

A Note on Decomposition Numbers for $SU(3, q^2)$

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This is a joint work with Prof. Okuyama. Let q be a power of a prime p and r is a prime which divides $q + 1$. So there are s and a such that $q + 1 = r^a s$ and s isn't divided by r any more. F is an algebraically closed field of the characteristic r .

We denote $G = SU(3, q^2) = \{A \in SL(3, q^2) \mid A\omega\bar{A}^t = \omega\}$ where $\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then the order of G is $q^3(q^3 + 1)(q^2 - 1)$.

In [1], Geck determined the decomposition numbers of the principal block of G as the following.

Theorem 0.1

		I_G	φ_S	φ_T
1	θ_1	1		
1	θ_{q^2-q}		1	
1	θ_{q^3}	1	α	1
$r^a - 1$	θ_{q^2-q+1}	1	1	
$r^a - 1$	θ_{q^2-q+1}		$\alpha - 1$	1
$(r^a - 1)(r^a - 2)/6$	θ_{q^2-q+1}	1	$\alpha - 2$	1

where $2 \leq \alpha \leq \frac{r^a+1}{3}$

Let we denote I_G, S, T simple FG -modules which are corresponding to above irreducible Brauer characters.

In this paper, we determine α in case the center of G is trivial.

1 Notation

For any elements a in the finite field $GF(q^2)$, $\bar{a} = a^q$. Then three kinds of elements $t, h(x), u(a, b)$ in G denote $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & x^{q-1} & 0 \\ 0 & 0 & x^{-q} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & -a & 1 \end{pmatrix}$. From these elements, we can construct subgroups of G .

- $H = \{h(x) \mid x \in GF(q^2)^\times\}$

- $U = \{u(a, b) | a\bar{a} + b + \bar{b} = 0\}$
- $U_0 = \{u(0, b) | b + \bar{b} = 0\}$
- $B = H \ltimes U$
- $B_0 = H \ltimes U_0$

And the order of each subgroups is $q^2 - 1$, q^3 , q , $q^3(q^2 - 1)$, $q(q^2 - 1)$. Let R be Sylow r -subgroup of B then the order of R is r^a .

2 About Subgroups

It is easy to check the following.

Lemma 2.1 $G = B \cup BtU$

Lemma 2.2 The center of U is U_0 .

Lemma 2.3 For any non-trivial subgroup R' of R , $N_B(R') = B_0$.

Lemma 2.4 Any subgroups R' of R is *TI* (Trivial Intersection) set.

Let L denotes $B_0 \cup B_0tU_0$, then the number of elements in L is $q(q^2 - 1) + q^2(q^2 - 1) = q(q + 1)(q^2 - 1)$ and a next lemma is followed.

Lemma 2.5 L is a subgroup of G and it is isomorphic to $U(2, q^2)$.

For any subsets S of G , we define $\hat{S} = \sum_{s \in S} s$. We fix the element b in $GF(q^2)$ with the condition $b + \bar{b} \neq 0$. For this b , we define $\gamma(b) = \sum_a \hat{B}tu(a, b)$ where a runs over $a\bar{a} + b + \bar{b} = 0$. Then we can get the following lemma.

Lemma 2.6 For the element b , let $a_0 \in GF(q^2)$ with $a_0\bar{a}_0 + b + \bar{b} = 0$.

i) If $b_0 = (a_0\bar{a}_0)^{-2}b$, then $a_0^{-1}\overline{a_0^{-1}} + b_0 + \bar{b}_0 = 0$.

ii) If $g = u(-\overline{a_0^{-1}}, b_0)tu(a_0, b)$, then $\gamma(b) = \widehat{BL}g$.

Proof : (i)

$$\begin{aligned} \overline{a_0^{-1}}a_0^{-1} + b_0 + \bar{b}_0 &= \overline{a_0^{-1}}a_0^{-1} + (a_0\bar{a}_0)^{-2}b + (a_0\bar{a}_0)^{-2}\bar{b} \\ &= (a_0\bar{a}_0)^{-2} \{a_0\bar{a}_0 + b + \bar{b}\} = 0 \end{aligned}$$

