

Two-point boundary-value problems with a discontinuous semilinear term

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1. Introduction

In this paper we consider the equation

$$(1.1) \quad \begin{cases} d^2v/dx^2 + g(v) = 0 & \text{on } I = (0, \ell_0) \\ dv/dx(0) = dv/dx(\ell_0) = 0, \end{cases}$$

where g is a function with a discontinuity point of the first kind. Our purpose is to present all the solutions of (1.1) under some appropriate conditions on g and v . At the same time we show that the cardinality of the set of all the solutions is $\aleph_0$ (countably infinite).

The equation (1.1) appears in the following situation. Consider the degenerate parabolic system of $u(x,t)$ and $v(x,t)$,

$$(1.2) \quad \begin{cases} \partial u / \partial t = f(u, v) & \text{in } I \times (0, +\infty) \\ \partial v / \partial t = d^2v/dx^2 + g(u, v) & \text{in } I \times (0, +\infty) \\ \partial v / \partial x(0, t) = \partial v / \partial x(\ell_0, t) = 0 & t > 0 \\ \text{initial conditions.} \end{cases}$$

In a biological point of view, it may be considered that u (resp. v) represents the density of a plant (resp. a herbivore) for example. Consider steady-state solutions of (1.2). Then the first equation of (1.2) reduces an algebraic equation. Hence, u can be expressed as a multi-valued function of v . If we fix v_* and assign each of two different parts of the multi-valued function on each side of v_* , u becomes a function of v with discontinuity at v_* . Substituting the function u of v into the second

equation of (1.2) and rewriting the composed function with the same symbol g , we obtain the equation (1.1). The existence of such a steady-state solution is recognized by some numerical experiments. (See Mimura [1]. Mimura also proved that stable steady-state solutions of (1.2) have a unique v_* determined by f and g under an appropriate assumption.)

2. Presentation of results

The semilinear term g we deal with in this paper is restricted to the one satisfying the following condition.

Condition 1.

- (i) g is a function defined on (v_1, v_2) with a discontinuity point of the first kind v_* and satisfies

$$-\infty < g(v_*-0) < 0 < g(v_*+0) < +\infty.$$
- (ii) $g < 0$ in (v_1, v_*) , $g > 0$ in (v_*, v_2) , and g is Lipschitz continuous in $(v_1, v_*-0]$ and $[v_*+0, v_2)$. (i.e., $\hat{g}$ is Lipschitz continuous in $[v_1+\varepsilon, v_*]$ for any $\varepsilon > 0$, where $\hat{g} = g(v)$ for $v \in (v_1, v_*)$ and $\hat{g} = g(v_*-0)$ at $v = v_*$. Similarly $[v_*+0, v_2)$ is considered.)
- (iii) g is monotone decreasing in (v_1, v_*) and (v_*, v_2) .

We rewrite (1.1) by the weak form:

$$(2.1) \quad \begin{cases} \text{Find } v \in H^1(I) \text{ such that} \\ (dv/dx, d\phi/dx) = (g(v), \phi) \quad \text{for all } \phi \in H^1(I), \end{cases}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(I)$. Our aim is to find all the solutions of (2.1) satisfying

$$(2.2) \quad v_1 < v(x) < v_2 \quad (0 \leq x \leq l_0).$$

Remark 1. (2.2) makes sense since $v \in H^1(I)$ implies $v \in C(I)$ by Sobolev's lemma. Furthermore, for the solution v of (2.1), we have $v \in C^1(I)$, since Condition 1 gives $g(v) \in L^2(I)$ which yields $v \in H^2(I)$.

Theorem 1. *In the case $g(v_*) \neq 0$, all the solutions of (2.1) and (2.2)*

are v_n^i , $i=1,2$ and $n \geq n_0$, where $v_n^i(x)$ are functions defined in the subsequent section (see (3.9)) and n_0 is a positive integer depending on g and ℓ_0 .

In the case $g(v_*) = 0$, v_* is added to the solutions.

Remark 2. If $g(v_1+0) = g(v_2-0) = 0$, and if g is Lipschitz continuous in $[v_1+0, v_*-0]$ and $[v_*+0, v_2-0]$, then $n_0 = 1$ for any $\ell_0 > 0$.

