

# Berezin Transforms and Laplace-Beltrami Operators on Homogeneous Siegel Domains

— commutativity, symmetry of the domain and a Cayley transform —

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## 1. Preliminaries

Homogeneous Siegel domains are described in terms of normal  $j$ -algebras (cf. [15]), of which we are going to give the definition. Let  $\mathfrak{g}$  be a split solvable Lie algebra,  $J$  a linear operator on  $\mathfrak{g}$  with  $J^2 = -I$  and  $\omega$  a linear form on  $\mathfrak{g}$ . Then the triple  $(\mathfrak{g}, J, \omega)$  is called a *normal  $j$ -algebra* if

$$(1.1) \quad [Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad (\text{for all } x, y \in \mathfrak{g}),$$

$$(1.2) \quad \langle x | y \rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines a } J\text{-invariant inner product on } \mathfrak{g}.$$

We describe here some basic facts about normal  $j$ -algebras following [15] and [17] (see also [16]). Let  $(\mathfrak{g}, J, \omega)$  be a normal  $j$ -algebra. Let  $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$  be the derived algebra of  $\mathfrak{g}$ , and  $\mathfrak{a}$  the orthogonal complement of  $\mathfrak{n}$  in  $\mathfrak{g}$  relative to the inner product  $\langle \cdot | \cdot \rangle_\omega$ . Evidently we have  $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ . Moreover,  $\mathfrak{a}$  is a commutative subalgebra of  $\mathfrak{g}$  such that  $\text{ad}(\mathfrak{a})$  consists of semisimple operators on  $\mathfrak{g}$ . For every  $\alpha \in \mathfrak{a}^*$  we set

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n} ; [h, x] = \langle h, \alpha \rangle x \text{ for all } h \in \mathfrak{a}\}.$$

Take all  $\alpha \in \mathfrak{a}^*$  such that  $\mathfrak{n}_\alpha \neq \{0\}$  and  $J\mathfrak{n}_\alpha \subset \mathfrak{a}$ , and number them as  $\alpha_1, \dots, \alpha_r$ . We have  $\dim \mathfrak{a} = r$  and  $\dim \mathfrak{n}_{\alpha_k} = 1$  for every  $k$ . The number  $r$  is called the *rank* of the normal  $j$ -algebra  $\mathfrak{g}$ . We can reorder  $\alpha_1, \dots, \alpha_r$ , if necessary, so that all the  $\alpha$  such that  $\mathfrak{n}_\alpha \neq \{0\}$  (such an  $\alpha$  is called a *root* of the normal  $j$ -algebra) are of the following form (some roots might be missing):

$$\begin{array}{llll} \frac{1}{2}(\alpha_m + \alpha_k) & (1 \leq k < m \leq r), & \frac{1}{2}(\alpha_m - \alpha_k) & (1 \leq k < m \leq r), \\ \frac{1}{2}\alpha_k & (1 \leq k \leq r), & \alpha_k & (1 \leq k \leq r). \end{array}$$

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We note that if  $\alpha, \beta$  are distinct roots, then  $\mathfrak{n}_\alpha$  is orthogonal to  $\mathfrak{n}_\beta$ . Put

$$\begin{aligned}\mathfrak{g}(0) &:= \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}, & \mathfrak{g}(1/2) &:= \sum_{i=1}^r \mathfrak{n}_{\alpha_i/2}, \\ \mathfrak{g}(1) &:= \sum_{i=1}^r \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m + \alpha_k)/2}.\end{aligned}$$

Understanding  $\mathfrak{g}(i) = 0$  for  $i > 1$ , we have  $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ . Moreover

$$J\mathfrak{n}_{(\alpha_m - \alpha_k)/2} = \mathfrak{n}_{(\alpha_m + \alpha_k)/2} \quad (m > k), \quad J\mathfrak{n}_{\alpha_i/2} = \mathfrak{n}_{\alpha_i/2} \quad (1 \leq i \leq r),$$

so that  $J\mathfrak{g}(0) = \mathfrak{g}(1)$  and  $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$ . Taking  $E_i \in \mathfrak{n}_{\alpha_i}$  ( $i = 1, \dots, r$ ) such that  $\alpha_k(JE_i) = \delta_{ki}$ , we put  $H_i := JE_i \in \mathfrak{a}$  and

