

On prime models of definably complete locally o-minimal theories

小峰 航 (Wataru Komine)

筑波大学 理工情報生命学術院 数理物質科学研究群
(Degree Programs in Pure and Applied Sciences,
Graduate School of Science and Technology,
University of Tsukuba)

Abstract

Pillay and Steinhorn showed that every o-minimal theory has a prime model and it is unique up to isomorphism. We consider whether its analogy holds in generalization of o-minimal theory. We give a partial answer of this question in case of definably complete locally o-minimal theory.

1 Introduction

We are interested in definably complete locally o-minimal structures and its theories. Pillay and Steinhorn showed the following:

Theorem 1 ([1]). *Every o-minimal theory has prime model over any sets, that are unique up to isomorphism.*

On the other hand, we gave an example of definably complete locally o-minimal theory which does not have any prime models. In this note, we consider uniqueness of prime models in definable complete locally o-minimal theories.

This work was supported by JST, the establishment of university fellowships towards the creation of science technology innovation, Grant Number JPMJFS2106.

2 Preliminaries

First, we define o-minimality and its generalization. Let a structure $\mathcal{M} = (M, <, \dots)$ be expansion of a linear ordered set without endpoints.

Definition 2. $\mathcal{M}$ is o-minimal if every definable subset of M is a finite union of points and open intervals.

$\mathcal{M}$ is locally o-minimal if for any points $a \in M$ and definable subset A of M , there is an open interval I containing a such that $A \cap I$ is a finite union of points and open intervals. $\mathcal{M}$ is definably complete if every definable subset of M has supremum and infimum in $M \cup \{+\infty, -\infty\}$.

Example 3. Dense linear ordered sets without endpoints, real closed fields and $(\mathbb{R}, 0, 1, +, -, \cdot, \exp(-))$ are o-minimal.

$(\mathbb{R}, <, 0, +, \sin(-))$ is not o-minimal, but it is definably complete locally o-minimal.

We say that an A -formula is a formula with parameters from A . An A -definable set is a set defined by A -formula. If it is an interval, it is called A -interval.

In definably complete locally o-minimal structure, analogy of monotonicity theorem holds as following (called strong local monotonicity theorem).

Theorem 4 ([2]). *Let I be a A -interval and $f : I \rightarrow M$ be a A -definable function. Then there exists a mutually disjoint A -definable partition $I = X_d \cup X_c \cup X_+ \cup X_-$ satisfying the following conditions:*

- (1) *the A -definable set X_d is discrete and closed;*
- (2) *the A -definable set X_c is open and f is locally constant on X_c ;*
- (3) *the A -definable set X_+ is open and f is locally strictly increasing and continuous on X_+ ;*
- (4) *the A -definable set X_- is open and f is locally strictly decreasing and continuous on X_- .*

In this note, a language $\mathcal{L}$ contains $<$ and we assume $<$ is linear ordering without endpoints in $\mathcal{L}$ -structures.

Definition 5. $\mathcal{L}$ -theory T is o-minimal (resp. definably complete locally o-minimal) if every model of T is o-minimal (resp. definably complete locally o-minimal) structure.

Remark 6. If $\mathcal{M}$ is o-minimal, its complete theory $\text{Th}(\mathcal{M})$ is o-minimal by uniform finiteness theorem. On the other hand, definable completeness and local o-minimality are represented by $\mathcal{L}$ -sentences. Therefore, if $\mathcal{M}$ is definably complete locally o-minimal, its complete theory $\text{Th}(\mathcal{M})$ is definably complete locally o-minimal.

We only consider complete theory and we work in monster model $\mathbb{M}$. Next, we define completeness for formulas and definable set.

Definition 7. Let A be a subset of $\mathbb{M}$. An A -complete formula is an A -formula which isolates some type over A . An A -complete set is A -definable set which is definable by some A -complete formula. Complete means $\emptyset$ -complete.

On existence and uniqueness of prime models, the following fact is well-known.

Fact 8 ([1][4]). *Let a complete theory T be given.*

- (1) *Let $A \subset \mathcal{M} \models T$. Any model $\mathcal{M}$ that is constructible over A also is prime over A .*
- (2) *Suppose that for any subset A of a model $\mathcal{M}$ of T and any formula φ having parameters A , there is a complete formula with parameters from A which, relative to $\text{Th}(\mathcal{M}, a)_{a \in A}$, implies φ . Then, for any $A \subset \mathcal{M}$, there is a model $\mathcal{M}$ of T that is constructible over A . (it is enough to show the assumption when φ has just one free variable)*
- (3) *Let $A \subset \mathcal{M} \models T$. Then any two models that are constructible over A are isomorphic over A .*

3 Main theorem

In this section, T is a definably complete locally o-minimal theory. We can show following theorems in a similar discussion in o-minimal theory (cf.[1])

Theorem 9. *If $\text{dcl}(\emptyset)$ is nonempty, then T has constructible model over emptyset.*

Theorem 10. *We assume that T has constructible model over emptyset and one variable complete formulas are points or open intervals. Then prime models of T are isomorphic.*

By using this theorem, it is easy to show the following corollaries.

