

Paris Harrington 原理とその周辺

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九大工学部 倉田令一郎 (Reijiro Kumata)

以下では Paris Harrington 原理 PH とその拡張 PH_n とその周辺を論ずる。周辺とは Reflection Principle から ω -帰納法のことである。左に考察の範囲は 1st order logic 内に限定される。

I Paris Harrington 原理

1. 定義 $M, e \in \omega$, M は $\{0, 1, \dots, M-1\}$ と同一視される。

$$[M]^e = \{X; X \subseteq M, \bar{X} = e\}$$

Partition とは map $P: [M]^e \rightarrow r (= \{0, 1, 2, \dots, r-1\})$ $q \in \tau$.

$X \subseteq M$; homogeneous for a Partition P とは $P \upharpoonright [X]^e = \text{const}$ $q \in \tau$ である。

$M \xrightarrow{*} (k)_r^e$ $k, e, r \in \omega$ ($k > e$) とは $\forall \Delta \exists q P: [M]^e \rightarrow r$ に対し、次のような $X \subseteq M$ が存在する $\Delta = \tau$ である。

(i) X is homogeneous for P , (ii) $\bar{X} \geq k$, (iii) $\bar{X} \geq \min X$.

PH (Paris Harrington Principle とは次の命題を意味する)。

$$\forall r \exists k \exists M (M \xrightarrow{*} (k)_r^e)$$

$$PH^e; \quad \forall r \forall k \exists M (M \xrightarrow{*} (k)_r^e)$$

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同様 $[a, b]^e \xrightarrow{*} (k)_r^e$ ($k > e$) は $\forall P: [a, b]^e \rightarrow r, \exists X \subseteq [a, b]$
 s. t. (i) X is P -homo (ii) $\bar{X} \geq k$ (iii) $\bar{X} \geq \min X$.

命題 PH は次の (1), (2), (3) の各は互に同値である。

$$(1) \forall a, k, r, e \exists b ([a, b] \xrightarrow{*} (k)_r^e)$$

$$(2) \forall x, z \exists y ([x, y] \xrightarrow{*} (z+1)_2^2)$$

$$(3) \forall z \exists y ([0, y] \xrightarrow{*} (z+1)_2^2)$$

2. PH は真である。

命題 (i) PH は真である。 (ii) 各 e に対し $PA \vdash PH^e$ である。

(iii) $PA, RFN_{\Sigma_1}(PA) \vdash PH$ ((ii) に依りては [17] 参照)

説明. (i) は König lemma と Infinite Ramsey theorem を用いて
 (c.f. 2 in [1]) (ii) (i) の証明より 各 e に対し $PA_2 \vdash PH^e$
 が得られる。 ことに PA_2 は PA の 2 階述語拡大, ことから
 各 e に対し $PA \vdash PH^e$, これを PA の中で形式化して (iii) を得る。

3. Lemma

PH を導入する諸定理のなかに次の補題をかくのが便利
 である。(von der Twer (2)) PH を仮定する。

与えられた $a, e, p, k (> e)$ が r primitive recursive function g
 に対し次のように b が存在する; 任意の $P: [a, b]^e \rightarrow 2$ に対し
 $X \subseteq [a, b]$ が存在して次を満足す (i) X is P -homo, (ii) $\bar{X} \geq k$
 (iii) $\min X \geq p$, (iv) $\bar{X} \geq g(\min X)$, (v) $x, y \in X, x < y \Rightarrow g(x) < y$

4. Theory T

T の言語: $0, 1, +, \cdot, <$, および無限個の constants $c_0, c_1, \dots$

T の公理 (i) $0, 1, +, \cdot, <$ の定義式は limited formula に与えられる induction (ii) $c_i^2 < c_{i+1}$

(iii) 任意の limited formula $\varphi(y_1, \dots, y_n, z_1, \dots, z_e)$ $i < k_1 < \dots < k_e$, $i < k'_1 < \dots < k'_e$, $\forall y < c_i [\varphi(y, c_{k_1}, \dots, c_{k_e}) \leftrightarrow \varphi(y, c_{k'_1}, \dots, c_{k'_e})]$

Theory T は Harrington に よる Σ_1^1 入 入 れ ば が なる こと により Σ_1^1 PH 原理の証明論的扱 ことが可能と なった.

定義 $\varphi := Q_0 x_0 \dots Q_{n-1} x_{n-1} \psi(x_0 \dots x_{n-1}) \in PA \cap \Sigma_n$ 又は Π_n formula とする. ψ は PR-formula ([6]) である. n とき φ^* と定義する; $\varphi^*(z_0 \dots z_{n-1}) := Q_0 x_0 < z_0 \dots Q_{n-1} x_{n-1} < z_{n-1} \psi(x_0 \dots x_{n-1})$ とし $z_0 \dots z_{n-1}$ は φ に なる variable とする.

