^-} \chi_- \right\} (1 + \tau),$$

where  $\tau f(x) = f(-x)$  is the reflection with respect to the origin and

$$k_1(x, y) = \psi(|x - y|^2)(x - y)^{-1}, \quad k_2(x, y) = \psi(|x - y|^2)(x \pm iy)^{-1}$$

so that  $T_{k_1}$  and  $T_{k_2^\pm}$  are Calderón–Zygmund operators. The abstract theorem for Calderón–Zygmund operators then shows, for any  $A_p$ -weight  $w_p$ ,

$$T_{k_1}, T_{k_2^\pm} \in \mathbb{B}(L^p(w_p)) \cap \mathbb{B}(L^1(w_1), L^{1,\infty}(w_1)).$$

If these weights are even, which means that they are invariant under the action  $f \mapsto \tau f$ , then the desired weighted  $L^p$ -boundedness in Theorem 1.1 follows.

### § 2.2. $\mathcal{H}^1$ - $L^1$ -boundedness

Let us consider again the operator  $T_k$ . Since  $\mathcal{H}^1$  is not invariant under the map  $f \mapsto \chi_{\pm} f$  (recall that any  $f \in \mathcal{H}^1$  must satisfy  $\int f dx = 0$ ), the above argument is not enough. Instead, we prove  $T_k \in \mathbb{B}(\mathcal{H}^1(\mathbb{R}), L^1(\mathbb{R}))$  directly. To this end, we observe that  $k$  satisfies

$$|k(x, y)| \lesssim \langle x - y \rangle^{-1}, \quad |k(x, y) - k(x, y')| \lesssim \frac{|y - y'|}{\min_{\pm} |x \pm y|^2}.$$

By virtue of these estimates, we can mimic a classical proof of the  $\mathcal{H}^1$ - $L^1$  boundedness for Calderón–Zygmund operators based on the atomic decomposition of  $\mathcal{H}^1$  even if  $T_k$  itself is not a Calderón–Zygmund operator.

### § 2.3. $W^{s,p}$ -boundedness

Once the  $L^p$ -boundedness of  $W_+$  is obtained, its  $W^{s,p}$ -boundedness easily follows from the intertwining property

$$\langle H \rangle^{s/4} W_+ = W_+ \langle H_0 \rangle^{s/4}$$

and the norm equivalence

$$\|\langle H_0 \rangle^{s/4} f\|_{L^p(\mathbb{R})} \lesssim \|\langle H \rangle^{s/4} f\|_{L^p}, \quad 1 < p < \infty,$$

This equivalence can be shown by an abstract method (see [Bl]) based on the generalized Gaussian bound for the semi-group  $e^{-t(H+E)}$  proved by [DDY].

### § 2.4. Counterexamples

All the operators in the class (I) are bounded on  $L^p(\mathbb{R})$  for all  $1 \leq p \leq \infty$ . Let  $T_1$  (resp.  $T_2$ ) be the integral operator in the class (II) associated with  $Q_1 A_1^0 Q_1$  (resp.  $\tilde{P}$ ). If we take a test function  $f_R = \chi_{[-R, R]}$  then

$$\sup_{R>0} \|T_1 f_R\|_{L^\infty} < \infty, \quad T_1 f_1 \in L^1(\mathbb{R}),$$

but

$$|(T_2 f_R)(R+2)| \rightarrow \infty \quad (R \rightarrow \infty), \quad T_2 f_1 \notin L^1(\mathbb{R}).$$

This shows  $W_+ \notin \mathbb{B}(L^1(\mathbb{R})) \cup \mathbb{B}(L^\infty(\mathbb{R}))$ . To disprove the  $L^\infty$ -BMO boundedness of  $W_+$ , we first observe that the same proof as for  $T_k$  implies  $T_2^* \in \mathbb{B}(\mathcal{H}^1(\mathbb{R}), L^1(\mathbb{R}))$ , so  $T_2 \in \mathbb{B}(L^\infty(\mathbb{R}), \text{BMO}(\mathbb{R}))$  by the duality. Moreover, if we take a test function

$$g_R(x) = \text{sgn}(x) \chi_{\{R \leq |x| \leq 2R\}}(x) \in \mathcal{H}^1(\mathbb{R})$$

then  $T_1^* g_R \notin L^1(\mathbb{R})$  for  $R \gg 1$ , which implies  $W_+^* \notin \mathbb{B}(\mathcal{H}^1(\mathbb{R}), L^1(\mathbb{R}))$ , and hence  $W_+ \notin \mathbb{B}(L^\infty(\mathbb{R}), \text{BMO}(\mathbb{R}))$ .

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