$$(1.3) \quad H := H_1 + \dots + H_r, \quad E := E_1 + \dots + E_r.$$

We write down here the constants used frequently in this note:

$$(1.4) \quad \begin{aligned}n_{mk} &:= \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m - \alpha_k)/2} = \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m + \alpha_k)/2} \quad (1 \leq k < m \leq r), \\ b_i &:= \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{n}_{\alpha_i/2} \quad (1 \leq i \leq r), \\ d_j &:= 1 + \frac{1}{2} \left( \sum_{k>j} n_{kj} + \sum_{i<j} n_{ji} \right) \quad (1 \leq j \leq r).\end{aligned}$$

Let  $G = \exp \mathfrak{g}$  be the connected and simply connected Lie group corresponding to  $\mathfrak{g}$ . Note that  $\mathfrak{g}(0)$  is a Lie subalgebra of  $\mathfrak{g}$ . We denote by  $G(0)$  the corresponding subgroup  $\exp \mathfrak{g}(0)$  of  $G$ . The group  $G(0)$  acts on  $V := \mathfrak{g}(1)$  by adjoint action. Let  $\Omega$  be the  $G(0)$ -orbit through  $E$ . By [17, Theorem 4.15]  $\Omega$  is a regular open convex cone in  $V$ , and  $G(0)$  acts on  $\Omega$  simply transitively. Being invariant under  $J$ , the subspace  $\mathfrak{g}(1/2)$  is considered as a *complex* vector space by means of  $-J$ . We shall write this complex vector space by  $U$ . We put  $W := V_{\mathbb{C}}$ , the complexification of  $V$ . The conjugation of  $W$  relative to the real form  $V$  is written as  $w \mapsto w^*$ . The real bilinear map  $Q$  defined by

$$Q(u, u') := \frac{1}{2} ([Ju, u'] - i[u, u']) \quad (u, u' \in \mathfrak{g}(1/2))$$

turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Hermitian map  $U \times U \rightarrow \mathbb{R}$  which is  $\Omega$ -positive. This means that

$$Q(u', u) = Q(u, u')^* \quad (u, u' \in U), \quad Q(u, u) \in \overline{\Omega} \setminus \{0\} \quad \text{for all } u \in U \setminus \{0\}.$$

With these data we define the *Siegel domain*  $D$  corresponding to the normal  $j$ -algebra  $(\mathfrak{g}, J, \omega)$  to be

$$D := \{(u, w) \in U \times W ; w + w^* - Q(u, u) \in \Omega\}.$$

Note that we take a generalized *right* half plane rather than a more familiar upper half plane.

Consider the Lie subalgebra  $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$ . It is at most 2-step nilpotent. Let  $N_D = \exp \mathfrak{n}_D$  be the corresponding connected and simply connected nilpotent Lie group contained in  $G$ . We write the elements of  $N_D$  by  $n(a, b)$  ( $a \in \mathfrak{g}(1)$ ,  $b \in \mathfrak{g}(1/2)$ ). The group  $N_D$  acts on  $D$  by

$$(1.5) \quad n(a, b) \cdot (u, w) = (u + b, w + ia + \tfrac{1}{2}Q(b, b) + Q(u, b)) \quad ((u, w) \in D).$$

On the other hand, the adjoint action of  $G(0)$  on  $\mathfrak{g}(1/2)$  commutes with  $J$ . This implies that  $G(0)$  acts on  $U$  complex-linearly. Moreover the adjoint action of  $G(0)$  on  $V = \mathfrak{g}(1)$  extends complex-linearly to  $W$ , so that  $G(0)$  acts on  $D$  complex-linearly. Hence  $G = N_D \rtimes G(0)$  acts on  $D$  simply transitively. To see this more explicitly, put  $e := (0, E) \in D$ . Then given  $z = (u, w) \in D$ , we can find a unique  $h \in G(0)$  satisfying  $hE = \operatorname{Re} w - Q(u, u)/2$ . Taking  $n = n(\operatorname{Im} w, u) \in N_D$ , we see by (1.5) that  $z = nh \cdot e$ .