(ii) Since $L = B_0 \cup B_0 t U_0$, $BL = B \cup B t U_0$. So $\widehat{BL} = \widehat{B} + \widehat{B t U_0} = \widehat{B} + \widehat{B t U_0}$. The equation : $tu(a, b)t = h(b)u(-a\overline{b^{-1}}b, \overline{b})tu(-a\overline{b^{-1}}, b^{-1})$ shows that

$$\begin{aligned}
 \widehat{BL}g &= \widehat{B}g + \widehat{B t U_0}g \\
 &= \widehat{B}u(-\overline{a_0^{-1}}, b_0)tu(a_0, b) + \widehat{B t U_0}u(-\overline{a_0^{-1}}, b_0)tu(a_0, b) \\
 &= \widehat{B}tu(a_0, b) + \widehat{B}t \sum_{b'} u(0, b')u(-\overline{a_0^{-1}}, b_0)tu(a_0, b) \\
 &= \widehat{B}tu(a_0, b) + \widehat{B}t \sum_{b'} u(-\overline{a_0^{-1}}, b_0 + b')tu(a_0, b) \\
 &= \widehat{B}tu(a_0, b) + \widehat{B} \sum_c tu(-\overline{a_0^{-1}}, c)tu(a_0, b) \\
 &= \widehat{B}tu(a_0, b) + \widehat{B} \sum_c tu(\overline{a_0^{-1}c^{-1}}, c^{-1})u(a_0, b) \\
 &= \widehat{B}tu(a_0, b) + \widehat{B} \sum_c tu(a_0 + \overline{a_0^{-1}c^{-1}}, b)
 \end{aligned}$$

where b' runs over $b' + \overline{b'} = 0$ and c runs over $\overline{a_0^{-1}a_0^{-1}} + c + \overline{c} = 0$. Now, if we put $a = a_0 + \overline{a_0^{-1}c^{-1}}$,

$$\begin{aligned}
 a\overline{a} &= (a_0 + \overline{a_0^{-1}c^{-1}})\overline{a_0 + \overline{a_0^{-1}c^{-1}}} \\
 &= (a_0 + \overline{a_0^{-1}c^{-1}})(\overline{a_0} + \overline{a_0^{-1}c^{-1}}) \\
 &= a_0\overline{a_0} + c^{-1} + \overline{c^{-1}} + \overline{a_0^{-1}c^{-1}a_0^{-1}c^{-1}} \\
 &= a_0\overline{a_0} + c^{-1}\overline{c^{-1}}(c + \overline{c} + \overline{a_0^{-1}a_0^{-1}}) \\
 &= a_0\overline{a_0}
 \end{aligned}$$

So we can check that a runs over $a\overline{a} + b + \overline{b} = 0$ and $a \neq a_0$. ■

3 Calculations of Modules

We denote k_H the trivial character of H and k_H^B induced character of k_H to B . Since the restriction of the irreducible character φ_S to the Borel subgroup B is irreducible as an ordinary character, this irreducible character $\varphi = \varphi_{S_B}$ is also irreducible as a Brauer character.

Let $\mathcal{B}$ be a block which contains φ . Let $\tilde{S}$ be a simple FB -module which is corresponding to the character φ . This $\tilde{S}$ is only simple module which belongs in block $\mathcal{B}$. The defect group $\delta(\mathcal{B})$ of block $\mathcal{B}$ is a cyclic group with its order r^a . So any indecomposable modules in block $\mathcal{B}$ is uniserial. Moreover the projective cover $P(\tilde{S})$ is uniserial of Loewy length r^a . Let we denote $\{\theta_{q^2-q}^{(0)}, \theta_{q^2-q}^{(s)}, \dots, \theta_{q^2-q}^{(s(r^a-1))}\}$ r^a ordinary irreducible characters in block $\mathcal{B}$.

Lemma 3.1 *For any non-projective indecomposable FB_0 -module M , M^B has only one non-projective indecomposable summand.*

Proof : This is Green correspondence of M . So this lemma is followed by lemma 2.4. ■

For FB -module M , $\mathcal{B}$ -part of M means direct summands of M which belong in $\mathcal{B}$.

Lemma 3.2 i) $I_H^B = I_B \oplus Y \oplus Z$ where

$$Y = \mathcal{B}\text{-part of } I_H^B = \begin{pmatrix} \tilde{S} \\ \tilde{S} \\ \vdots \\ \tilde{S} \end{pmatrix}$$

is uniserial with Loewy length $r^a - 1$ and Z is projective.

ii) Let Y_i be a submodule of Y with Loewy length i , then $\dim \text{Inv}_H(Y_i) = i$

iii) $\dim \text{Inv}_H(Z) = q + 2 - r^a$

Proof : From calculations of characters,

- $k_H^{B_0} = k_{B_0} + \theta_0$ (θ_0 is an irreducible Brauer character.)
- $k_{B_0}^B = k_B + \theta_{q^2-1}$ (θ_{q^2-1} is an irreducible projective character.)
- $\theta_0^B = \sum_{u=1}^q \theta_{q^2-q}^{(u)}$

Since $\theta_{q^2-q}^{(0)}$ isn't in θ_0^B , (i) is followed by lemma 3.1.

The restriction of short exact sequence

$$0 \rightarrow \tilde{S} \rightarrow P(\tilde{S}) \rightarrow Y \rightarrow 0$$

to B_0 shows that the Green correspondence of $\tilde{S}$ is uniserial FB_0 -module with Loewy length $r^a - 1$ and all composition factors are isomorphic to simple module $\tilde{S}_0$ corresponding to θ_0 . Let B_0 be a block which contains θ_0 .

