

Infinite Pre-dominant Integral Weights for Affine Types

By

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Abstract

We study infinite pre-dominant integral weights for affine Kac-Moody Lie algebras. Furthermore, we determine the W -orbit of an infinite pre-dominant integral weight.

§ 1. Introduction

The purpose of the present paper is to study *infinite pre-dominant* integral weights for affine Kac-Moody Lie algebras. The notion of *pre-dominant* integral weights is introduced in [4]. Our motivation of this study is the following:

Let λ be a partition of d , and Y_λ the Young (or Ferrers) diagram of shape λ . In [6], R. P. Stanley studied the generating function $U(Y_\lambda; q)$ for the reverse plane partitions of shape Y_λ , and proved the q -hook formula:

$$(1.1) \quad U(Y_\lambda; q) = \prod_{v \in Y_\lambda} \frac{1}{1 - q^{h_v}},$$

where h_v denotes the hook-length at $v \in Y$. In a forthcoming paper [5], we have succeeded in proving a generalization of (1.1) by using a root system for a Kac-Moody Lie algebra. Let λ be a *pre-dominant* integral weight, and $D(\lambda)^\vee$ be the *shape* of λ (a certain subset of positive real coroots). These are defined in section 3. A map $\sigma : D(\lambda)^\vee \rightarrow \mathbb{Z}_{\geq 0}$ is called a $D(\lambda)^\vee$ -partition [5] if

1. If $\beta^\vee \triangleleft \gamma^\vee$ ($\beta^\vee, \gamma^\vee \in D(\lambda)^\vee$), then $\sigma(\beta^\vee) \geq \sigma(\gamma^\vee)$,

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2. There exists at most finitely many $\beta^\vee \in D(\lambda)^\vee$ such that $\sigma(\beta^\vee) \neq 0$,

where $\beta^\vee \triangleleft \gamma^\vee$ means $\beta^\vee < \gamma^\vee$ and $\langle \gamma, \beta^\vee \rangle \geq 1$. The generating function $U(D(\lambda)^\vee; q)$ for $D(\lambda)^\vee$ -partitions is defined by

$$U(D(\lambda)^\vee; q) := \sum_{\sigma: D(\lambda)^\vee\text{-partition}} q^{\sum_{\beta^\vee \in D(\lambda)^\vee} \sigma(\beta^\vee)}.$$

In [5], the following statement is shown:

Theorem 1.1 (*q-hook formula* [5]). *Let λ be a pre-dominant integral weight satisfying $\#D(\lambda)^\vee < \infty$. Then we have:*

$$(1.2) \quad U(D(\lambda)^\vee; q) = \prod_{\beta^\vee \in D(\lambda)^\vee} \frac{1}{1 - q^{\text{ht}(\beta)}},$$

where $\text{ht}(\beta)$ denotes the height of β .

We stress that *almost all* part of the proof [5] of the q -hook formula (1.2) is true even if $\#D(\lambda)^\vee = \infty$. (The condition “ $\#D(\lambda)^\vee < \infty$ ” is necessary only for certain inductive argument.) The author thinks that, by a suitable modification of formulation of (1.2), the q -hook formula should hold also for *infinite* pre-dominant integral weights.

In the present paper, we study several properties of *infinite* pre-dominant integral weights for affine Kac-Moody Lie algebras. Let $P_{\geq -1}^{\text{inf}}(I)$ denote the set of infinite pre-dominant integral weights. Here, I denotes the index set of simple roots as in TABLE 2. Then, as is shown in Proposition 5.8, we have:

$$(1.3) \quad P_0(I) \sqcup P_{\geq -1}^{\text{inf}}(I) = P_{\text{sig}}(I),$$

where the sets $P_0(I)$ and $P_{\text{sig}}(I)$ are defined in Section 3. An element of $P_{\text{sig}}(I)$ is called a *signature* integral weight. The equation (1.3) shows that classification of infinite pre-dominant integral weights reduces to that of signature integral weights. It is easy to see the set $P_{\text{sig}}(I)$ is closed under the Weyl group action.

In Section 5, we study the set $P_{\text{sig}}(I)$ for a Cartan matrix of affine type and, our main result (Theorem 5.15) gives the classification of signature integral weights in the form of $W(I)$ -orbit decomposition of $P_{\text{sig}}(I)$:

$$P_{\text{sig}}(I) = \bigsqcup_{i \in (I)_1} W(I) \cdot P(i, *, I),$$

where $W(I)$ denotes the Weyl group, $*$ is a distinguished index in I , $(I)_1$ is a certain subset of I , $P(i, *, I)$ is a set of integral weights with a certain condition. See Section 5 for unexplained notion and further details.

In Section 4, we study the set $P_{\text{sig}}(I)$ for a Cartan matrix of finite type. In Proposition 4.6, we determine the $W(I)$ -orbit decomposition of $P_{\text{sig}}(I)$:

$$P_{\text{sig}}(I) = P_0(I) \sqcup \bigsqcup_{i \in (I)_0} W(I) \cdot P(i, I).$$

This result is used in the proof of the main result. See Section 4, for the definition of $(I)_0$ and $P(i, I)$.

§ 2. Preliminaries

Let $A = (a_{i,j})_{i,j \in I}$ be a Cartan matrix of a Levi subalgebra $\mathfrak{g}_I$ of a Kac-Moody Lie algebra [2][3] defined over $\mathbb{R}$. We denote the Cartan subalgebra by $\mathfrak{h}$. Then, we have two sets of linearly independent elements $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

The set of integral weights is denoted by $P(I)$. An integral weight $\lambda \in P(I)$ is said to be *dominant* if

$$\langle \lambda, \alpha_i^\vee \rangle \geq 0, \quad i \in I.$$

The set of dominant integral weights is denoted by $P_{\geq 0}(I)$. For each $i \in I$, we define $s_i \in GL(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The group $W = W(I)$ generated by $\{s_i \mid i \in I\}$ is called the *Weyl group*, which acts on $\mathfrak{h}$ by:

$$\langle w(\lambda), w(h) \rangle = \langle \lambda, h \rangle, \quad w \in W, \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

We define the *real root system* (resp. *real coroot system*) by $\Phi := W\Pi$ (resp. $\Phi^\vee := W\Pi^\vee$) (or, expressing the index set explicitly, $\Phi(I)$ (resp. $\Phi^\vee(I)$)). We denote by Φ_+ and Φ_- the sets of positive and negative roots of Φ , respectively. The *dual* $\beta^\vee \in \Phi^\vee$ of a root $\beta \in \Phi$ is defined by the property

$$w(\beta^\vee) = w(\beta)^\vee, \quad w \in W.$$

For a subset S of Φ , we define a subset $S^\vee$ of $\Phi^\vee$ by:

$$S^\vee := \{\beta^\vee \in \Phi^\vee \mid \beta \in S\}.$$

For each $\beta \in \Phi$, we define $s_\beta \in W$ by:

$$s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*.$$

Then, s_β acts on $\mathfrak{h}$ by:

$$s_\beta(h) = h - \langle \beta, h \rangle \beta^\vee, \quad h \in \mathfrak{h}.$$

Let $\alpha_i, \alpha_j \in \Pi$.

