

Note on shape theory II : Problems

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Let $\mathcal{M}$ be the class consisting of metrizable spaces. We define a shape category in $\mathcal{M}$ called a fine shape category as follows. Let M and N be spaces in $\mathcal{M}$ and let X and Y be closed sets in M and N respectively. According to [7] a continuous map $f : M-X \rightarrow N$ is said to be a fine map from $M-X$ into N rel. X, Y if for every neighborhood V of Y in N there is a neighborhood U of X in M such that $f(U-X) \subset V$. Two fine maps $f, g : M-X \rightarrow N$ rel. X, Y are fine homotopic (notation: $f \underset{F}{\simeq} g$ rel. X, Y) if there is a homotopy $H : (M-X) \times I \rightarrow N$ satisfying the following condition: $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for $x \in M-X$, and for every neighborhood V of Y in N there is a neighborhood U of X in M such that $H((U-X) \times I) \subset V$. The following lemmas are obvious.

Lemma 1. If Y is an unstable closed set of N and $f : M-X \rightarrow N$ is a fine map rel. X, Y , then there is a fine map $f' : M-X \rightarrow N$ rel. X, Y such that $f'(M-X) \subset N-Y$ and $f \underset{F}{\simeq} f'$ rel. X, Y .

Lemma 2. Let $L \in \mathcal{M}$ and Z a closed set of L . If $f : M-X \rightarrow N$ is a fine map rel. X, Y such that $f(M-X) \subset N-Y$ and $g : N-Y \rightarrow L$ is a fine map rel. Y, Z , then $gf : M-X \rightarrow L$ is a fine map rel. X, Z .

A fine map $f : M-X \rightarrow N$ rel. X, Y is said to be a fine equivalence if there are fine maps $f' : M-X \rightarrow N$ rel. X, Y and $g : N-Y \rightarrow M$ rel. Y, X such that $f'(M-X) \subset N-Y$, $f' \underset{F}{\simeq} f$ rel. X, Y , and

$$(1) \quad gf' \underset{F}{\simeq} 1_{M-X} \text{ rel. } X, X,$$

$$(2) \quad f'g \underset{F}{\simeq} l_{N-Y} \text{ rel. } Y, Y,$$

where l_{M-X} and l_{N-Y} are the identity fine maps in $M-X$ and $N-Y$ respectively. If only (1) is satisfied, then f is said to be a fine domination.

For $X, Y \in \mathcal{M}$, let M and N be AR's containing X, Y as unstable closed sets respectively. If there is a fine equivalence $f : M-X \rightarrow N \text{ rel. } X, Y$, then we say that X and Y are fine shape equivalent or have the same fine shape and write $Sh_F(X) = Sh_F(Y)$. If there is a fine domination $f : M-X \rightarrow N$, then X fine shape dominates Y and we write $Sh_F(X) \geq Sh_F(Y)$. By Lemmas 1 and 2, it is easy to see that there is a shape category consisting of spaces in M whose morphisms are the fine homotopy equivalence classes of fine maps. We call it a fine shape category. If spaces are compact then the fine shape defined here is the same as one defined in [9].

Problem 1. For $X \in \mathcal{M}$, what relation is there between $Sh(X)$ and $Sh_F(X)$ or $Sh_W(X)$ and $Sh_F(X)$? If X and Y are locally compact and $Sh(X) \geq Sh(Y)$ or $Sh_W(X) \geq Sh_W(Y)$, is $Sh_F(X) \geq Sh_F(Y)$ true? Here $Sh(X)$ is the shape of X in the sense of Fox [4] and $Sh_W(X)$ is the weak shape in the sense of Borsuk [1].

Problem 2. Characterize the space X such that $Sh_F(X) = 1$, that is, X is fine shape equivalent with a one point space.

The following are problems concerning the shape $Sh(X)$ however many of them are interesting one's concerning the fine shape $Sh_F(X)$ too.

Problem 3. Let $X, Y, X', Y' \in \mathbb{M}$ and let Y, Y' be locally compact. If $\text{Sh}(X) = \text{Sh}(X')$ and $\text{Sh}(Y) = \text{Sh}(Y')$, is $\text{Sh}(X \times Y) = \text{Sh}(X' \times Y')$ true ?

Problem 4. Are there $X \in \mathbb{M}$ and a locally compact space $Y \in \mathbb{M}$ such that $\text{Fd}(X \times Y) < \text{Fd}(X) + \text{Fd}(Y)$? Here $\text{Fd}(X) = \min\{\dim Y : Y \in \mathbb{M} \text{ and } \text{Sh}(X) \leq \text{Sh}(Y)\}$.

Cf. [5] and [6].

Problem 5. (S. Mardešić) Let $X \in \mathbb{M}$ and let Y be a locally finite simplicial polytope.

(1) Is there the product $\text{Sh}(X) \times \text{Sh}(Y)$?

(2) If $\text{Sh}(X) \times \text{Sh}(Y)$ exists, is $\text{Sh}(X) \times \text{Sh}(Y) = \text{Sh}(X \times Y)$?

Problem 6. Let $X, Y \in \mathbb{M}$ and $f : X \rightarrow Y$. If there is a locally finite open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ such that for any finite set $\{\alpha_0, \dots, \alpha_n\} \subset \Lambda$ $f|_{f^{-1}(U_{\alpha_0} \cap \dots \cap U_{\alpha_n})} : f^{-1}(U_{\alpha_0} \cap \dots \cap U_{\alpha_n}) \rightarrow U_{\alpha_0} \cap \dots \cap U_{\alpha_n}$ is a shape equivalence, is f a shape equivalence ?

By the same way as in the proof of [8], the problem is solved affirmatively for the fine shape Sh_F .

Problem 7. If X is a compactum which is a 1-1 continuous image of a locally compact ANR, then:

(1) Is X an FANR ?

(2) Is X movable ?

Problem 8. Let X be a locally compact space in $\mathbb{M}$. If X is movable in the sense of Kozłowski and Segal [10], is every metrizable compactification of X movable ?

Problem 9. Let X be a compactum. If there is a countable

number of movable compacta X_i , $i=1,2,\dots$, such that $X = \bigcup_{i=1}^{\infty} X_i$, is X movable? (Here X_i and X_j , $i \neq j$, are not necessarily disjoint.)

Problem 10. Let (X, x_0) and (Y, y_0) be pointed compacta. If $\underline{f} : (X, x_0) \rightarrow (Y, y_0)$ is a shaping such that $\underline{f}_* : \underline{\pi}_n(X, x_0) \xrightarrow{\cong} \underline{\pi}_n(Y, y_0)$ for $n = 0, 1, 2, \dots$, is $\text{Sh}_F(X, x_0) = \text{Sh}_F(Y, y_0)$ true? Here $\underline{\pi}_*$ is the shape group defined by Quigley [11].

Problem 11. (Chapman) Is every weak proper homotopy equivalence a proper homotopy equivalence?

If Problem 11 has an affirmative solution, the second part of Problem 1 is so. Cf. [2] and [3].

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