Corollary 11. *If $dcl(\emptyset)$ is nonempty, then T has prime model over emptyset and it is unique up to isomorphism.*

Corollary 12. *Let A be a nonempty set. Then T has prime model A and it is unique up to isomorphism.*

The following is the main theorem.

Theorem 13. *We assume that a definably complete locally o-minimal $\mathcal{L}$ -theory T has a complete formula. Then, if T has a prime model over $\emptyset$, any prime models of T are isomorphic.*

sketch of proof. We can assume T has an one variable complete formula $\theta_0(v)$ by taking existence symbol several times and we define $\mathcal{L}_c = \mathcal{L} \cup \{c\}$ where c is a new constant symbol.

If $\theta_0(v)$ is $v = v$, one variable complete formula is open interval and isolated types are dense in $S^1(T)$. So T has the constructible model. By theorem 10, we get theorem in this case.

We assume T has non-trivial complete formula $\theta_0(v)$. Then the boundary of $\theta_0(\mathbb{M})$ is complete set or union of two complete sets. So we take complete formula $\theta(v)$ that defines a discrete set. Let $\mathcal{N} \models T$ be prime over $\emptyset$ and take $a \in \theta(\mathcal{N})$. It is sufficient to show the following claim.

Claim A. *there are exists $a' \in \mathcal{N}$ such that $tp_{\mathcal{N}}(a) = tp_{\mathcal{N}}(a')$ and $(\mathcal{N}, a')$ is prime model of T' .*

By corollary 12, we get constructible model $\mathcal{M} \models T$ over a , see $\mathcal{M}$ as $\mathcal{L}_c$ -structure $(\mathcal{M}, a)$ and define $T' = \text{Th}_{\mathcal{L}_c}(\mathcal{M}, a)$. Then $(\mathcal{M}, a)$ is prime model of T' . Since $\mathcal{N}$ is prime, we get an elementary embedding $j : \mathcal{N} \hookrightarrow \mathcal{M}$. If $j(a) = a$, $(\mathcal{N}, a)$ is prime model of T' since $(\mathcal{M}, a)$ is prime.

We consider in case $j(a) \neq a$. Since $\mathcal{M}$ is constructible over a and $\theta(v) \in \text{tp}_{\mathcal{M}}(j(a)/a)$, $j(a)$ is defined by some formula $\varphi(v, a)$. It means $\mathcal{M} \models \exists b(\varphi(\mathcal{M}, b) = \{j(a)\})$ and this is represented by formulas.

We remark that points from $\theta(\mathcal{N})$ have same type over $\emptyset$ and they are isolated by points. Since $j(a)$ and a have same type, a is defined by $\varphi(v, b')$ for some $b' \in \theta(\mathcal{M})$. Similarly, b' is defined by $\psi(v, a)$. So $\mathcal{M} \models \forall b[\varphi(\mathcal{M}, b') = \{b\} \rightarrow \psi(\mathcal{M}, b) = \{b'\}]$. Since b' and a have same type, we get $\mathcal{M} \models \forall b[\varphi(\mathcal{M}, a) = \{b\} \rightarrow \psi(\mathcal{M}, b) = \{a\}]$. So a is defined by $\psi(v, j(a))$. Since j is elementary embedding, $\psi(v, b)$ defines some element $c' \in \mathcal{N}$ and then $j(c') = a$. Since $\mathcal{M}$ is prime over $\{a\}$, $(\mathcal{N}, c')$ is prime model of T' . $\square$

References

- [1] Pillay Anand, and Charles Steinhorn, Definable sets in ordered structures. I., Transactions of the American Mathematical Society 295.2 (1986): 565-592.
- [2] Fujita Masato, Kawakami Tomohiro, Komine Wataru, Tameness of definably complete locally o-minimal structures and definable bounded multiplication, Mathematical Logic Quarterly, Volume 68, Issue 4, November 2022, Pages 496-515.
- [3] Lou van den Dries, Tame Topology and O-minimal Structures, Cambridge University Press (1998)
- [4] David Marker, Model Theory: An Introduction, Springer (2002)