Remark 2 as well as Theorem 1 is proved at the next section.

3. Proof of Theorem 1.

Let $v(x; c)$ be a solution of the initial value problem:

$$(3.1) \quad \begin{cases} d^2v/dx^2 = -\tilde{g}(v) & (x > 0) \\ dv/dx(0) = 0 \\ v(0) = c, \end{cases}$$

where $c > v_1$ and

$$\tilde{g}(v) = \begin{cases} g(v) & (v_1 < v < v_*) \\ g(v_*-0) & (v_* \leq v). \end{cases}$$

Such $v(x; c)$ exists uniquely since $\tilde{g}$ is Lipschitz continuous in $[c - \varepsilon, +\infty)$

for some $\varepsilon > 0$ where any solution of (3.1) lies since $\tilde{g} < 0$.

Lemma 1. Let $v(x; c)$ be as above. Then, we have

$$(3.2) \quad v(x; c_1) < v(x; c_2) \quad (x \geq 0)$$

for $v_1 < c_1 < c_2$.

Proof. Let $x_0 > 0$ be the first intersecting point of $v(x; c_1)$ and $v(x; c_2)$.

Obviously it holds that

$$dv/dx(x_0; c_1) \geq dv/dx(x_0; c_2).$$

On the other hand, by (iii) of Condition 1, we have

$$d^2v/dx^2(x; c_1) = -\tilde{g}(v(x; c_1)) \leq -\tilde{g}(v(x; c_2)) = d^2v/dx^2(x; c_2) \quad (0 < x < x_0).$$

Integrating both sides from 0 to x_0 , we have

$$dv/dx(x_0; c_1) \leq dv/dx(x_0; c_2).$$

Hence we obtain

$$(d/dx)^i v(x_0; c_1) = (d/dx)^i v(x_0; c_2) \quad (i=0,1).$$

By the uniqueness of the initial value problem, we have

$$v(x; c_1) = v(x; c_2) \quad (x \geq 0).$$

This is a contradiction. Hence we obtain (3.2). Q.E.D.

Define a mapping ϕ from (v_1, v_*) into $(0, +\infty)$ by

$$v(\phi(c); c) = v_*.$$

Since $d^2v/dx^2(x; c) \geq -g(c) > 0$, $\phi(c)$ is well-defined. By Lemma 1, $\phi(c)$ is monotone decreasing. Set

$$\bar{l} = \lim_{c \downarrow v_1} \phi(c).$$

Lemma 2. ϕ is homeomorphic from (v_1, v_*) onto $(0, \bar{l})$ and strictly decreasing.

Proof. ϕ is strictly decreasing and therefore injective by Lemma 1. We show ϕ is continuous. Let $\{c_j\}$ be any sequence in (v_1, v_*) converging to $c \in (v_1, v_*)$. It is sufficient for us to show that there exists a subsequence $\{c_{j_k}\}$ such that $\phi(c_{j_k})$ converges to $\phi(c)$. Let $\bar{c}$ (resp. $\underline{c}$) be the supremum (resp. infimum) of $\{c_j\}$. Since $\phi(c_j) \in [\phi(\bar{c}), \phi(\underline{c})]$, there exists a subsequence $\{c_{j_k}\}$ which converges to some $d \in [\phi(\bar{c}), \phi(\underline{c})]$. Then it holds that

$$\begin{aligned} & |v(d; c) - v_*| \\ & \leq |v(d; c) - v(\phi(c_{j_k}); c)| + |v(\phi(c_{j_k}); c) - v(\phi(c_{j_k}); c_{j_k})| \\ & \leq |v(d; c) - v(\phi(c_{j_k}); c)| + \max_{0 \leq x \leq \phi(\underline{c})} |v(x; c) - v(x; c_{j_k})| \\ & \rightarrow 0 \quad \text{as } j_k \rightarrow +\infty, \end{aligned}$$

since $v(x; c)$ is continuous and $\tilde{g}$ is Lipschitz continuous in $[\underline{c}, v_* - 0]$.