命題 (Uesaka) $\theta(y) := \theta(y_1, \dots, y_n)$ は Π_n 又は Σ_n -formula とし $y_1, \dots, y_n$ 以外の自由変項は あり得ないとする

$PA \vdash \theta(y) \quad i < k_1 < \dots < k_n$ ならば $T \vdash y < c_i \rightarrow \theta^*(y, c_{k_1}, \dots, c_{k_n})$

これは [1] の 2.4 に対応するが証明論的証明は上記 [1] に 依る

(奥村 [5] 2.2 による).

系 $PA \vdash \text{Con } T \rightarrow \text{Con } PA$

5. $PA \vdash \text{Con } PA \rightarrow \text{Con } T \quad PA \not\vdash PH$

[1] では $PA, PH \Rightarrow \vdash \text{Con } T$ を導き上の系から $PA \not\vdash PH$ を導くことができるがもう少し精密な結果が得られる

定義 $S = S(c_0 \dots c_{n-1})$ は T の 文 である T の constants は $\{c_0, \dots, c_{n-1}\}$

にあるものを示す。 $\hat{S} := \exists x_0 \dots \exists x_{n-1} S(x_0 \dots x_{n-1})$ とおく。

$FS_T(x)$ は " x は T の有限個の公理の conjunction の $\neg$ -閉包である" であるとする formula, $FC_\omega(T)$ は " T の $\wedge$ の $\neg$ の有限部分集合は ω 上 satisfiable である" であるとする formula とする。

よって $FC_\omega(T) := \forall x (FS_T(x) \rightarrow Tr_{\Sigma_1}(\hat{x}))$ (T の公理は Σ_1 -文)

Lemma (1) $S = S(x_0 \dots x_{n-1})$ は T の有限部分集合 である。

と仮定すると d があつて $PH^d \Rightarrow \hat{S}$ は真。

(2) 上の S に対し $PA \vdash \hat{S}$

(3) $PA \vdash PH \rightarrow FC_\omega(T)$

Theorem (1) $PA \not\vdash PH$, (2) $PA \vdash \text{Con } PA \rightarrow \text{Con } T$ ($\text{Con } PA \leftrightarrow \text{Con } T$)

(1) は [1] の主要結果の一つである。 $PA \vdash PH \xrightarrow{\text{Lemma 3}} FC_\omega(T) \xrightarrow{\text{Prop 4}} \text{Con } T \rightarrow \text{Con } PA$

PA より出る。 (2) は Lemma (2) から出る。

6. $PH \leftrightarrow \text{RFN}_{\Sigma_1}(PA)$ 等

定義 $PA(n) := PA + \text{Th } \Pi_n(M)$ ($= \{\text{true } \Pi_n\text{-sentences}\}$)

Theorem $PH, FC_\omega(T), \text{Con}(PA(1)), \text{RFN}_{\Sigma_1}$ は PA の同値である。

[証明] $PH \xrightarrow{(1)} FC_\omega(T) \xrightarrow{(2)} \text{Con}(PA(1)) \xrightarrow{(3)} \text{RFN}_{\Sigma_1} \xrightarrow{(4)} \text{RFN}_{\Sigma_1} \xrightarrow{(5)} PH$

(1) は Lemma 4. (3), (4) は命題, (5) は Prop. 2 (iii)

(2) は informal に ω の $\neg$ の有限部分集合は ω 上 satisfiable と仮定し

$PA \vdash \theta_1 \dots \theta_n \rightarrow \perp$ (θ_i は true Π_1 sentences) とする。 と仮定すると $T \vdash \theta_1^* \dots \theta_n^* \rightarrow \perp$

(Prop 4) $\theta_1^* \dots \theta_n^*$ は satisfiable on ω ($\theta_i = \forall x \psi_i(x)$ なら $\theta_i^* := \forall x < c_0 \psi_i(x)$)

であるから $FC_\omega(T)$ は矛盾である。

系 $PA(1) \nVdash PH$

7. モデル論的方法

[2], [3]等のモデル論的方法は基本的には a lemma を出発点にある。

Lemma M is countable nonstandard model of PA である。

$c > M$ に対して $M \models [a, b] \xrightarrow{*} (c+1)_c$ なる sequence $(c_i)_{i \in \mathbb{N}}$,

$c_i \in [a, b]$ が存在して $(M, (c_i)_{i \in \mathbb{N}}) \models T$.