If $\langle \alpha_i, \alpha_j^\vee \rangle = -1$, and $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, then we denote $\overset{\circ}{i} \text{---} \overset{\circ}{j}$

If $\langle \alpha_i, \alpha_j^\vee \rangle = -2$, and $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, then we denote $\overset{\circ}{i} \text{---} \text{---} \overset{\circ}{j}$

If $\langle \alpha_i, \alpha_j^\vee \rangle = -3$, and $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, then we denote $\overset{\circ}{i} \text{---} \text{---} \text{---} \overset{\circ}{j}$

§ 3. Pre-dominant Integral Weights and Signature Integral Weights

Definition 3.1. An integral weight λ is said to be *pre-dominant* if:

$$\langle \lambda, \beta^\vee \rangle \geq -1, \text{ for each } \beta^\vee \in \Phi_+^\vee.$$

The set of pre-dominant integral weights is denoted by $P_{\geq -1}(I)$.

Definition 3.2. For $\lambda \in P_{\geq -1}$, the set $D(\lambda)^\vee$ defined by

$$D(\lambda)^\vee := \left\{ \beta^\vee \in \Phi_+^\vee \mid \langle \lambda, \beta^\vee \rangle = -1 \right\}$$

is called the *shape* of λ . We say that a pre-dominant integral weight λ is *finite* if $\#D(\lambda)^\vee < \infty$, *infinite* otherwise. The set of finite (resp. infinite) pre-dominant integral weights is denoted by $P_{\geq -1}^{\text{fin}}(I)$ (resp. $P_{\geq -1}^{\text{inf}}(I)$).

Lemma 3.3 (See [4]Lemma 4.1). *Let $\lambda \in P_{\geq -1}(I)$ and $\beta^\vee \in D(\lambda)^\vee$. Then we have $s_\beta(\lambda) \in P_{\geq -1}(I)$.*

Definition 3.4. An integral weight λ is said to be a *signature* integral weight if:

$$\langle \lambda, \gamma^\vee \rangle \in \{1, 0, -1\}, \text{ for each } \gamma \in \Phi.$$

The set of signature integral weights is denoted by $P_{\text{sig}}(I)$. We note $P_{\text{sig}}(I) \subseteq P_{\geq -1}(I)$.

The following is clear.

Lemma 3.5. *Let $\lambda \in P_{\text{sig}}(I)$ and $\beta \in \Phi(I)$. Then we have $s_\beta(\lambda) \in P_{\text{sig}}(I)$.*

Since the Weyl group $W(I)$ is generated by $\Pi = \{s_i \mid i \in I\}$, we get:

Corollary 3.6. *The set $P_{\text{sig}}(I)$ of signature integral weights is closed under the $W(I)$ -action.*

Definition 3.7. We define a set $P_0(I)$ by

$$P_0(I) := \left\{ \lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle = 0, \text{ for each } i \in I \right\}.$$

We note that $D(\lambda)^\vee = \emptyset$ for $\lambda \in P_0(I)$ so that we have $P_0(I) \subseteq P_{\geq -1}^{\text{fin}}(I)$ and $P_0(I) \cap P_{\geq -1}^{\text{inf}}(I) = \emptyset$.

Proposition 3.8. *The set $P_0(I)$ is closed under the $W(I)$ -action.*

Proof. Let $\lambda \in P_0(I)$, $w \in W(I)$, and $i \in I$. Then we have $\langle w(\lambda), \alpha_i^\vee \rangle = \langle \lambda, w^{-1}(\alpha_i)^\vee \rangle$. Since $w^{-1}(\alpha_i)^\vee$ is a linear combination of elements of $\Pi^\vee = \left\{ \alpha_j^\vee \mid j \in I \right\}$, we have $\langle \lambda, w^{-1}(\alpha_i)^\vee \rangle = 0$. Hence, $w(\lambda) \in P_0(I)$. $\square$

§ 4. Classification for Finite Types

Through this section, we suppose the Cartan matrix $A = (a_{i,j})_{i,j \in I}$ is of finite type. The index set I is given as in TABLE 1. The purpose of this section is to determine the $W(I)$ -orbit decomposition of $P_{\text{sig}}(I)$.

Definition 4.1. Let $\beta_{hs}^\vee \in \Phi^\vee$ denote the *highest coroot*. The integers $d_i^\vee$ ($i \in I$) are defined by $\beta_{hs}^\vee = \sum_{i \in I} d_i^\vee \alpha_i^\vee$.

Definition 4.2. Let $\beta_{hl}^\vee \in \Phi^\vee$ denote the *dual of the highest root*. The integers $c_i^\vee$ ($i \in I$) are defined by $\beta_{hl}^\vee = \sum_{i \in I} c_i^\vee \alpha_i^\vee$.

We note that if the Cartan matrix A is simply-laced, then we have $\beta_{hs}^\vee = \beta_{hl}^\vee$, and that if multiply-laced, then $\beta_{hs}^\vee > \beta_{hl}^\vee$.

When the Cartan matrix A is of type B_l, C_l, F_4 , or G_2 , the coroot system $\Phi^\vee$ is decomposed as:

$$\Phi^\vee = \Phi_\ell^\vee \sqcup \Phi_s^\vee, \quad (\text{disjoint union}),$$

where Φ_ℓ is the set of long roots and Φ_s is the set of short roots. We note that we have $\beta_{hl}^\vee \in \Phi_\ell^\vee$ and $\beta_{hs}^\vee \in \Phi_s^\vee$, and that $\Phi_\ell^\vee$ and $\Phi_s^\vee$ are the dual of Φ_ℓ and Φ_s , respectively.