We show ϕ is surjective. It is sufficient for us to show that

$$\phi(c) \rightarrow 0 \quad \text{as } c \rightarrow v_*.$$

Suppose that $\lim_{c \downarrow v_*} \phi(c) = d > 0$. Fix $x_0 \in (0, d)$. Then it holds that

$$v(x_0; c) \rightarrow v(x_0; v_*) > v_* \quad \text{as } c \rightarrow v_*,$$

since $v(x;c)$ converges to $v(x;v_*)$ uniformly in $[0, d]$. On the other hand, by $v(x_0;c) < v(d;c) < v_*$, we have

$$\limsup_{c \uparrow v_*} v(x_0;c) \leq v_*.$$

This is a contradiction. Hence, $d = 0$.

That ϕ^{-1} is continuous is easily proved. Q.E.D.

For any $\ell \in (0, \bar{\ell})$, $v(x; \phi^{-1}(\ell))$ is a unique solution of

$$(3.3) \quad \begin{cases} d^2 v / dx^2 = -g(v) & (0 < x < \ell) \\ dv/dx(0) = 0 \\ v(\ell) = v_* \end{cases}$$

satisfying

$$(3.4) \quad v_1 < v(x) < v_* \quad (0 \leq x \leq \ell).$$

Uniqueness is proved easily by making use of Lemma 1. Define a mapping α from $(0, \bar{\ell})$ into $(0, +\infty)$ by

$$\alpha(\ell) = dv/dx(\ell; \phi^{-1}(\ell)).$$

Lemma 3. α is strictly increasing and homeomorphic from $(0, \bar{\ell})$ onto $(0, \bar{\alpha})$ and satisfies

$$(3.5) \quad 0 < \alpha(\ell) \leq -g(v_* - 0)\ell \quad (0 < \ell < \bar{\ell}),$$

where $\bar{\alpha} = \limsup_{\ell \uparrow \bar{\ell}} \alpha(\ell)$.

Proof. We first show that α is strictly increasing. Fix ℓ_i , $i=1,2$, such that

$$0 < \ell_1 < \ell_2 < \bar{\ell} \quad \text{and} \quad \ell_2 < 2\ell_1.$$

Set for $s \in [0, \ell_2]$

$$w_1(s) = v(\ell_1 - s; \phi^{-1}(\ell_1)) \quad \text{and} \quad w_2(s) = v(\ell_2 - s; \phi^{-1}(\ell_2)),$$

where $v(x; \phi^{-1}(\ell_1))$ is assumed to be extended in $x < 0$ to the even function.

Then w_i , $i=1,2$, satisfies

$$(3.6) \quad \begin{cases} d^2 w_i / ds^2 = -g(w_i(s)) & (0 < s < \ell_2) \\ w_i(0) = v_* \end{cases}$$

$$\left\{ \begin{array}{l} dw_i/ds(0) = -\alpha(\ell_i) \end{array} \right\}.$$

For the purpose of an indirect proof, assume that $\alpha(\ell_1) \geq \alpha(\ell_2)$. If $\alpha(\ell_1) = \alpha(\ell_2)$, (3.6) implies that $w_1(s) = w_2(s)$ ($0 \leq s \leq \ell_2$), which contradicts that

$$w_1(\ell_2) > \phi^{-1}(\ell_1) > \phi^{-1}(\ell_2) = w_2(\ell_2).$$

Assume $\alpha(\ell_1) > \alpha(\ell_2)$. Let s_0 be the first intersecting point of w_1 and w_2 . Obviously it holds that

$$dw_1/ds(s_0) \geq dw_2/ds(s_0).$$

On the other hand, we have

$$d^2w_2/ds^2(s) = -g(w_2(s)) \geq -g(w_1(s)) = d^2w_1/ds^2(s) \quad (0 \leq s \leq s_0),$$

since $w_2 \geq w_1$ on $[0, s_0]$ and g is monotone decreasing. Hence we obtain

$$\begin{aligned} dw_2/ds(s_0) &= -\alpha(\ell_2) + \int_0^{s_0} d^2w_2/ds^2 ds \\ &\geq -\alpha(\ell_2) + \int_0^{s_0} d^2w_1/ds^2 ds \\ &> dw_1/ds(s_0), \end{aligned}$$

which is a contradiction. Therefore we get $\alpha(\ell_1) < \alpha(\ell_2)$, which also shows α is injective.