$I = \{d \in M \mid \exists i \in \mathbb{N} \ d < c_i\}$ である。 $I \subseteq_e M$ (end extension) である。

$I \models PA$, 更にこの場合 $I \subseteq_{\Sigma_0} M$, かつ Σ_0 -formula φ と $\vec{a} \in I$ に対して $I \models \varphi(\vec{a}) \iff M \models \varphi(\vec{a})$ (2.3 (ii) [2])

$PA(1) \nVdash PH$ のモデル論的証明; $M \models Th(M)$ である。

$a \in M \setminus \mathbb{N}$, $N \models PH$ である; $M \models [a, b] \xrightarrow{*} (a+1)_a$ である最小の

$b \in M$ である。上の lemma に従って $I \subseteq_e M$ である, $I \models PA$

$I \models Th_{\pi_1}(M)$ $a < I < b$. 又 $PA(1) \vdash PH$ なる $b' \in I$ がある。

$I \models [a, b'] \xrightarrow{*} (a+1)_a$ であることは $M \models [a, b'] \xrightarrow{*} (a+1)_a$ ($I \subseteq_e M$)

であるから b の最小性に矛盾する。

2 Provably recursive function

$f(a) = \mu b ([a, b] \xrightarrow{*} (a+1)_a)$ であることは recursive

Theorem g が provably recursive in PA ならば f は g を dominate する。すなわち $\exists n \in \mathbb{N} \ \forall m \in \mathbb{N} \ (m \geq n \rightarrow f(m) > g(m))$

この model 論的証明は主わの2簡單である (c.f. [2]). [1] へ

の2が命題1の $PH \leftrightarrow (2)$ と明確にして存いののである。
証明論的証明はむづかしい。これは事実上 Ketonen Solovay
[16] にある。

von der Twer は [2] の序で「PA における証明の概念を完全に抹殺した」と述べているが、奥村 [5], 小野角田 [17] 倉田 [4] は「モデルの概念を完全に抹殺した」と述べている。

9. T_n, T_∞ についての注意

定義 Theory T_n ($n=0, 1, \dots$) を T_∞ は T の公理 (i) (iii) の limited formula と Π_n -formula をいしは任意の formula にかまかえて得られるものとする。 $FC_\omega(T_n)$ は PA の文になる ω 次が成立。

(1) T_∞ は PA の conservative extension である。

(2) $PA \vdash Con PA \leftrightarrow Con T_n \leftrightarrow Con T_\infty$

(3) $PA \vdash PH \leftrightarrow FC_\omega(T_n)$

したがって $\{Con(T_n)\}_{n \in \omega}$ と $\{FC_\omega(T_n)\}_{n \in \omega}$ と $RFN_{Z_n}(PA)$

($n=1, 2, \dots$) に対応する hierarchy を与えることができる。

II Reflection Principle と PH , # と PH_n

10. Reflection principle

$Rfn(PA)$ (local reflection principle) $Pr_{PA}(\ulcorner \varphi \urcorner) \rightarrow \varphi$, φ は文 (PA)。

$RFN(PA)$ (uniform reflection P.) $\forall x Pr_{PA}(\ulcorner \varphi(x) \urcorner) \rightarrow \forall x \varphi(x)$

$RFN'(PA)$ (second uniform r. P.) $\forall x [Pr_{PA}(\ulcorner \varphi(x) \urcorner) \rightarrow \varphi(x)]$

$z := \ulcorner \varphi \urcorner$ is x and φ is free in z . $\ulcorner \varphi(x) \urcorner$ is $\ulcorner \varphi(z) \urcorner$ and so

$S(\ulcorner \varphi \urcorner, x)$ is formal expression.

$Rf_{\Sigma_k(\pi_k)}(PA)$, $RFN_{\Sigma_k(\pi_k)}(PA)$, $RFN_{\Sigma_k(\pi_k)}(PA)$ is Rf_n , RFN
 RFN' of $\varphi \in \Sigma_k(\pi_k)$ -sentence or formula is $\#1116$ is 49.