Proposition 4.3. *Let $\lambda \in P(I)$. Then there exists a unique element $\Lambda \in W(I) \cdot \lambda$ such that $\Lambda \in P_{\geq 0}(I)$. Furthermore, if $\lambda \in P_{\text{sig}}(I)$, then such a $\Lambda \in W(I) \cdot \lambda$ is also a signature integral weight.*

Proof. Note that A is of finite type in this section. The first part of the proposition is a well-known fact. The second part follows from Corollary 3.6. $\square$

Definition 4.4. We suppose that a Cartan matrix $A = (a_{i,j})_{i,j \in I}$ is of finite type. We define a subset $(I)_0$ of I by:

$$(I)_0 := \left\{ i \in I \mid d_i^\vee = 1 \right\}.$$

The highest coroot and the set $(I)_0$ for each finite type are listed in TABLE 1.

Definition 4.5. Let $i \in I$. We define a subset $P(i; I)$ of $P(I)$ by:

$$P(i; I) := \left\{ \lambda \in P(I) \mid \langle \lambda, \alpha_j^\vee \rangle = \delta_{i,j} \ (j \in I) \right\},$$

where $\delta_{i,j}$ denotes the Kronecker's delta. An element of $P(i; I)$ is called an *i-th fundamental weight (over I)*.

Remark. If $\dim \mathfrak{h} = \#I$, then the set $P(i; I)$ contains a unique element. In Section 5, we deal with a Cartan matrix $A = (a_{i,j})_{i,j \in I}$ of affine type, and we restrict the index set I to a subset J , which is of finite type. In such a situation, we have $\dim \mathfrak{h} > \#J$. Hence, we have $\#P(i; J) \neq 1$.

Remark. When a Cartan matrix $A = (a_{i,j})_{i,j \in I}$ is of finite type, a dominant integral weight Λ is said to be *minuscule* if $\langle \Lambda, \beta^\vee \rangle \leq 1$ for all $\beta \in \Phi_+$ and there exists $i \in I$ such that $\langle \Lambda, \alpha_j^\vee \rangle = \delta_{i,j}$ for $j \in I$. Equivalently, Λ is minuscule if

$$\text{there exists an index } i \in (I)_0 \text{ such that } \Lambda \in P(i; I).$$

Let Λ be a minuscule weight. By Corollary 3.6, if $\lambda \in W(I) \cdot \Lambda$, then $\lambda \in P_{\text{sig}}(I)$.

The following proposition shows the converse of the previous remark.

Proposition 4.6. *We have:*

$$P_{\text{sig}}(I) = P_0(I) \sqcup \bigsqcup_{i \in (I)_0} W(I) \cdot P(i; I) \quad (\text{disjoint union}),$$

Proof. Let $\lambda \in P_{\text{sig}}(I)$. By Proposition 4.3, we may assume $\lambda \in P_{\text{sig}}(I)$ is dominant. Since $\langle \lambda, \beta_{hs}^\vee \rangle \in \{0, 1\}$, we have $\sum_{i \in I} d_i^\vee \langle \lambda, \alpha_i^\vee \rangle \in \{0, 1\}$. Since $\langle \lambda, \alpha_i^\vee \rangle \in \{0, 1\}$, we have either:

1. $\langle \lambda, \alpha_i^\vee \rangle = 0 \ (i \in I)$, or
2. there exists a unique index $i \in (I)_0$ such that $\langle \lambda, \alpha_j^\vee \rangle = \delta_{i,j} \ (j \in I)$.

Hence, we have:

$$(4.1) \quad P_{\text{sig}}(I) \subseteq P_0(I) \sqcup \bigsqcup_{i \in (I)_0} W(I) \cdot P(i; I).$$

The opposite inclusion is by the previous remark. □

$A_{l-1} \quad (l \geq 2)$	$B_l \quad (l \geq 2)$
$\alpha_1^\vee + \cdots + \alpha_{l-1}^\vee$	$\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_l^\vee \quad 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_l^\vee$ $\Phi_\ell^\vee := W\{\alpha_1^\vee, \dots, \alpha_{l-1}^\vee\}, \quad \Phi_s^\vee := W\{\alpha_l^\vee\}$
$I = \{1, \dots, l-1\}$	$\{l\}$
${}_l C_i \quad (1 \leq i \leq l-1)$	2^l
$D_{l+2} \quad (l \geq 2)$	$C_l \quad (l \geq 2)$
$\alpha_0^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_{l+1}^\vee + \alpha_{l+2}^\vee$	$\alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_{l-2}^\vee + \alpha_{l-1}^\vee \quad 2\alpha_0^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-2}^\vee + \alpha_{l-1}^\vee$ $\Phi_\ell^\vee := W\{\alpha_0^\vee\}, \quad \Phi_s^\vee := W\{\alpha_1^\vee, \dots, \alpha_{l-1}^\vee\}$
$\{0, +, -\}$	$\{l-1\}$
$2(l+2), \quad 2^{l+1}, \quad 2^{l+1}$	$2 \cdot l$
E_6	F_4
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_{2'}^\vee + \alpha_{1'}^\vee + 2\alpha_{2''}^\vee$	$2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee \quad 2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$ $\Phi_\ell^\vee := W\{\alpha_1^\vee, \alpha_2^\vee\}, \quad \Phi_s^\vee := W\{\alpha_3^\vee, \alpha_4^\vee\}$
$\{1, 1'\}$	$\emptyset$
$27, 27$	$-$
E_7	G_2
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_{3'}^\vee + 2\alpha_{2'}^\vee + 2\alpha_{2''}^\vee$	$2\alpha_1^\vee + \alpha_2^\vee \quad 3\alpha_1^\vee + 2\alpha_2^\vee$ $\Phi_\ell^\vee := W\{\alpha_1^\vee\}, \quad \Phi_s^\vee := W\{\alpha_2^\vee\}$
$\{1\}$	$\emptyset$
56	$-$
E_8	The type
	Dynkin diagram with indexes of vertices
$2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 5\alpha_5^\vee + 6\alpha_6^\vee + 4\alpha_{4'}^\vee + 2\alpha_{2'}^\vee + 3\alpha_{3''}^\vee$	the dual of the highest root definition of $\Phi_\ell^\vee$ the highest coroot definition of $\Phi_s^\vee$
$\emptyset$	$(I)_0$
$-$	$\#(W \cdot \lambda) \quad (\lambda \in (i; I), i \in (I)_0)$

Table 1. Finite Types

Remark (How to read TABLE 1). In TABLE 1, we list the following information from the top:

- the type of Cartan matrix A ,
- the Dynkin and our indexing of the nodes,
- the dual $\beta_{hl}^\vee$ of the highest root and the highest coroot $\beta_{hs}^\vee$,
- the set $(I)_0$,
- the cardinality $\#(W(I) \cdot \lambda)$ of $W(I)$ -orbit for $\lambda \in P(i; I)$ ($i \in (I)_0$),

as indicated at the end of the table. The cardinality of $W(I)$ -orbit for a minuscule λ is equal to the dimension of the irreducible highest weight module $V(\lambda)$ so that the Weyl dimension formula gives the cardinality.