For every  $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$  let  $\chi_{\mathbf{s}}$  be the one-dimensional representation of  $A := \exp \mathfrak{a}$  defined by

$$\chi_{\mathbf{s}} \left( \exp \sum_k t_k H_k \right) = \exp \left( \sum_k s_k t_k \right) \quad (t_1, \dots, t_r \in \mathbb{R}).$$

Let  $N := \exp \mathfrak{n}$ . It is clear that  $G = N \rtimes A$ . We extend  $\chi_{\mathbf{s}}$  to a one-dimensional representation of  $G$  by defining  $\chi_{\mathbf{s}}(n) = 1$  for  $n \in N$ . Let us define functions  $\Delta_{\mathbf{s}}$  ( $\mathbf{s} \in \mathbb{C}^r$ ) on  $\Omega$  by  $\Delta_{\mathbf{s}}(hE) = \chi_{\mathbf{s}}(h)$  ( $h \in G(0)$ ). Evidently it holds that

$$(1.6) \quad \Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h) \Delta_{\mathbf{s}}(x) \quad (h \in G(0), x \in \Omega).$$

We know that  $\Delta_{\mathbf{s}}$  extends to a holomorphic function on the tube domain  $\Omega + iV$  (cf. for example [7, Corollary 2.5]).

For  $h \in G(0)$ , let  $\operatorname{Ad}_{\mathfrak{g}(1)}(h) := (\operatorname{Ad} h)|_{\mathfrak{g}(1)}$ . Moreover let  $\operatorname{Ad}_U(h)$  stand for the *complex* linear operator on  $U$  defined by the adjoint action of  $h \in G(0)$  on  $\mathfrak{g}(1/2)$ , and  $\det \operatorname{Ad}_U(h)$  its determinant as a complex linear operator. Then, with  $\mathbf{d} := (d_1, \dots, d_r)$  and  $\mathbf{b} := (b_1, \dots, b_r)$ , we have for  $h \in G(0)$

$$(1.7) \quad \det \operatorname{Ad}_{\mathfrak{g}(1)}(h) = \chi_{\mathbf{d}}(h), \quad |\det \operatorname{Ad}_U(h)|^2 = \chi_{\mathbf{b}}(h).$$

By [6, §5] or [18, §II.6], it is known that  $D$  has a Bergman kernel  $\kappa$ . If  $\text{Hol}(D)$  denotes the Lie group of the holomorphic automorphisms of  $D$ , then  $\kappa$  satisfies

$$(1.8) \quad \kappa(z_1, z_2) = \kappa(g \cdot z_1, g \cdot z_2) \det g'(z_1) \overline{\det g'(z_2)} \quad (g \in \text{Hol}(D), z_1, z_2 \in D),$$

where  $g'(z)$  is the complex Jacobian map of  $g$  at  $z \in D$ . The description of the simple transitive action of  $G$  on  $D$  together with the property (1.7) and (1.8) shows

$$(1.9) \quad \kappa(z_1, z_2) = C \cdot \Delta_{-2d-b}(w_1 + w_2^* - Q(u_1, u_2)) \quad (z_j = (u_j, w_j) \in D)$$

with  $C = \kappa(e, e) \Delta_{2d+b}(2E) > 0$ . We put  $\eta := \Delta_{-2d-b}$  in what follows for simplicity.

## 2. Cayley transform

Let  $D_v$  be the directional derivative in the direction  $v \in V$  given by

$$D_v f(x) = \left. \frac{d}{dt} f(x + tv) \right|_{t=0}.$$

For every  $x \in \Omega$  we define  $\mathcal{I}(x) \in V^*$  to be  $-\nabla \log \eta(x)$ , that is,

$$\langle v, \mathcal{I}(x) \rangle = -D_v \log \eta(x) \quad (v \in V).$$

$\mathcal{I}$  is called the *pseudoinverse map*. By [3, §2],  $\mathcal{I}$  gives a diffeomorphism of  $\Omega$  onto the dual cone  $\Omega^*$  in  $V^*$ , where

$$\Omega^* := \{ \xi \in V^* ; \langle x, \xi \rangle > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\} \}.$$