$RFN_{\Sigma_k(\pi_k)}^G$ (Global Reflection Principle) is

$$\forall x (St_{\Sigma_k(\pi_k)}(x) \wedge Pr_{PA}(x) \rightarrow Tr_{\Sigma_k(\pi_k)}(x))$$

Prop (1) $PA \vdash RFN_{\Sigma_k(\pi_k)}^G \rightarrow RFN'_{\Sigma_k(\pi_k)}$

(2) $PA \vdash RFN_{\Sigma_k}^G \leftrightarrow Con(PA(k))$

$$PA(k) = PA + Th_{\pi_k}(M)$$

(2) $RFN_{\Sigma_k}^G(PA) \Leftrightarrow \forall \varphi \in \Sigma_k (Pr_{PA}(\ulcorner \varphi \urcorner) \rightarrow Tr_{\Sigma_k}(\ulcorner \varphi \urcorner))$

$\Leftrightarrow \forall \varphi \in \Sigma_k (\neg Tr_{\Sigma_k}(\ulcorner \varphi \urcorner) \rightarrow \neg Pr_{PA}(\ulcorner \varphi \urcorner))$

$\Leftrightarrow \forall \varphi \in \pi_k (Tr_{\pi_k}(\ulcorner \varphi \urcorner) \rightarrow \neg Pr_{PA}(\ulcorner \neg \varphi \urcorner)) \Leftrightarrow Con(PA(k))$

Theorem (6) (i) $PA \vdash Con_{PA} \leftrightarrow RFN_{\pi_1}(PA)$

(ii) $PA \vdash RFN_{\Sigma_k(\pi_k)} \leftrightarrow RFN'_{\Sigma_k(\pi_k)}$

(iii) $PA \vdash RFN_{\pi_{k+1}}(PA) \leftrightarrow RFN_{\Sigma_k}(PA) \quad k \geq 0$

(iv) $PA \vdash RFN_{\pi_k} \nleftrightarrow RFN_{\Sigma_k} \quad k \geq 1$

($PA + RFN_{\pi_k}$ is consistent is fixed is

(iii) (iv) is $\vdash \vdash RFN_{\Sigma_k} (k=1, 2, \dots)$ is hierarchy of $\vdash$.

$PH \leftrightarrow RFN_{\Sigma_1}$ in PA is $\vdash \vdash$ $PH_n \leftrightarrow RFN_{\Sigma_n}$ in PA is $\vdash \vdash$

so that PH is stronger than PH_n is obvious.

11 PH_n is defined

(1) $\varphi \in \Pi_n$ -sentence $\forall v_0 \exists v_1 \dots \exists v_{n-1} \psi(v_0 \dots v_{n-1}) - \psi$ is PR-formula — is it 1, $\varphi^*(u_0 \dots, u_{n-1})$ is

$\forall v_0 < u_0 \exists v_1 < u_1 \dots \exists v_{n-1} < u_{n-1} \psi(v_0 \dots v_{n-1})$ is it 2, 2 is " $u_0 \dots u_{n-1}$ is φ is a variable is it 3.

$X = \{x_0 x_1 \dots\} (x_0 < x_1 < \dots) \in \bar{X} \geq n$ is it 4, 4 is "有限部分集合 is it 5.

$$X \models_0 \varphi^* \iff \varphi^*(x_0, \dots, x_{n-1}) \text{ is true}$$

$$X \models \varphi^* \iff \varphi^*(x_{i_0} \dots x_{i_{n-1}}) \text{ is true for all } i_0 < i_1 < \dots < i_{n-1} < \bar{X}$$

(2) $M \xrightarrow[\star]{\varphi} (k)_r^e (k \geq n)$ φ is Π_n -sentence is it 6, 6 is "意味するもの is it 7, 7 is "partition $P: [M]^e \rightarrow r$ is it 8

$X \leq M$ is it 9, 9 is "1 is it 10, 10 is "i) X is homogeneous for P ,

ii) $\bar{X} \geq k$ iii) $\bar{X} \geq \min X$ iv) $X \models_0 \varphi^*$

(3) $PH_n \forall k \geq n \in, \text{rand of all true } \Pi_n\text{-sentence } \varphi, M \text{ is it 11}$
 11 is $M \xrightarrow[\star]{\varphi} (k)_r^e$

Prop 1 PH_n is Π_{n+1} -sentence is it 12.

Π_n -formula $z_n(x)$ is it 13, 13 is "partition P is it 14, 14 is "is it 15, 15 is " $z_n(P) \in \mathbb{N}$ " is it 16

$$PH_n \text{ is } \forall k \geq n \forall r \forall e \forall p [z_n(p) \rightarrow \exists M (M \xrightarrow[\star]{z_n(p)} (k)_r^e)]$$

