

Some remarks on a cohomology for foliated structures.

1. Let L be a topological vector space with a continuous linear map d into itself so that $d \circ d = 0$, then we see that

$$\text{Im } d \subset \text{Ker } d, \quad \overline{\text{Im } d} \subset \overline{\text{Ker } d}$$

and so that

$$\overline{\text{Ker } d} = \text{Ker } d,$$

therefore we have

$$\overline{\text{Im } d} \subset \text{Ker } d.$$

The following notion is suggested by Prof. K. Shiga,

Definition We set

$$\overline{H}(L, d) = \text{Ker } d / \overline{\text{Im } d}.$$

Let V be a normed vector space with a linear map ∂ into itself so that $\partial \circ \partial = 0$ and denote the conjugate object with

. Introducing the weak topology into V^ , we see that ∂^* turns out to be continuous linear.

Then it is obtained that ([S])

$$\text{Prop. 1.} \quad \dim \overline{H}(V^*, \partial^*) \leq \dim \overline{H}(V, \partial).$$

In case that ∂ itself is continuous, Prop. 1 is sharpened as follows:

Theorem 1 (Suzuki-Shikata-Sakai)

$$\dim \overline{H}(V^*, \partial^*) = \dim \overline{H}(V, \partial) \leq \dim H(V, \partial)$$

Proof. Take linearly independent elements $a_1, \dots, a_m$ in $\text{Ker } \partial$ on $\overline{\text{Im } \partial}$, and define V_m to be a subspace in V generated by $a_1, \dots, a_m$ and by $\text{Im } \partial$. Since $V_m / \overline{\text{Im } \partial}$ is canonically homeomorphic to the euclidean m space, we can define continuous linear functionals $\tilde{a}_1, \dots, \tilde{a}_m$ on V_m so that

$$\tilde{a}_i(a_j) = \delta_{ij}, \quad \tilde{a}_i = 0 \quad \text{on } \overline{\text{Im } \partial}.$$

The functionals $\tilde{a}_1 \dots \tilde{a}_m$ extend to functionals $a_1^* \dots a_m^*$ on V^* by Hahn-Banach's theorem and satisfy that

$$a_i^*(a_j) = \delta_{ij}, \quad a_i^* = 0 \quad \text{on } \overline{\text{Im } \partial}.$$

Thus we have that

$$(\partial^* a_i^*)(x) = a_i^*(\partial x) = 0, \quad \text{for any } x \in V,$$

$$(\lim \partial^* \xi_i)(a_j) = \lim (\xi_i(\partial a_j)) = 0, \quad \text{for } (\lim \partial^* \xi_i) \in \overline{\text{Im } \partial^*}$$

$$(\sum c_i a_i^* + \mu)(a_j) = c_j, \quad \text{for } \mu \in \overline{\text{Im } \partial^*}$$

Hence we see that $a_1^* \dots a_m^*$ define linearly independent elements in $H(V^*, \partial^*)$ and that

$$\dim \overline{H(V^*, \partial^*)} \geq \dim \overline{H(V, \partial)}$$

Conversely, take linearly independent elements $a_1 \dots a_m$ in $\text{Ker } \partial^*$ over $\overline{\text{Im } \partial^*}$ and define similarly $\alpha_1^* \dots \alpha_m^*$ to be (weak) continuous functionals on V^* satisfying that

$$\alpha_i^*(\alpha_j) = \delta_{ij}, \quad \alpha_i^* = 0 \quad \text{on } \overline{\text{Im } \partial^*}.$$

Then by Mackey's weak duality theorem, we find $a_1 \dots a_m \in V$ so that

$$\alpha_i^*(x) = x(a_i), \quad \text{for any } x \in V^*,$$

thus we have that

$$x(\partial a_i) = (\partial^* x)(a_i) = \alpha_i^*(\partial^* x) = 0, \quad \text{for } x \in V^*$$

$$a_j(\lim \partial b_i) = \lim a_j(\partial b_i) = \lim (\partial^* \alpha_j)(b_i) = 0, \quad \text{for } \lim \partial b_i \in \overline{\text{Im } \partial}$$

$$a_j(\sum c_i a_i + \mu) = c_j, \quad \text{for } \mu \in \overline{\text{Im } \partial}.$$