The group  $G(0)$  acts also on  $V^*$  by the coadjoint action:  $h \cdot \xi = \xi \circ h^{-1}$ , where  $h \in G(0)$  and  $\xi \in V^*$ . It is easy to show by using (1.6) that  $\mathcal{I}$  is  $G(0)$ -equivariant:

$$\mathcal{I}(hx) = h \cdot \mathcal{I}(x) \quad (h \in G(0), x \in \Omega).$$

In particular,  $\mathcal{I}(\lambda x) = \lambda^{-1} \mathcal{I}(x)$  for all  $\lambda > 0$ , and  $G(0)$  acts on  $\Omega^*$  simply transitively. Moreover,  $\mathcal{I}$  can be extended to a rational map  $W \rightarrow W^*$  [4, Satz I.2.3].

In order to find an inverse map of  $\mathcal{I}$ , we need to dualize the above matters concerning  $\mathcal{I}$ . First we define  $E_1^*, \dots, E_r^* \in V^*$  by

$$\left\langle \sum_{j=1}^r x_j E_j + \sum_{m>k} X_{mk}, E_i^* \right\rangle = x_i \quad (x_j \in \mathbb{R}, X_{mk} \in \mathfrak{n}_{(\alpha_m + \alpha_k)/2}),$$

and for every  $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$ ,

$$E_{\mathbf{s}}^* := s_1 E_1^* + \dots + s_r E_r^* \in V^*.$$

We can show that  $\mathcal{I}(E) = E_{2d+b}^*$ . Next we put  $\mathbf{s}^* := (s_r, \dots, s_1)$  and set

$$\chi_{\mathbf{s}}^* := \chi_{-\mathbf{s}^*}, \quad \Delta_{\mathbf{s}}^*(h \cdot E_{2d+b}^*) := \chi_{\mathbf{s}}^*(h) \quad (h \in G(0)).$$

$\Delta_s^*$  is a function on  $\Omega^*$  such that  $\Delta_s^*(h \cdot \xi) = \chi_s^*(h)\Delta_s^*(\xi)$  for  $h \in G(0)$  and  $\xi \in V^*$ . We define  $\eta^* := \Delta_{-2d^*-b^*}^*$  and

$$\langle \mathcal{I}^*(\xi), f \rangle := -D_f \log \eta^*(\xi) \quad (\xi \in \Omega^*, f \in V^*).$$

Thus  $\mathcal{I}^*(\xi) \in V$  and  $\mathcal{I}^*$  gives a diffeomorphism of  $\Omega^*$  onto  $\Omega$ . Moreover,  $\mathcal{I}^*$  is  $G(0)$ -equivariant, that is,  $\mathcal{I}^*(h \cdot \xi) = h(\mathcal{I}^*(\xi))$  for any  $h \in G(0)$ . We can prove that  $\mathcal{I}^*$  is extended to a rational map  $W^* \rightarrow W$ .

**Proposition 2.1.**  $\mathcal{I}^* = \mathcal{I}^{-1}$ .

**Theorem 2.2** ([11]). (1)  $\mathcal{I}$  is holomorphic on  $\Omega + iV$ , and  $\mathcal{I}^*$  is holomorphic on  $\Omega^* + iV^*$ .

(2)  $\mathcal{I}(\Omega + iV)$  is contained in the holomorphic domain of  $\mathcal{I}^*$ , and  $\mathcal{I}^*(\Omega^* + iV^*)$  is contained in the holomorphic domain of  $\mathcal{I}$ .

*Remark 2.3.* In general we cannot have  $\mathcal{I}(\Omega + iV) \subset \Omega^* + iV^*$  if  $\Omega$  is no longer selfdual. This failure is given by an example where  $\Omega$  is the Vinberg cone. See [11] for details.