Prop 2 PH_n is it 17, 17 is "1 is it 18, 18 is "2 is it 19, 19 is "3 is it 20, 20 is "4 is it 21, 21 is "5 is it 22, 22 is "6 is it 23, 23 is "7 is it 24, 24 is "8 is it 25, 25 is "9 is it 26, 26 is "10 is it 27, 27 is "11 is it 28, 28 is "12 is it 29, 29 is "13 is it 30, 30 is "14 is it 31, 31 is "15 is it 32, 32 is "16 is it 33, 33 is "17 is it 34, 34 is "18 is it 35, 35 is "19 is it 36, 36 is "20 is it 37, 37 is "21 is it 38, 38 is "22 is it 39, 39 is "23 is it 40, 40 is "24 is it 41, 41 is "25 is it 42, 42 is "26 is it 43, 43 is "27 is it 44, 44 is "28 is it 45, 45 is "29 is it 46, 46 is "30 is it 47, 47 is "31 is it 48, 48 is "32 is it 49, 49 is "33 is it 50, 50 is "34 is it 51, 51 is "35 is it 52, 52 is "36 is it 53, 53 is "37 is it 54, 54 is "38 is it 55, 55 is "39 is it 56, 56 is "40 is it 57, 57 is "41 is it 58, 58 is "42 is it 59, 59 is "43 is it 60, 60 is "44 is it 61, 61 is "45 is it 62, 62 is "46 is it 63, 63 is "47 is it 64, 64 is "48 is it 65, 65 is "49 is it 66, 66 is 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$$(3) \forall \text{true } \Pi_n\text{-}\varphi \forall a \exists b ([0, b] \xrightarrow{\varphi} (a+1)_a^a)$$

12 PH_n is true

Prop (i) PH_n is true

$$(ii) \text{ 各 } e \text{ に } \exists \tau \subseteq PA \vdash PH_n^e$$

$$(iii) PA + RFN_{\Sigma_n}(PA) \rightarrow PH_n$$

$$: \vdash PH_n^e \text{ 且 } \forall k \geq n \forall r \forall P [\tau_n(P) \rightarrow \exists M (M \xrightarrow{\tau_n(P)} (k)_r^e)]$$

13 Lemma

PH_n は仮定 7.3. 3 にいう $n \in a, e, P, k \geq n$, primitive recursive function g , true Π_n -sentence φ , に $\exists \tau \subseteq b$ が存在して
 $\forall P : [a, b]^e \rightarrow 2, \exists X \subseteq [a, b]$ s.t. i) X is P -homo, ii) $\min X \geq P$,
 iii) $\bar{X} \geq k$, (iv) $g(\min X) \leq \bar{X} \quad \forall x, y, x < y \Rightarrow g(x) < y$
 vi) $X \models \varphi^*$

14 Theory $T(n)$

$T(n)$ は 4 の T に true Π_n -sentence φ に $\exists \tau \subseteq \tau \varphi^*(c_0 \dots c_{n-1})$
 と axiom τ 1.7 の τ を加えたもの

15 $FC_\omega(T(n))$

$FC_\omega(T(n))$ は $T(n)$ の有限部分集合は ω 上 satisfiable であることを示すことを証明する formula (PA の formula τ 1.7 を用いる)。

Lemma $S = S(c_0 \dots c_{n-1})$ は T の finite axioms の conjunction
 τ 1. constants は $c_0 \dots c_{n-1}$ の中にあり $n \geq \tau$ である。 φ は
 τ の true Π_n -sentence である。

2.9 2.3 (1) e が存在し

$$PH_n^e \Rightarrow \exists x_0 \cdots \exists x_{n-1} (S(x_0 \cdots x_{n-1}) \wedge \varphi^*(x_0 \cdots x_{n-1}))$$

$$(2) PA \vdash \exists x_0 \cdots \exists x_{n-1} (S(x_0 \cdots x_{n-1}) \wedge \varphi^*(x_0, \dots, x_{n-1}))$$

$$(3) PA \vdash PH_n \rightarrow FC_\omega(T(n)).$$

16 Theorem $PH_n, FC_n(T(n)), Con(T(n)), Con(PA(n)),$
 $RFN_{\Sigma_n}^Q, RFN_{\Sigma_n}$ は $\pi_1 \sim PA$ におい同値である。

$$\text{系} \quad PA(n) \not\vdash PH_n$$

17 Simple form of PH_n (von der Twer, Uesu)

$k \geq 0, e, r, M \in \omega, f: \omega \rightarrow \omega$ とする。

$$M \xrightarrow[f]{f} (k)_r^e \text{ は } \forall n \in \omega \text{ に対して } \exists x \in M \text{ かつ } x \geq n \text{ かつ } \forall p: [M]^e \rightarrow r$$

$$\exists X \subseteq M \text{ s.t. (i) } X \text{ is homo for } P, \text{ ii) } \bar{X} \geq k, \text{ iii) } \bar{X} \geq f(\min X)$$

$$PH(f) := \forall k \in r \exists M (M \xrightarrow[f]{f} (k)_r^e)$$

Theorem f_n は単調増大 Δ_n -function であり $\pi_1 \sim \Delta_n$ -function
 ε dominate する (n ≥ 1). 2.9 2.3

$$PA \vdash PH(f_n) \leftrightarrow PH_n \text{ (cf. [4])}$$

$$\text{系 1. } f_n \text{ は上と同値であること } PA(n) \not\vdash PH(f_n)$$

(von der Twer [2], 4.5)