§ 5. Classification for Affine Types

In this section, we suppose a Cartan matrix $A = (a_{i,j})_{i,j \in I}$ is of affine type. We choose the index set I as in TABLE 2. In particular, the distinguished index $*$ is specified as in the table. The purpose of this section is to determine the $W(I)$ -orbit decomposition of $P_{\text{sig}}(I)$.

§ 5.1. Coroots

Definition 5.1. Let K denote the *null coroot* (the *canonical central element*) defined by

$$\langle \alpha_i, K \rangle = 0, \quad i \in I.$$

The integers $a_i^\vee$ ($i \in I$) are defined by $K = \sum_{i \in I} a_i^\vee \alpha_i^\vee$.

Remark. Put $J := I \setminus \{*\}$. Note that we have $a_*^\vee = 1$. Through this section, we always fix this subset J of I . Then we note that the Cartan matrix $A(J) := (a_{i,j})_{i,j \in J}$ is of finite type.

If the Cartan matrix A is of type $A_{l-1}^{(1)}$ (resp. $D_{l+2}^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $D_{l+1}^{(2)}$, $A_{2l-1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$), then the Cartan matrix $A(J) := (a_{i,j})_{i,j \in J}$ is of type A_{l-1} (resp. D_{l+2} , E_6 , E_7 , E_8 , B_l , C_l , F_4 , G_2). We note that we have $a_i^\vee = d_i^\vee$ for each $i \in J$.

If the Cartan matrix A is of type $B_l^{(1)}$, $C_l^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$, then the Cartan matrix $A(J) := (a_{i,j})_{i,j \in J}$ is of type B_l , C_l , F_4 , G_2 . We note that we have $a_i^\vee = c_i^\vee$ for each $i \in J$.

Observe that $1 \leq c_i^\vee \leq d_i^\vee$ for $i \in I$. Thus, if $d_i^\vee = 1$, then $a_i^\vee = 1$ for all types except $A_{2l}^{(2)}$.

If the Cartan matrix A is of type $A_{2l}^{(2)}$, then the Cartan matrix $A(J) := (a_{i,j})_{i,j \in J}$ is of type B_l . In this case, we have $a_i^\vee = 2$ for each $i \in J$.

Lemma 5.2 (See[2]). *The set $\Phi_+^\vee(I)$ is decomposed as follows:*

1. *If I is of type $A_{l-1}^{(1)}, D_{l+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}, D_4^{(3)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n \geq 1} (\Phi^\vee(J) + nK), \quad (\text{disjoint union}).$$

2. *If I is of type $B_l^{(1)}, C_l^{(1)}, F_4^{(1)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n \geq 1} (\Phi_\ell^\vee(J) + 2nK) \sqcup \bigsqcup_{m \geq 1} (\Phi_s^\vee(J) + mK).$$

3. *If I is of type $G_2^{(1)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n \geq 1} (\Phi_\ell^\vee(J) + 3nK) \sqcup \bigsqcup_{m \geq 1} (\Phi_s^\vee(J) + mK).$$

4. *If I is of type $A_{2l}^{(2)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n \geq 1} \frac{1}{2} (\Phi_\ell^\vee(J) + (2n-1)K) \sqcup \bigsqcup_{n \geq 1} (\Phi_s^\vee(J) + nK) \sqcup \bigsqcup_{n \geq 1} (\Phi_\ell^\vee(J) + 2nK).$$

Lemma 5.3. *We have:*

1. *If A is not of type $A_{2l}^{(2)}$ ($l \geq 1$), then:*

$$\Phi_+^\vee(I) \subseteq \bigsqcup_{n \geq 0} (\Phi^\vee(J) + nK).$$

2. *If A is of type $A_{2l}^{(2)}$ ($l \geq 1$), then:*

$$\Phi_+^\vee(I) \subseteq \bigsqcup_{n \geq 0} \frac{1}{2} (\Phi^\vee(J) + (2n-1)K) \sqcup \bigsqcup_{n \geq 0} (\Phi^\vee(J) + nK).$$

Proof. This follows from Lemma 5.2. □

Definition 5.4. We define an integer $r^\vee$ by:

$$r^\vee := \begin{cases} 1 & \text{if } I \text{ is of type } A_{l-1}^{(1)}, D_{l+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}, D_4^{(3)} \\ 2 & \text{if } I \text{ is of type } B_l^{(1)}, C_l^{(1)}, F_4^{(1)}, A_{2l}^{(2)} \\ 3 & \text{if } I \text{ is of type } G_2^{(1)} \end{cases}$$

Proposition 5.5. *Let $\beta^\vee \in \Phi_+^\vee(I)$ and $n \geq 0$. Then we have $\beta^\vee + r^\vee nK \in \Phi_+^\vee(I)$.*

Proof. This follows from Lemma 5.2. $\square$

Proposition 5.6. *Let $\lambda \in P_{\geq -1}$. Then we have:*

1. $\langle \lambda, K \rangle \geq 0$.
2. If $\langle \lambda, K \rangle \geq 1$, then we have $\lambda \in P_{\geq -1}^{\text{fin}}(I) \setminus P_0(I)$.
3. If $\langle \lambda, K \rangle = 0$, then we have $\lambda \in P_{\geq -1}^{\text{inf}}(I) \sqcup P_0(I)$.