(3.5) is proved easily by making use of

$$\begin{aligned} \alpha(\ell) &= \int_0^\ell d^2v/dx^2(x; \phi^{-1}(\ell)) dx \\ &= -\int_0^\ell g(v(x; \phi^{-1}(\ell))) dx. \end{aligned}$$

We next show that α is continuous. We have

$$\begin{aligned} |\alpha(\ell_1) - \alpha(\ell_2)| &\leq |dv/dx(\ell_1; \phi^{-1}(\ell_1)) - dv/dx(\ell_2; \phi^{-1}(\ell_1))| \\ &\quad + |dv/dx(\ell_2; \phi^{-1}(\ell_1)) - dv/dx(\ell_2; \phi^{-1}(\ell_2))| \\ &\leq |dv/dx(\ell_1; \phi^{-1}(\ell_1)) - dv/dx(\ell_2; \phi^{-1}(\ell_1))| \\ &\quad + \max_{0 \leq x \leq \ell_1 + \epsilon} |dv/dx(x; \phi^{-1}(\ell_1)) - dv/dx(x; \phi^{-1}(\ell_2))|, \\ &\quad \text{for } 0 < \ell_1 < \ell_2 < \ell_1 + \epsilon < \bar{\ell}. \end{aligned}$$

Letting $\ell_2 \rightarrow \ell_1$, we obtain $\alpha(\ell_2) \rightarrow \alpha(\ell_1)$ since $\phi^{-1}(\ell_2) \rightarrow \phi^{-1}(\ell_1)$.

That α is surjective and that α^{-1} is continuous are easily proved.

Q.E.D.

Define $s_1 = \alpha^{-1}$. Then, s_1 is homeomorphic from $(0, \bar{\alpha})$ onto $(0, \bar{\ell})$ and strictly increasing and satisfies

$$-\alpha/g(v_*-0) \leq s_1(\alpha) \quad (0 < \alpha < \bar{\alpha}).$$

$v(x; \phi^{-1}(s_1(\alpha)))$ is a unique solution of

$$(3.7) \quad \begin{cases} d^2v/dx^2 = -g(v) & (0 < x < s_1(\alpha)) \\ dv/dx(0) = 0 \\ v(s_1(\alpha)) = v_* \\ dv/dx(s_1(\alpha)) = \alpha \\ v_1 < v(x) < v_* & (0 \leq x \leq s_1(\alpha)). \end{cases}$$

Similarly we can define s_2 such that s_2 is homeomorphic from $(0, \bar{\alpha})$ onto $(0, \bar{\ell})$ and strictly increasing and satisfies

$$\alpha/g(v_*+0) \leq s_2(\alpha) \quad (0 < \alpha < \bar{\alpha}).$$

For any $\alpha \in (0, \bar{\alpha})$ there exists a unique solution of

$$(3.8) \quad \begin{cases} d^2v/dx^2 = -g(v) & (0 < x < s_2(\alpha)) \\ dv/dx(0) = 0 \\ v(s_2(\alpha)) = v_* \\ dv/dx(s_2(\alpha)) = -\alpha \\ v_* < v(x) < v_2 & (0 \leq x \leq s_2(\alpha)). \end{cases}$$

Set $\hat{\alpha} = \min(\bar{\alpha}, \bar{\alpha})$ and $\hat{\ell} = \lim_{\alpha \uparrow \hat{\alpha}} (s_1 + s_2)(\alpha)$. For $\alpha \in (0, \hat{\alpha})$ we denote the unique solution of (3.7) (resp. (3.8)) by $v(x; \alpha, 1)$ (resp. $v(x; \alpha, 2)$). Then, $s_1 + s_2$ is homeomorphic from $(0, \hat{\alpha})$ onto $(0, \hat{\ell})$ and strictly increasing and satisfies

$$\{-1/g(v_*-0) + 1/g(v_*+0)\}\alpha \leq (s_1 + s_2)(\alpha).$$

Let n_0 be the smallest positive integer greater than $\ell_0/\hat{\ell}$. Define α_n and v_n^i , $i=1,2$ and $n \geq n_0$, by