$PH \equiv PH(f_n)$ に拡張し系 1 を証明した最初の人 von der
 Twer である。倉田は独立に PH_n を見出し Theorem 16 を証
 明した (von der Twer には RFN の誤植が 17 の意図がなかった
 といわれる) 上は [1] は [2] を知る前に PH_n が上記の $PH(f_n)$

に同値であることは容易に示唆し得る。

系 2 $RFN(PA) \Leftrightarrow$ 大抵の単調増大算術的関数 f に対し $PH(f)$ が成立。

III $RFN(PA)$ と ε_0 -Induction

18 $RFN \rightarrow Ind \varepsilon_0$

定義 $Ind(n, \varphi) : \omega_n$ 上での transfinite induction of PA の formula φ に対して成立することは言明する PA の formula, すると

$$\forall x (\forall y \leq x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x) \quad (\text{cf. [8]})$$

$$Ind(\varepsilon_0, \varphi) := \forall x Ind(x, \varphi), \quad Ind \varepsilon_0 : Ind(\varepsilon_0, \varphi) \text{ for every } \varphi.$$

Prop $PA \vdash RFN \rightarrow Ind \varepsilon_0$

実際.. PA の任意の formula $\varphi(x)$ に対して, n に対し

$$PA \vdash Ind(n, \varphi) \quad ([7], [8], [9], [10])$$

$$\text{よって } PA \vdash \forall x Pr_{PA}(\ulcorner Ind(x, \varphi) \urcorner).$$

19 $Ind \varepsilon_0 \rightarrow RFN$ (1st proof)

PA^w : axiom は true quantifier free formula of PA

inference rule は LK の ω -rule と ω .

$\varphi(x)$ は x 以外に free var. を持たない PA の formula とする,

$$PA \vdash \varphi(\bar{x}) \Rightarrow PA^w \vdash \varphi(\bar{x}) \Rightarrow PA^w \vdash \varphi(\bar{x}) \quad (\text{カットなし, } \Rightarrow)$$

よって $Ind \varepsilon_0 \models \varphi(\bar{x}) \Rightarrow \varphi(\bar{x})$ is true, すると PA の任意の formula $\varphi(x)$ に対し

$$PA \vdash Ind \varepsilon_0 \vdash Pr_{PA}(\ulcorner \varphi(\bar{x}) \urcorner) \rightarrow Tr_p(\ulcorner \varphi(\bar{x}) \urcorner)$$

$$\text{" } \vdash Pr_{PA}(\ulcorner \varphi(\bar{x}) \urcorner) \rightarrow \varphi(x)$$

$\vdash \vdash \tau$ p is $\varphi(x)$ の unbounded quantifier の 数.

(cf. Schwichtenberg [15])

20 Ind $\varepsilon_0 \rightarrow \text{RFN}(\text{PA})$ (2nd Proof)

$$\varphi(x) := \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \psi(x_1 \dots x_n, y_1 \dots y_n) \quad \psi \in \Sigma_0$$

$\tau \vdash \vdash \tau$. $\vdash \vdash \tau$ のように $\varphi_H(x)$ を 定義する. $\varphi(x) \rightarrow \varphi_H(x)$ in LK τ の 2.

$$\varphi_H(x) := \exists x_1 \dots \exists x_n \psi(x, x_1 \dots x_n, f_1(x_1) \dots f_n(x_1 \dots x_n))$$

No counter example interpretation by functional $F_1 \dots F_n$ へ

$$\varphi^N(x) := \varphi^N(x, F_1 \dots F_n)$$

$$= \psi(x, \bar{F}_1 \dots \bar{F}_n, f_1(\bar{F}_1) \dots f_n(\bar{F}_1 \dots \bar{F}_n)) \quad \text{「定」の 3.}$$

$$\vdash \vdash \vdash \bar{F}_0 = F_0(x, f_1 \dots f_n) \quad (\text{Kreisel [11]}).$$

$$\vdash \vdash \text{PA} \vdash \varphi(\bar{x}) \quad (\vdash \vdash \tau \text{ は } \text{PA} \vdash \varphi_H(\bar{x}))$$