Proof. (1) Fix an arbitrary $\beta^\vee \in \Phi_+^\vee(I)$. By Proposition 5.5, for each $n \geq 0$, we have:

$$\beta^\vee + r^\vee nK \in \Phi_+^\vee(I).$$

If $\langle \lambda, K \rangle \leq -1$, then, for a sufficiently large n , we have $\langle \lambda, \beta^\vee + r^\vee nK \rangle \leq -2$. This contradicts $\lambda \in P_{\geq -1}$. This proves Part (1).

(2) If $\lambda \in P_0(I)$, then we have $\langle \lambda, K \rangle = \sum_{i \in I} a_i^\vee \langle \lambda, \alpha_i^\vee \rangle = 0$. Hence, we have $\lambda \notin P_0(I)$. If the Cartan matrix A is not of type $A_{2l}^{(2)}$ ($l \geq 1$), then, by Lemma 5.3 (1), we have:

$$D(\lambda)^\vee \subseteq \bigsqcup_{n \geq 0} (D(\lambda)^\vee \cap (\Phi^\vee(J) + nK)),$$

Since $\langle \lambda, K \rangle \geq 1$, we have $D(\lambda)^\vee \cap (\Phi^\vee(J) + nK) = \emptyset$ for $n \gg 0$. Since the set $\Phi^\vee(J)$ is finite, $D(\lambda)^\vee$ is finite.

If the Cartan matrix A is of type $A_{2l}^{(2)}$ ($l \geq 1$), then, by Lemma 5.3 (2), we have:

$$D(\lambda)^\vee \subseteq \bigsqcup_{n \geq 0} \left(D(\lambda)^\vee \cap \frac{1}{2} (\Phi^\vee(J) + (2n-1)K) \right) \sqcup \bigsqcup_{n \geq 0} (D(\lambda)^\vee \cap (\Phi^\vee(J) + nK)).$$

Since $\langle \lambda, K \rangle \geq 1$, we have $D(\lambda)^\vee \cap \frac{1}{2} (\Phi^\vee(J) + (2n-1)K) = \emptyset$ and $D(\lambda)^\vee \cap (\Phi^\vee(J) + nK) = \emptyset$ for $n \gg 0$. Since the set $\Phi^\vee(J)$ is finite, $D(\lambda)^\vee$ is finite.

(3) Suppose $\lambda \notin P_0(I)$. Then, there exists an index $i \in I$ such that $\langle \lambda, \alpha_i^\vee \rangle \neq 0$. Since $\langle \lambda, K \rangle = 0$, there exists an index $i_0 \in I$ such that $\langle \lambda, \alpha_{i_0}^\vee \rangle < 0$. Since $\langle \lambda, \alpha_{i_0}^\vee \rangle \geq -1$, we have $\langle \lambda, \alpha_{i_0}^\vee \rangle = -1$. Hence, we have $\alpha_{i_0}^\vee \in D(\lambda)^\vee$. By Proposition 5.5, for each $n \geq 0$, we have $\alpha_{i_0}^\vee + r^\vee nK \in \Phi_+^\vee(I)$. Since

$$\langle \lambda, \alpha_{i_0}^\vee + r^\vee nK \rangle = \langle \lambda, \alpha_{i_0}^\vee \rangle + r^\vee n \langle \lambda, K \rangle = -1 + r^\vee n \cdot 0 = -1,$$

we have $\alpha_{i_0}^\vee + r^\vee nK \in D(\lambda)^\vee$, for each $n \geq 0$. Hence, we have $\#D(\lambda)^\vee = \infty$. $\square$

Proposition 5.7. *Let $\lambda \in P_{\geq -1}^{\text{inf}}(I)$. Then we have $\lambda \in P_{\text{sig}}(I)$.*

Proof. Let $\lambda \in P_{\geq -1}$ and $\beta \in \Phi$. By Proposition 5.6, we have $\langle \lambda, K \rangle = 0$. First, suppose $\beta \in \Phi_+$. Then we have $\langle \lambda, \beta^\vee \rangle \geq -1$. Since there exists a positive integer m such that $\beta^\vee < r^\vee mK$, we have $r^\vee mK - \beta^\vee \in \Phi_+^\vee$. Since $\lambda \in P_{\geq -1}$, we have:

$$-1 \leq \langle \lambda, r^\vee mK - \beta^\vee \rangle = r^\vee m \langle \lambda, K \rangle - \langle \lambda, \beta^\vee \rangle = -\langle \lambda, \beta^\vee \rangle.$$

Hence, we have $\langle \lambda, \beta^\vee \rangle \leq 1$. This proves $\langle \lambda, \beta^\vee \rangle \in \{1, 0, -1\}$ for $\beta \in \Phi_+$.

Suppose, on the other hand, $\beta \in \Phi_-$, then we have $-\beta \in \Phi_+$. By the above argument, we have $\langle \lambda, (-\beta)^\vee \rangle \in \{1, 0, -1\}$. This proves $\langle \lambda, \beta^\vee \rangle \in \{1, 0, -1\}$ for $\beta \in \Phi_-$.

Thus, we always have $\langle \lambda, \beta^\vee \rangle \in \{1, 0, -1\}$, for $\beta \in \Phi$. $\square$

Proposition 5.8. *We have $P_0(I) \sqcup P_{\geq -1}^{\text{inf}}(I) = P_{\text{sig}}(I)$.*

Proof. It is trivial that $P_0(I) \subseteq P_{\text{sig}}(I)$. By Proposition 5.7, we have $P_{\geq -1}^{\text{inf}}(I) \subseteq P_{\text{sig}}(I)$. Hence, we have $P_0(I) \sqcup P_{\geq -1}^{\text{inf}}(I) \subseteq P_{\text{sig}}(I)$. Now, we prove the converse inclusion. Let $\lambda \in P_{\text{sig}}(I)$. Suppose $\langle \lambda, K \rangle \neq 0$. If $\beta^\vee \in \Phi_+^\vee$, then, by Proposition 5.5, we have $\beta^\vee + r^\vee nK \in \Phi^\vee$, for each $n \geq 0$. Hence, for a sufficient large n , we have $\langle \lambda, \beta^\vee + r^\vee nK \rangle \notin \{1, 0, -1\}$. This contradicts $\lambda \in P_{\text{sig}}(I)$. Hence, we have $\langle \lambda, K \rangle = 0$. By Proposition 5.6 (3), we have $P_0(I) \sqcup P_{\geq -1}^{\text{inf}}(I) \supseteq P_{\text{sig}}(I)$. $\square$