Now considering  $E_{2d+b}^*$  naturally as an element of  $W^*$ , we define

$$C(w) := E_{2d+b}^* - 2\mathcal{I}(w + E) \in W^* \quad (w \in W).$$

It is evident that  $C$  is a rational mapping  $W \rightarrow W^*$  which is holomorphic on  $\Omega + iV$ . Let  $U^\dagger$  denote the space of all antilinear forms on  $U$ . We set for  $z = (u, w) \in U \times W$

$$\mathcal{C}(z) := (2\mathcal{I}(w + E) \circ Q(u, \cdot), C(w)) \in U^\dagger \times W^*.$$

Clearly  $\mathcal{C}$  is a rational map  $U \times W \rightarrow U^\dagger \times W^*$ . It should be noted that if  $z = (u, w) \in D$ , then we have  $w \in \Omega + iV$ , so that  $\mathcal{C}(z)$  is holomorphic on  $D$ . We call  $\mathcal{C}$  a *Cayley transform*. This is a slight modification of Penney's [14]. By a verbal translation of Penney's proof [14] we have

**Proposition 2.4.** *The image  $\mathcal{C}(D)$  of  $D$  is bounded.*

To give the inverse map of  $\mathcal{C}$  explicitly we note first that

$$(2.1) \quad \langle v_1 | v_2 \rangle_\eta := D_{v_1} D_{v_2} \log \eta(E) \quad (v_1, v_2 \in V)$$

defines an inner product on  $V$  (see [3, §2]). Extending this inner product to a complex bilinear form (denoted by the same symbol  $\langle \cdot | \cdot \rangle_\eta$ ) on  $W \times W$ , we define  $\tilde{f} \in W$  and  $\hat{w} \in W^*$  for  $f \in W^*$  and  $w \in W$  respectively by

$$\langle w' | \tilde{f} \rangle_\eta = \langle w', f \rangle, \quad \langle w', \hat{w} \rangle = \langle w' | w \rangle_\eta \quad (w' \in W).$$

Next we put

$$(2.2) \quad (u_1 | u_2)_\eta := \langle Q(u_1, u_2) | E \rangle_\eta \quad (u_1, u_2 \in U).$$

It is easy to see that this is a Hermitian inner product on  $U$ . Now define linear maps  $F \mapsto \tilde{F}$  from  $U^\dagger$  to  $U$  and  $u \mapsto \hat{u}$  from  $U$  to  $U^\dagger$  by

$$(\tilde{F} | u')_\eta = \langle u', F \rangle, \quad \langle u', \hat{u} \rangle = (u | u')_\eta \quad (u' \in U).$$

Obviously they are inverse to one another. Moreover, for every  $w \in W$ , let  $\varphi(w)$  be the complex linear operator on  $U$  determined through

$$(2.3) \quad (\varphi(w)u_1 | u_2)_\eta = \langle Q(u_1, u_2) | w \rangle_\eta \quad (u_1, u_2 \in U).$$

Clearly  $\varphi(E)$  is the identity operator, and it is easy to see that  $\varphi(w^*) = \varphi(w)^*$ . Let us set

$$\begin{aligned} B(f) &:= 2\mathcal{I}^*(E_{2d+b}^* - f) - E \in W \quad (f \in W^*), \\ \mathcal{B}(F, f) &:= (\varphi(E - \tilde{f})^{-1}\tilde{F}, B(f)) \in U \times W \quad ((F, f) \in U^\dagger \times W^*). \end{aligned}$$

It is evident that both  $B$  and  $\mathcal{B}$  are rational mappings.

**Theorem 2.5** ([11]).  $\mathcal{C} : D \rightarrow \mathcal{C}(D)$  is biholomorphic and birational with  $\mathcal{C}^{-1} = \mathcal{B}$ .

*Remark 2.6.* Suppose that  $D$  is *quasisymmetric* in this remark. This means that  $\Omega$  is selfdual with respect to the inner product  $\langle \cdot | \cdot \rangle_\eta$  defined by (2.1). We identify  $V^*$  with  $V$  and  $W$  with  $W^*$  by  $\langle \cdot | \cdot \rangle_\eta$ . Then by [1, Proposition 3] the product  $\circ$  defined by

$$\langle v_1 \circ v_2 | v_3 \rangle_\eta := -\frac{1}{2} D_{v_1} D_{v_2} D_{v_3} \log \eta(E) \quad (v_1, v_2, v_3 \in V)$$