Hilbert's Substitution method に よる $\varphi_H(\bar{x})$ の 証明 は functional

$F_1 \dots F_n$ を 構成 する $\vdash \vdash \tau$ が τ まで $\varphi^N(x, F_1 \dots F_n)$ が 成立 する

$\vdash \vdash \tau$ が Σ_0 -induction を 用いて 証明 される (Ackermann [2] Tait [13])

$\vdash \vdash \tau$ を 形式化 する

Prop(1) x の 4 は free に なる 任意 の formula $\varphi(x)$ と $\varphi(\bar{x})$ の 証明 p に 対し. Functional $F_1, \dots, F_n$ ($\bar{F}_1 \dots \bar{F}_n$ は PA' の term) が 存在 する

$$\tau \vdash \vdash \text{PA}' \vdash \text{Ind}' \varepsilon_0 \vdash \text{Pr}(\text{tp}_A(\varphi(\bar{x})), p) \rightarrow \varphi^N(x, F_1 \dots F_n) \quad (*)$$

$\vdash \vdash \tau$ PA' は PA に free function $f_1, \dots, f_n$ と μ -symbol を 追加 した

$\text{Ind}' \varepsilon_0$ は PA' の formula に 対し τ は Σ_0 -induction

Prop(2) PA' の term $e_1, \dots, e_n$ が ある

$$PA' \vdash \forall x_1 \dots \forall x_n \psi(x_1, \dots, x_n, e_1(x_1), \dots, e_n(x_1, \dots, x_n)) \rightarrow \varphi(x)$$

$$\text{E} \quad \text{と} \quad \text{E} \quad \text{は} \quad e_1(x_1) = \mu y_1 \rightarrow \exists x_2 \forall y_2 \dots \exists x_n \forall y_n \psi(x, x_1, \dots, x_n, y_1, \dots, y_n)$$

$$e_2(x_1, x_2) = \mu y_2 \rightarrow \exists x_3 \forall y_3 \dots \exists x_n \forall y_n \psi(x, x_1, \dots, x_n, y_1, \dots, y_n)$$

----- と 同様に "E" 。

$$(x) = \text{すなわち} \quad \varphi^N(x, F_1, \dots, F_n) \text{ の } f_1, \dots, f_n \in e_1, \dots, e_n \text{ によって}$$

prop (2) を用いて "E" 。

$$PA' + \text{Ind}' \varepsilon_0 \vdash \text{Pr}_{PA}(\ulcorner \varphi(x) \urcorner) \rightarrow \varphi(x)$$

を得る。すなわち $PA' + \text{Ind}' \varepsilon_0$ は $PA + \text{Ind} \varepsilon_0$ の conservative extension である。

$$\text{よって} \quad PA + \text{Ind} \varepsilon_0 \not\vdash \text{Pr}_{PA}(\ulcorner \varphi(x) \urcorner) \rightarrow \varphi(x)$$

以上は Kreisel-Lévy [14] の証明である。

$$\text{Theorem } PA \vdash \text{RFN}(PA) \leftrightarrow \text{Ind} \varepsilon_0$$

$$\text{IV} \quad PH_n \quad n=1, 2, \dots \quad \text{と} \quad \text{Ind} \varepsilon_0$$

PH_n は RFN を証明する RFN は $\text{Ind} \varepsilon_0$ を証明する
 ことが知られている。すなわち PH_n は $\text{Ind} \varepsilon_0$ を証明するに等しい。
 直接証明は事実上 Ketonen-Solovay ([17]) による。以下にこれを整理した。

$$1. \quad \lambda \leq \varepsilon_0, \quad n \in \omega \quad \text{に対して} \quad \{\lambda\}(n)$$

$\lambda < \varepsilon_0$ に対して $\lambda = \omega^{\alpha_1} m_1 + \dots + \omega^{\alpha_k} m_k$ と λ の Cantor normal form とする。すなわち $\alpha_1 > \dots > \alpha_k$ m_i は正整数。

$$\lambda = \omega^{\alpha}(\beta+1) \quad \text{と} \quad \text{書ける}$$

$$(1) \quad \lambda = \omega^{\alpha+1}(\beta+1) \quad \text{ならば} \quad \{\lambda\}(n) = \omega^{\alpha+1} \beta + \omega^{\alpha} \cdot n$$

$$(2) \lambda = \omega^{\beta+1} \text{ is a limit ordinal; } \{\lambda\}(n) = \omega^{\beta} \cdot \beta + \omega^{\beta}(n)$$

$$(3) \lambda = \varepsilon_0 \text{ is a limit ordinal; } \{\varepsilon_0\}(0) = \omega, \{\varepsilon_0\}(n+1) = \omega^{\{\varepsilon_0\}(n)}$$

$$(4) \{\beta+1\}(n) = \beta, \{0\}(n) = 0$$

$$\text{Lemma 1. } 0 < \alpha \leq \varepsilon_0 \Rightarrow \{\alpha\}(n) < \alpha, \lim_{n \rightarrow \infty} \{\alpha\}(n) = \alpha.$$