Lemma 3.1 shows that the $\mathcal{B}_0$ -part of the restriction of Y_i to B_0 is a direct sum of the uniserial module Y'_i with Loewy length $r^a - i$ and $i - 1$ projective indecomposable modules which are isomorphic to the projective cover $P(\tilde{S}_0)$.

From $k_H^{B_0} = k_{B_0} + \theta_0$,

$$\begin{aligned} \dim \text{Inv}_H(Y_i) &= \dim \text{Hom}_H(I_H, Y_{iH}) \\ &= \dim \text{Hom}_{B_0}(I_H^{B_0}, Y_{iB_0}) \\ &= \dim \text{Hom}_{B_0}(\tilde{S}_0, Y_{iB_0}) \\ &= \dim \text{Hom}_{B_0}(\tilde{S}_0, Y'_i) + (i - 1) \dim \text{Hom}_{B_0}(\tilde{S}_0, P(\tilde{S}_0)) \\ &= 1 + (i - 1) = i \end{aligned}$$

So (ii) is proved. Finally,

$$\begin{aligned} \dim \text{Inv}_H(I_H^B) &= \dim \text{Hom}_H(I_H, I_H^B) \\ &= \dim \text{Hom}_{B_0}(I_H^B, I_H^B) \\ &= q + 2 \end{aligned}$$

So from (i),

$$\begin{aligned}\dim \operatorname{Inv}_H(Z) &= \dim \operatorname{Inv}_H(I_H^B) - \dim \operatorname{Inv}_H(Y) - 1 \\ &= (q+2) - (r^a - 1) - 1 = q+2 - r^a\end{aligned}$$

■

4 The Number α

Theorem 4.1 *If the center of $SU(3, q^2)$ is trivial, then α in Theorem 0.1 is 2.*

Proof : From theorem 0.1, composition factors of I_B^G is $2 \times I_G + \alpha \times S + T$. Remember the homomorphism $f : I_L^G \rightarrow I_B^G$ in section 5 of [1], the composition factors of $\operatorname{Im}(f)$ is $I_G + \alpha/2 \times S + T$. The correspondence between notations of [1] and one of this paper about $I_L^G = \widehat{L}FG$, $I_B^G = \widehat{B}FG$ are the following. $v_\infty \leftrightarrow \widehat{B}$, $v_{0,0} \leftrightarrow \widehat{B}t$, $\delta(v_\infty, v_{0,0}) \leftrightarrow \widehat{L}$, and a set $\delta(v_\infty, v_{0,0})$ has elements $\{\langle v_\infty \rangle, \langle v_\infty tu \rangle | u \in U_0\}$. This correspondence shows that

$$\begin{aligned}f(\widehat{L}) &\leftrightarrow f(\delta(v_\infty, v_{0,0})) \\ &= v_\infty + \sum_{u \in U_0} v_\infty tu \\ &\leftrightarrow \widehat{B} + \sum_{u \in U_0} \widehat{B}tu \\ &= \widehat{B} + \widehat{B}t\widehat{U}_0 \\ &= \widehat{B}\widehat{L}\end{aligned}$$

Thus, $\operatorname{Im}(f) = f(I_L^G) = f(\widehat{L})FG = \widehat{B}\widehat{L}FG$. From lemma 3.2 i), $\widehat{B}FG_B = I_B^G_B = I_B \oplus I_H^B = I_B \oplus I_B \oplus Y \oplus Z$. Since the composition factors of $(\widehat{B}FG/\operatorname{Im}(f))_B$ are I_B and $\alpha/2 \tilde{S}$,

$$\operatorname{Im}(f)_B = I_B \oplus Y_{r^a-1-\alpha/2} \oplus Z.$$

From lemma 2.6, both $\widehat{B}\widehat{L}$ and $\sum_{h \in H_0 \setminus H} \gamma(b)h$ are in $\operatorname{Im}(f) = \widehat{B}\widehat{L}FG$. Since the action of H on these linearly independent elements is trivial, $\dim \operatorname{Inv}_H(\operatorname{Im}(f)) \geq q+1$. So from lemma 3.2 ii) and iii),

$$1 + r^a - 1 - \alpha/2 + q + 2 - r^a \geq q + 1.$$

We can get $\alpha = 2$ from $\alpha \geq 2$ in theorem 0.1. ■

References

- [1] Meinolf Geck, Irreducible Brauer characters of the 3-dimensional special unitary groups in non-defining characteristic, *Communications in Algebra* 18(2), 563–584, 1990
- [2] Tetsuro Okuyama, Katsushi Waki, Decomposition Numbers of $Sp(4, q)$ *J. Alg.*, 1997