is a Jordan algebra product, so that  $V$  is a Euclidean Jordan algebra in the sense of [5]. The identity element is  $E$ , and by the above identification we have  $\mathcal{I}(x) = x^{-1}$ , the Jordan algebra inverse of  $x$ . Identifying further  $U^\dagger$  with  $U$  by means of  $(\cdot | \cdot)_\eta$  in (2.2), we get

$$\mathcal{C}(u, w) = (2\varphi(w + E)^{-1}u, (w - E)(w + E)^{-1}).$$

Thus our  $\mathcal{C}$  coincides with Dorfmeister's in [2, (2.8)] for quasisymmetric  $D$ . We note that the map  $w \mapsto \varphi(w)$  with  $\varphi(w)$  as in (2.3) is a representation of the complex Jordan algebra  $W = V_{\mathbb{C}}$  in the present case (cf. [2, Theorem 2.1]).

### 3. A characterization of symmetric Siegel domains

By definition, the spaces  $\mathfrak{g}(1/2)$  and  $V = \mathfrak{g}(1)$  have the real inner product  $\langle \cdot | \cdot \rangle_\omega$  of (1.2). We first export this inner product to  $V^*$  canonically by identifying  $V^*$  with  $V$  by  $\langle \cdot | \cdot \rangle_\omega$ . Note that this identification is not quite the same as in Remark 2.6 in general. The real inner product on  $V^*$  obtained this way is again denoted by  $\langle \cdot | \cdot \rangle_\omega$ , which is extended naturally to a Hermitian inner product  $(\cdot | \cdot)_\omega$  on  $W^*$ . On the other hand the complex vector space  $U$  has a Hermitian inner product  $(\cdot | \cdot)_\omega$  defined by

$$(3.1) \quad (u_1 | u_2)_\omega := 2 \langle Q(u_1, u_2), \omega \rangle = \langle [Ju_1, u_2], \omega \rangle - i \langle [u_1, u_2], \omega \rangle.$$

We note that  $\operatorname{Re}(u_1 | u_2)_\omega = \langle u_1 | u_2 \rangle_\omega$  for  $u_1, u_2 \in U$ . By a procedure similar to the above we introduce a Hermitian inner product  $(\cdot | \cdot)_\omega$  on  $U^\dagger$  by importing the Hermitian inner product (3.1) from  $U$ .

Let  $\beta \in \mathfrak{g}^*$  be the *Koszul form* given by

$$\langle x, \beta \rangle := \operatorname{tr}(\operatorname{ad}(Jx) - J \circ (\operatorname{ad} x)) \quad (x \in \mathfrak{g}).$$

It is known by [10] (see also [9, §5]) that  $\langle [Jx, y], \beta \rangle$  is (the real part of) the inner product on  $\mathfrak{g}$  induced by the Bergman metric of the corresponding Siegel domain  $D$  up to a positive multiple. Indeed we can show that  $\beta|_{\mathfrak{n}}$  is equal to  $E_{2d+b}^*$  extended to  $\mathfrak{n}$  by zero-extension.

**Theorem 3.1** ([12]). *One has  $\|\mathcal{C}(g \cdot e)\|_\omega = \|\mathcal{C}(g^{-1} \cdot e)\|_\omega$  for all  $g \in G$  if and only if the following two conditions are satisfied:*

- (1)  *$D$  is symmetric,*
- (2)  *$\omega|_{\mathfrak{n}}$  is equal to a positive number multiple of  $\beta|_{\mathfrak{n}}$ .*

*Remark 3.2.* Since  $\mathcal{C} : D \rightarrow \mathcal{C}(D)$  is biholomorphic with  $\mathcal{C}(e) = 0$ , we have

$$\begin{aligned} \|\mathcal{C}(g \cdot e)\|_\omega &= \|\mathcal{C}(g^{-1} \cdot e)\|_\omega \quad \text{for all } g \in G \\ \iff \|h \cdot 0\|_\omega &= \|h^{-1} \cdot 0\|_\omega \quad \text{for all } h \in \mathcal{G} := \mathcal{C} \circ G \circ \mathcal{C}^{-1}. \end{aligned}$$