2. α -large set

$\alpha \leq \varepsilon_0$. $S = \langle s_0, \dots, s_{n-1} \rangle \in \omega$ a finite sequence $\in \mathbb{N}$.

$\{\alpha\}\langle S \rangle \in \mathbb{N}$ is defined as follows. $\{\alpha\}\langle \emptyset \rangle = \alpha$ ($\emptyset$ is empty set).

$$\{\alpha\}\langle s_0, \dots, s_{n-1} \rangle = \{\{\alpha\}\langle s_0, \dots, s_{n-2} \rangle\}(s_{n-1})$$

ω a finite set X is defined as

$$\{\alpha\}\langle X \rangle = \{\alpha\}\langle x_0, x_1, \dots, x_{n-1} \rangle$$

$$\text{where } X = \{x_0, x_1, \dots, x_{n-1}\}, x_0 < x_1 < \dots < x_{n-1} \in \mathbb{N}.$$

Define X is α -large $\Leftrightarrow \{\alpha\}\langle X \rangle = 0$ or $\in \mathbb{N}$

Lemma 1. The properties are as follows.

(i) X is ω -large iff X is a finite set and X is α -large $\Leftrightarrow \{\alpha\}\langle X \rangle = 0$.

$$(ii) X \text{ is } m\text{-large } (m \in \omega) \Leftrightarrow \bar{X} \geq m$$

$$(iii) X \text{ is } \omega\text{-large} \Leftrightarrow \bar{X} > \min X$$

$$(iv) X \text{ is } \omega+n\text{-large} \Leftrightarrow X \text{ is relatively large and } \bar{X} \geq n$$

$$(v) A \subset B, A \text{ is } \alpha\text{-large} \Rightarrow B \text{ is } \alpha\text{-large}$$

$F: X \rightarrow \mathbb{N}$ is a function $\Rightarrow \exists j \in \mathbb{N}, f^{-1}(j)$ is α -large

3 Ketonen Solovay's Lemma

$\alpha \geq \omega$ $2 \leq n$ $r < \omega$ ならば

$$\theta_r(\alpha) = \omega^\alpha + \omega^3 + \max(r, \|\alpha\|) + 3 < \alpha <.$$

すなわち $\|\alpha\|$ は $\alpha = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} n_k$ (Cantor normal form)

のとき $\|\alpha\| = \sum_{j=1}^k \|\alpha_j + 1\| \cdot \omega_j^n$ ならば $\alpha > 2$ は数である。

Lemma, X が $\theta_r(\alpha)$ -large set ならば $P: [X]^{n+1} \rightarrow r$ ならば

$\exists Y \subseteq X$; Y は α -large prehomogeneous set, すなわち

α ならば $x_0 < \dots < x_{n+1} < y$, $x_0 < \dots < x_{n+1} < z$ from Y ならば

$$P(x_0, x_1, \dots, x_{n+1}, y) = P(x_0, x_1, \dots, x_{n+1}, z)$$

$$\theta_r^1 = \alpha, r, \quad \theta_r^{n+1}(\alpha) = \theta_r(\theta_r^n(\alpha)) \quad n \geq 1 \text{ ならば.} \quad \alpha \text{ ならば.}$$

Theorem X が $\theta_r^n(\alpha)$ -large ならば α ならば $\exists$ a partition

$P: [X]^n \rightarrow r$ は α -large homogeneous set for P ならば.

Corollary M が $\theta_r^n(\omega + n + 1)$ -large $\Rightarrow M \xrightarrow[n]{*} (n+1)_r^n$

f は 帰強単調増大関数: $\omega \rightarrow \omega$ ならば.

ω の有限集合 X が α -large(f) ならば $f(X)$ が α -large $n = 2$ ならば.

Corollary 2 M が $\theta_r^n(\omega + n + 1)$ -large(f) $\Rightarrow$

$$M \xrightarrow[n]{f} (n+1)_r^n$$

$\text{Ind} \mathcal{E}_0$ と PH_n $n=1, 2, \dots$ への直接証明の方法は等価に

は等価である。

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