$$(s_1+s_2)(\alpha_n) = \ell_0/n \quad ,$$

and

$$(3.9) \quad v_n^i(x) = \begin{cases} v(x; \alpha_n, i) & (0 \leq x \leq s_i(\alpha_n)) \\ v((s_1+s_2)(\alpha_n) - x; \alpha_n, i+1) & (s_i(\alpha_n) < x \leq (s_1+s_2)(\alpha_n)) \\ v_n^i(2(s_1+s_2)(\alpha_n) - x) & ((s_1+s_2)(\alpha_n) < x \leq 2(s_1+s_2)(\alpha_n)) \\ \text{periodic with period } 2(s_1+s_2)(\alpha_n) & (2(s_1+s_2)(\alpha_n) < x \leq \ell_0), \end{cases}$$

where $v(x; \alpha_n, 3)$ is equal to $v(x; \alpha_n, 1)$. It is easy to observe that v_n^i , $i=1, 2$ and $n \geq n_0$, satisfy (2.1) and (2.2). To complete the proof of Theorem 1 we must show that there exist no other solutions of (2.1) and (2.2). Let $v \neq v_*$ be a solution of (2.1) and (2.2). By Remark 1 and (2.1), we have

$$(3.10) \quad v \in C^1[0, \ell_0] \quad \text{and} \quad \int_0^{\ell_0} g(v(x)) \, dx = 0.$$

We first show that there exists $x_0 \in (0, \ell_0)$ such that

$$(3.11) \quad v(x_0) = v_* \quad \text{and} \quad dv/dx(x_0) \neq 0.$$

Take z_0 such that $v(z_0) \neq v_*$. Let x_0 be the nearest point to z_0 satisfying $v(x_0) = v_*$. Such x_0 is well-defined since $\{x; x \in (0, \ell_0), v(x) = v_*\}$ is not empty by (3.10). Without loss of generality, we may assume that

$$x_0 < z_0 \quad \text{and} \quad v(z_0) < v(x_0) (= v_*) .$$

From (2.1), we observe v satisfies the first equation of (1.1) in (x_0, z_0) .

Integrating the equation from x_0 to y_0 , where $y_0 \in (x_0, z_0)$ is a point satisfying $dv/dx(y_0) < 0$, we have

$$\begin{aligned} dv/dx(x_0) &= dv/dx(y_0) + \int_{x_0}^{y_0} g(v(s)) \, ds \\ &\leq dv/dx(y_0) \\ &< 0 . \end{aligned}$$

Hence x_0 satisfies (3.11). Set $\alpha = -dv/dx(x_0) > 0$. While v is lying in (v_1, v_*) , v satisfies the first equation of (1.1). Therefore, v can be extended until v reaches v_* or $x = \ell_0$. In the former case there exists

$x_1 = x_0 + 2s_1(\alpha) \in I$ and satisfies

$$v(x_1) = v_* \quad \text{and} \quad dv/dx(x_1) = \alpha.$$

Since $v \in C^1(I)$ and $\alpha > 0$, $v(x)$ transverses $v = v_*$. While v is lying in (v_*, v_2) , v satisfies the first equation of (1.1). Hence v can be extended until v reaches v_* or $x = x_0$. In the former case there exists $x_2 = x_1 + 2s_2(\alpha) \in I$ and satisfies

$$v(x_2) = v_* \quad \text{and} \quad dv/dx(x_2) = -\alpha.$$

Repeating this process on both sides of x_0 , and noting the boundary condition, we observe that α must be equal to some α_n and that $v = v_n^1$ or v_n^2 . This completes the proof of Theorem 1.

Proof of Remark 2. Under the conditions of Remark 2, the equation (3.1) with $c = v_1$ and $\tilde{g}(v_1) = 0$ has a unique solution $v = v_1$. Since $\tilde{g}$ is Lipschitz continuous in $[v_1, v_*]$, we have $\bar{\ell} = +\infty$. Similarly $\bar{\ell} = +\infty$ is obtained. Therefore, we get $n_0 = 1$. Q.E.D.

Reference

- [1] Mimura, M., Striking patterns in a diffusion system related to the Gierer and Meinhardt model, in preparation.