### 4. Berezin transforms

For simplicity we set

$$\lambda_0 := \max_{1 \leq j \leq r} \frac{b_j + d_j + p_j/2}{b_j + 2d_j},$$

where  $p_j := \sum_{k > j} n_{kj}$ . Let  $\lambda > \lambda_0$ . This is the condition for the non-triviality of certain Hilbert spaces  $H_\lambda^2(D)$  of holomorphic functions on  $D$  (cf. [17] or [7]). Let  $\kappa$

be the Bergman kernel of  $D$  (see (1.9)). The *Berezin kernel*  $A_\lambda$  on  $D$  is given by

$$A_\lambda(z_1, z_2) := \left( \frac{|\kappa(z_1, z_2)|^2}{\kappa(z_1, z_1) \kappa(z_2, z_2)} \right)^\lambda \quad (z_1, z_2 \in D).$$

We put  $a_\lambda(g) := A_\lambda(g \cdot e, e)$  ( $g \in G$ ). Then it is easy to see that  $a_\lambda(g) = a_\lambda(g^{-1})$ . We know that  $a_\lambda$  is integrable on  $G$  with respect to the left Haar measure. Consider the space  $L^2(G)$  on  $G$  for the left Haar measure. The *Berezin transform*  $B_\lambda$ , when transferred to  $L^2(G)$ , is given by the convolution operator

$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x) dy = f * a_\lambda(x) \quad (f \in L^2(G)).$$

On the other hand, the inner product  $\langle \cdot | \cdot \rangle_\omega$  on  $\mathfrak{g}$  defines a left invariant Riemannian metric on  $G$ , relative to which we have the Laplace-Beltrami operator  $\mathcal{L}_\omega$  on  $G$ . In order to express  $\mathcal{L}_\omega$  in terms of the elements of the enveloping algebra  $U(\mathfrak{g})$ , we set for  $X \in \mathfrak{g}$

$$Xf(x) := \frac{d}{dt} f((\exp -tX)x) \Big|_{t=0}, \quad \tilde{X}f(x) := \frac{d}{dt} f(x \exp tX) \Big|_{t=0}.$$

These are extended to  $U(\mathfrak{g})$  by homomorphisms. Though the following lemma holds for any connected Lie group, we write it down here in our situation. See [19, Theorem 1] for a proof.

**Lemma 4.1.** *Take  $\Psi \in \mathfrak{g}$  for which one has  $\langle X | \Psi \rangle_\omega = \text{tr ad}(X)$  for all  $x \in \mathfrak{g}$ . Then  $\mathcal{L}_\omega = -\tilde{\Lambda} + \tilde{\Psi}$ , where  $\Lambda := X_1^2 + \cdots + X_{2N}^2$  with an orthonormal basis  $\{X_j\}_{j=1}^{2N}$  of  $\mathfrak{g}$  relative to  $\langle \cdot | \cdot \rangle_\omega$ .*

We note that  $\Psi \in \mathfrak{a}$  in our case.

**Theorem 4.2** ([13]). *Let  $\lambda > \lambda_0$  be fixed. Then,  $B_\lambda$  commutes with  $\mathcal{L}_\omega$  if and only if  $D$  is symmetric and  $\omega|_{\mathfrak{n}}$  is equal to a positive number multiple of  $\beta|_{\mathfrak{n}}$ .*

We indicate here how Theorem 4.2 is derived from Theorem 3.1.

- (1)  $B_\lambda$  commutes with  $\mathcal{L}_\omega \iff (-\tilde{\Lambda} + \tilde{\Psi})a_\lambda = (-\Lambda + \Psi)a_\lambda$ .
- (2) Since  $a_\lambda(g) = a_\lambda(g^{-1})$ , we have  $\tilde{X}a_\lambda(g) = Xa_\lambda(g^{-1})$  for all  $X \in U(\mathfrak{g})$  and  $g \in G$ .
- (3)  $(\Lambda - \Psi)a_\lambda(g) = \lambda a_\lambda(g)(\lambda \|\mathcal{C}(g \cdot e)\|_\omega^2 - \langle \Psi, \alpha \rangle)$  for some  $\alpha \in \mathfrak{a}^*$ .

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