§ 5.2. An invariant of W -Orbit

Definition 5.9. Let Γ be an abelian group. A map $\gamma : I \rightarrow \Gamma$ is said to *satisfy Condition (A)* if:

$$\prod_{j \in I} \gamma(j)^{\langle \alpha_i, \alpha_j^\vee \rangle} = 1, \text{ for each } i \in I.$$

Let $\lambda \in P(I)$. We define an element $a(\lambda) \in \Gamma$ as:

$$a(\lambda) := \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle}.$$

Proposition 5.10. *Suppose that $\gamma : I \rightarrow \Gamma$ satisfies Condition (A). Then, for $\lambda \in P(I)$ and $w \in W(I)$, we have $a(\lambda) = a(w(\lambda))$.*

Proof. We may assume that w is a simple reflection s_i . Then we have:

$$\begin{aligned} a(s_i(\lambda)) &= \prod_{j \in I} \gamma(j)^{\langle s_i(\lambda), \alpha_j^\vee \rangle} = \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle - \langle \lambda, \alpha_i^\vee \rangle \langle \alpha_i, \alpha_j^\vee \rangle} \\ &= \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle} \cdot \left(\prod_{j \in I} \gamma(j)^{\langle \alpha_i, \alpha_j^\vee \rangle} \right)^{-\langle \lambda, \alpha_i^\vee \rangle} = \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle} = a(\lambda). \end{aligned}$$

This proves the statement. $\square$

This proposition shows that the map $a : P(I) \ni \lambda \mapsto a(\lambda) \in \Gamma$ is a $W(I)$ -invariant function on $P(I)$.

Recall that $J = I \setminus \{*\}$ and $(J)_0 = \{i \in J \mid d_i^\vee = 1\}$. We explained that if $i \in (J)_0$ then $a_i^\vee = 1$. Note that if A is of type $A_{2l}^{(2)}$ then $(J)_0 = \emptyset$.

Definition 5.11. We define a subset $(I)_1$ of I by:

$$(I)_1 := \begin{cases} \{*\} \sqcup (J)_0 & \text{if } A \text{ is not of type } A_{2l}^{(2)}, \\ \{*\} & \text{if } A \text{ is of type } A_{2l}^{(2)}. \end{cases}$$

Definition 5.12. A map $\gamma : I \rightarrow \Gamma$ is said to *satisfy Condition (B)* if

$$\gamma(i) \neq \gamma(j), \quad \text{for } i, j (i \neq j) \in (I)_1.$$

We shall construct $\gamma_1 : I \rightarrow \Gamma_1$ which satisfies Condition (A) and (B).

5.2.1. Case of $A_{l-1}^{(1)}$ ($l \geq 2$) Suppose that the Cartan matrix A is of type $A_{l-1}^{(1)}$. We denote the cyclic group $C_l = \{1, g, \dots, g^{l-1}\}$ of degree l by Γ_1 . We define a map $\gamma_1 : I \rightarrow \Gamma_1$ by:

$$\begin{aligned} * &\mapsto 1, \\ i \ (1 \leq i \leq l-1) &\mapsto g^i. \end{aligned}$$

5.2.2. Case of $D_{l+2}^{(1)}$ ($l \geq 2, l : \text{even}$) Suppose that the Cartan matrix A is of type $D_{l+2}^{(1)}$ ($l : \text{even}$).

We denote the Klein four-group $K_4 = \{1, x, y, xy\}$ by Γ_1 . We define a map $\gamma_1 : I \rightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1, i : \text{odd}) &\mapsto 1, \\ + &\mapsto x, \\ - &\mapsto y, \\ 0, i \ (1 \leq i \leq l-1, i : \text{even}) &\mapsto xy. \end{aligned}$$

5.2.3. Case of $D_{l+2}^{(1)}$ ($l \geq 2, l : \text{odd}$) Suppose that the Cartan matrix A is of type $D_{l+2}^{(1)}$ ($l : \text{odd}$).

We denote the cyclic group $C_4 = \{1, g, g^2, g^3\}$ by Γ_1 . We define a map $\gamma_1 : I \rightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1, i : \text{odd}) &\mapsto 1, \\ + &\mapsto g, \\ 0, i \ (1 \leq i \leq l-1, i : \text{even}) &\mapsto g^2, \\ - &\mapsto g^3. \end{aligned}$$

5.2.4. Case of $E_6^{(1)}$ Suppose that the Cartan matrix A is of type $E_6^{(1)}$. We denote the cyclic group $C_3 = \{1, g, g^2\}$ by Γ_1 . We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, 2'', 3 &\longmapsto 1, \\ 1', 2 &\longmapsto g, \\ 1, 2' &\longmapsto g^2. \end{aligned}$$

5.2.5. Case of $E_7^{(1)}$ Suppose that the Cartan matrix A is of type $E_7^{(1)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1 . We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, 2', 3', 4, 2 &\longmapsto 1, \\ 1, 3, 2'' &\longmapsto g. \end{aligned}$$

5.2.6. Case of $B_l^{(1)}$ Suppose that the Cartan matrix A is of type $B_l^{(1)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1 . We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1) &\longmapsto 1, \\ l &\longmapsto g. \end{aligned}$$

5.2.7. Case of $D_{l+1}^{(2)}$ Suppose that the Cartan matrix A is of type $D_{l+1}^{(2)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1 . We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1) &\longmapsto 1, \\ l &\longmapsto g. \end{aligned}$$

5.2.8. Case of $C_l^{(1)}$ Suppose that the Cartan matrix A is of type $C_l^{(1)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1 . We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (0 \leq i \leq l-1, l-i : \text{even}) &\longmapsto 1, \\ i \ (0 \leq i \leq l-1, l-i : \text{odd}) &\longmapsto g. \end{aligned}$$

5.2.9. Case of $A_{2l-1}^{(2)}$ Suppose that the Cartan matrix A is of type $A_{2l-1}^{(2)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1 . We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (0 \leq i \leq l-1, i : \text{even}) &\longmapsto 1, \\ i \ (0 \leq i \leq l-1, i : \text{odd}) &\longmapsto g. \end{aligned}$$

In each case written above, it is straightforward to see that γ_1 satisfies Condition (A) and (B).