Hence we also see that $a_1 \dots a_m$ define linearly independent elements in $\overline{H(V, \partial)}$ and that

$$\dim \overline{H(V^*, \partial^*)} \leq \dim \overline{H(V, \partial)}$$

finishing the proof. As a corollary to Th. 1, we have that

Corollary 1,

$$\dim H(V^*, \partial^*) \geq \dim \bar{H}(V, \partial).$$

2. Let M be a smooth manifold with two foliated structures F, G which are dual to each other. The flat norm $\| \cdot \|_b$ in the cosheaf $\Psi_q(R^S)$ of q -chain in $R^S (s = \dim F)$ ~~space~~ extends naturally to that in the cosheaf $\Psi_q(F)$ through the (local) identification of the leaf of F to R^S . Let $\hat{\Psi}_q$ denote the completion of $\Psi_q = \Psi_q(F)$ under the flat norm, then the Čech chain $C_p(\{U\}; \oplus_q)(\oplus_q \Psi_q, \hat{\Psi}_q)$ for a finite open covering $\{U\}$ turns out to be a normed linear space with the norm given by

$$\|a\| = \sup \left\{ \|a_{j_0} \dots j_p\|_b / (j_0 \dots j_p) : p\text{-Čech simplex} \right\}$$

for $a \in C_p(\{U\}; \oplus_q)$.

Since the completion $\hat{\Psi}_q$ and Ψ_q have the same dual space F^q , we have that

Prop. 2 For the sheaf F^q of flat q -cochain on R^S , it holds that

$$\dim \bar{H}_{\oplus}^{\bar{H}}(C^p(\{U\}; F^q), \Delta^*) = \dim \bar{H}(C_p(\{U\}; \oplus_q), \Delta)$$

where $\bar{H}_{\oplus}$ indicates the difference of the weak topology in F^q .

Corollary 3

$$\begin{aligned} \dim \bar{H}_{\Psi}^{\bar{H}}(C^p(\{U\}; F^q), \Delta^*) &\leq \dim H(C_p(\{U\}; \Psi_q), \Delta) \\ \dim \bar{H}_{\hat{\Psi}}^{\bar{H}}(C^p(\{U\}; F^q), \Delta^*) &\leq \dim H(C_p(\{U\}; \hat{\Psi}_q), \Delta) \end{aligned}$$

Corollary 4

$$\begin{aligned} \dim \bar{H}(C_p(\{U\}; \Psi_q), \Delta) &\leq \dim H_{\Psi}^{\bar{H}}(C^p(\{U\}; F^q), \Delta^*) \\ \dim \bar{H}(C_p(\{U\}; \hat{\Psi}_q), \Delta) &\leq \dim H_{\hat{\Psi}}^{\bar{H}}(C^p(\{U\}; F^q), \Delta^*) \end{aligned}$$

Taking $H_{p,q}$ (resp $\hat{H}_{p,q}$) to be (resp. the completion of) the

$\bigwedge$, the Weil's chain gives isomorphisms between homologies
cosheaf of (p,q) -chains

of cosheaves;

Prop. 3. For a finite simple (admissible) covering U it holds that

$$L(C_p(\{U\}, \oplus_q): \Delta) = L(\oplus_{p,q}; \partial_x),$$

where

$$L = H \text{ or } \bar{H}^{\oplus}, \quad \oplus_q = \bar{\Phi}_q \text{ or } \hat{\Phi}_q, \quad \oplus_{p,q} = H_{p,q} \text{ or } \hat{H}_{p,q}.$$

Since we know that

$$\dim H(H_{p,o}; \partial_x) = \# \left\{ \begin{array}{l} \text{independent } p\text{-cycles} \\ \text{in the leaf of } F \end{array} \right\},$$

we have that

Corollary 4

$$\begin{aligned} \# \{ \text{indep. } p\text{-cycles in the leaf} \} &= \dim H(C_p(\{U\}; \bar{\Phi}_0), \Delta) \\ &\geq \dim \bar{H}_{\Psi}(C^p(\{U\}; F^0), \Delta) \end{aligned}$$

Introducing the notion of pseudo closed cycles in the leaf, we also have that

$$\dim H(\hat{H}_{p,o}; \partial_x) = \# \left\{ \begin{array}{l} \text{indep. pseudo closed cycles} \\ \text{in the leaf of } F \end{array} \right\}$$

and that

Corollary 5

$$\# \{ \text{indep. pseudo } p\text{-cycles in the leaf} \} \geq \dim \bar{H}_{\Psi}(C^p(\{U\}; F^0), \Delta)$$

[S] Y. Shikata On the cohomology of bigraded forms associated with foliated structures.