5.2.10. Other Cases : $E_8^{(1)}, F_4^{(1)}, E_6^{(2)}, G_2^{(1)}, D_4^{(3)}$, or $A_{2l}^{(2)}$ We do not need construct γ_1 because $(I)_1 = \{*\}$.

Definition 5.13. Let $i, j \in I$. We define a set $P(i, j; I)$ by:

$$P(i, j; I) := \left\{ \lambda \in P(I) \mid \langle \lambda, \alpha_k^\vee \rangle = \delta_{i,k} - \delta_{j,k}, \text{ for each } k \in I \right\}.$$

Note that if $i = j$ then $P(i, j; I) = P_0(I)$.

Proposition 5.14. Let γ_1 as above, and let $a : P(I) \longrightarrow \Gamma_1$ be the invariant function on $P(I)$ associated with γ_1 . Let $i \in (I)_1$. Then, for any $\lambda \in W(I) \cdot P(i, *; I)$, we have $a(\lambda) = \gamma_1(i)$.

Proof. By our assumption, there exists $w \in W(I)$ such that $w^{-1}(\lambda) \in P(i, *; I)$. Hence, by Proposition 5.10, we have $a(\lambda) = a(w^{-1}(\lambda))$. Then, since $w^{-1}(\lambda) \in P(i, *; I)$, we have $a(w^{-1}(\lambda)) = \gamma_1(i)\gamma_1(*)^{-1}$. Observe that $\gamma_1(*) = 1$ in all cases. Then we get $a(\lambda) = \gamma_1(i)$. $\square$

§ 5.3. Main Theorem

We now state the main result of this paper.

Theorem 5.15. We suppose that the Cartan matrix A is of affine type. Then we have

$$P_{\text{sig}}(I) = \bigsqcup_{i \in (I)_1} W(I) \cdot P(i, *; I).$$

Theorem 5.15 follows from Proposition 5.17 and Proposition 5.18 below.

First, we suppose that A is not of type $A_{2l}^{(2)}$.

Lemma 5.16. We have:

1. $P_0(I) = P_0(J) \cap P_{\text{sig}}(I)$.
2. $P(i, *; I) = P(i; J) \cap P_{\text{sig}}(I)$, for each $i \in (J)_0$.

Proof. (1) It is trivial that we have $P_0(I) \subseteq P_0(J) \cap P_{\text{sig}}(I)$. Now, we prove $P_0(I) \supseteq P_0(J) \cap P_{\text{sig}}(I)$. Let $\lambda \in P_0(J) \cap P_{\text{sig}}(I)$. Since $\lambda \in P_0(J)$, we have $\langle \lambda, \alpha_j^\vee \rangle = 0$ ($j \in J$). Since $\lambda \in P_{\text{sig}}(I)$, we have $\langle \lambda, K \rangle = 0$. (This follows from Proposition 5.6 and Proposition 5.8.) Since

$$0 = \langle \lambda, K \rangle = \sum_{i \in I} a_i^\vee \cdot \langle \lambda, \alpha_i^\vee \rangle = 1 \cdot \langle \lambda, \alpha_*^\vee \rangle,$$

we have $\langle \lambda, \alpha_*^\vee \rangle = 0$. Hence, we have $\lambda \in P_0(I)$.

(2) We show that the required equality holds when $a_i^\vee = 1$. First, we prove $P(i, *; I) \subseteq$

$P(i; J) \cap P_{\text{sig}}(I)$. Let $i \in (J)_0$ and $\lambda \in P(i, *; I)$. Since $i \in J$ and $* \notin J$, we have $\lambda \in P(i; J)$. Since

$$\langle \lambda, K \rangle = \sum_{j \in I} a_j^\vee \cdot \langle \lambda, \alpha_j^\vee \rangle = a_i^\vee \cdot 1 + a_*^\vee \cdot (-1) + \sum_{j \in I \setminus \{i, *\}} a_j^\vee \cdot 0 = 0,$$

we have $\langle \lambda, K \rangle = 0$. By Proposition 5.6 and Proposition 5.7, we have $\lambda \in P_{\text{sig}}(I)$.

Next, we prove $P(i, *; I) \supseteq P(i; J) \cap P_{\text{sig}}(I)$. Let $\lambda \in P(i; J) \cap P_{\text{sig}}(I)$. Since $\lambda \in P(i; J)$, we have $\langle \lambda, \alpha_i^\vee \rangle = 1$ and $\langle \lambda, \alpha_j^\vee \rangle = 0$ ($j \in J \setminus \{i\}$). Since $\lambda \in P_{\text{sig}}(I)$, we have $\langle \lambda, K \rangle = 0$. Since

$$0 = \langle \lambda, K \rangle = \sum_{j \in I} a_j^\vee \cdot \langle \lambda, \alpha_j^\vee \rangle = 1 \cdot \langle \lambda, \alpha_*^\vee \rangle + 1 \cdot 1,$$

we have $\langle \lambda, \alpha_*^\vee \rangle = -1$. Hence, we have $\lambda \in P(i, *; I)$. $\square$

Proposition 5.17. *We have*

$$P_{\text{sig}}(I) = \bigsqcup_{i \in (I)_1} W(I) \cdot P(i, *; I).$$

Proof. First, we prove $P_{\text{sig}}(I) \supseteq \bigcup_{i \in (I)_1} W(I) \cdot P(i, *; I)$. Let $\lambda \in W(I) \cdot P(i, *; I)$ for $i \in (I)_1$. If $i = *$, then we have $\lambda \in P_0(I) \subseteq P_{\text{sig}}(I)$. If $i \in (J)_0$, then, by Lemma 5.16 (2), we have $\lambda \in W(I) \cdot (P(i; J) \cap P_{\text{sig}}(I)) \subseteq W(I) \cdot P_{\text{sig}}(I) = P_{\text{sig}}(I)$. Thus, we always have $\lambda \in P_{\text{sig}}(I)$.

Next, we prove $P_{\text{sig}}(I) \subseteq \bigcup_{i \in (I)_1} W(I) \cdot P(i, *; I)$. Let $\lambda \in P_{\text{sig}}(I)$. Then we have $\lambda \in P_{\text{sig}}(J)$. Hence, by Proposition 4.6, we have $\lambda \in P_0(J) \sqcup \bigsqcup_{i \in (J)_0} W(J) \cdot P(i; J)$. Since $\lambda \in P_{\text{sig}}(I)$, we have $\lambda \in (P_0(J) \cap P_{\text{sig}}(I)) \sqcup \bigsqcup_{i \in (J)_0} W(J) \cdot (P(i; J) \cap P_{\text{sig}}(I))$. By Lemma 5.16 (1) and (2), we have $\lambda \in P_0(I) \sqcup \bigsqcup_{i \in (J)_0} W(J) \cdot P(i, *; I)$. Hence, we have $\lambda \in P_0(I) \cup \bigcup_{i \in (J)_0} W(I) \cdot P(i, *; I) = \bigcup_{i \in (I)_1} W(I) \cdot P(i, *; I)$.

Finally, we prove the disjointness of the decomposition. Let $i, j \in (I)_1$ ($i \neq j$). Let $\lambda \in W(I) \cdot P(i, *; I)$ and $\mu \in W(I) \cdot P(j, *; I)$. Then, by Proposition 5.14, we have $a(\lambda) = \gamma_1(i)$ and $a(\mu) = \gamma_1(j)$. Since γ_1 satisfies Condition (B), we have $\gamma_1(i) \neq \gamma_1(j)$. Hence, we have $(W(I) \cdot P(i, *; I)) \cap (W(I) \cdot P(j, *; I)) = \emptyset$. This proves the statement. $\square$

Next, we suppose that A is of type $A_{2l}^{(2)}$.

Proposition 5.18. *We have $P_{\text{sig}}(I) = P_0(I)$.*

Proof. We prove $P_{\text{sig}}(I) \subseteq P_0(I)$. The opposite inclusion is clear. Let $\lambda \in P_{\text{sig}}(I)$. Then the Cartan matrix $A(J) := (a_{i,j})_{i,j \in J}$ is of type B_l . We have $\lambda \in P_{\text{sig}}(J)$. By Proposition 4.3, we have $w(\lambda) \in P_{\text{sig}}(J) \cap P_{\geq 0}(J)$ for some $w \in W(J)$. Hence, by Proposition 4.6, we have either:

(a) $w(\lambda) \in P_0(J)$.

(b) $w(\lambda) \in P(l; J)$.

Suppose that the case (a) holds. Since $w(\lambda) \in P_{\text{sig}}(I)$, we have $\langle w(\lambda), K \rangle = 0$. Since

$$0 = \langle w(\lambda), K \rangle = a_*^\vee \cdot \langle w(\lambda), \alpha_*^\vee \rangle + \sum_{i \in J} a_i^\vee \cdot \langle w(\lambda), \alpha_i^\vee \rangle = \langle w(\lambda), \alpha_*^\vee \rangle,$$

we have $\langle w(\lambda), \alpha_*^\vee \rangle = 0$. Since $\langle w(\lambda), \alpha_i^\vee \rangle = 0$ for each $i \in I$, we have $w(\lambda) \in P_0(I)$. Hence, we have $\lambda \in P_0(I)$. Suppose, on the other hand, the case (b) holds. Since $w(\lambda) \in P_{\text{sig}}(I)$, we have $\langle w(\lambda), K \rangle = 0$. Since

$$\begin{aligned} 0 &= \langle w(\lambda), K \rangle = a_*^\vee \cdot \langle w(\lambda), \alpha_*^\vee \rangle + \sum_{i \in J} a_i^\vee \cdot \langle w(\lambda), \alpha_i^\vee \rangle = \langle w(\lambda), \alpha_*^\vee \rangle + 2\langle w(\lambda), \alpha_l^\vee \rangle \\ &= \langle w(\lambda), \alpha_*^\vee \rangle + 2, \end{aligned}$$

we have $\langle w(\lambda), \alpha_*^\vee \rangle = -2$. This contradicts $\lambda \in P_{\geq -1}(I)$. Hence, we get $P_{\text{sig}}(I) = P_0(I)$. $\square$

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$A_{l-1}^{(1)} \quad (l \geq 2)$	$B_l^{(1)} \quad (l \geq 3)$	$D_{l+1}^{(2)} \quad (l \geq 2)$
$\alpha_*^\vee + \alpha_1^\vee + \cdots + \alpha_{l-1}^\vee$	$\alpha_*^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_l^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-2}^\vee + 2\alpha_{l-1}^\vee + \alpha_l^\vee$
I C_l	$\{*, l\}$ C_2	$\{*, l\}$ C_2
$D_{l+2}^{(1)} \quad (l \geq 2)$	$C_l^{(1)} \quad (l \geq 2)$	$A_{2l-1}^{(2)} \quad (l \geq 3)$
$\alpha_*^\vee + \alpha_0^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_+^\vee + \alpha_-^\vee$	$\alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_{l-2}^\vee + \alpha_{l-1}^\vee + \alpha_*^\vee$	$2\alpha_0^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-2}^\vee + \alpha_{l-1}^\vee + \alpha_*^\vee$
$\{*, 0, +, -\}$ $K_4(l : \text{even})$ or $C_4(l : \text{odd})$	$\{*, l-1\}$ C_2	$\{*, l-1\}$ C_2
$E_6^{(1)}$	$F_4^{(1)}$	$E_6^{(2)}$
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_{2'}^\vee + \alpha_{1'}^\vee + 2\alpha_{2''}^\vee + \alpha_*^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$	$2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee + \alpha_*^\vee$
$\{*, 1, 1'\}$ C_3	$\{*\}$ e	$\{*\}$ e
$E_7^{(1)}$	$G_2^{(1)}$	$D_4^{(3)}$
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_{3'}^\vee + 2\alpha_{2'}^\vee + \alpha_*^\vee + 2\alpha_{2''}^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + \alpha_2^\vee$	$3\alpha_1^\vee + 2\alpha_2^\vee + \alpha_*^\vee$
$\{*, 1\}$ C_2	$\{*\}$ e	$\{*\}$ e
$E_8^{(1)}$	$A_{2l}^{(2)} \quad (l \geq 1)$	The type
		Dynkin diagram with indexes of vertices
$\alpha_*^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 5\alpha_5^\vee + 6\alpha_6^\vee + 4\alpha_{4'}^\vee + 2\alpha_{2'}^\vee + 3\alpha_{3''}^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + 2\alpha_l^\vee$	the null coroot
$\{*\}$ e	$\{*\}$ e	$(I)_1$ Γ_1

Table 2. Affine Types