

Propagation of the analyticity for the solution to the Euler equations with non-decaying initial velocity

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1 Introduction

This note is a survey of our paper [12] on the initial value problems for the Euler equations in $\mathbb{R}^n$ with $n \geq 2$, describing the motion of perfect incompressible fluids,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \quad (\text{E})$$

where the unknown functions $u = u(x, t) = (u^1(x, t), \dots, u^n(x, t))$ and $p = p(x, t)$ denote the velocity field and the pressure of the fluid, respectively, while $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ is the given initial velocity field satisfying the compatibility condition $\operatorname{div} u_0 = 0$.

The main purpose of this note is to prove the propagation properties of the real analyticity in spatial variables for the solution of (E) with non-decaying initial velocity. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing functions, and let $\mathcal{S}'(\mathbb{R}^n)$ be the space of all tempered distributions. We first recall the definition of the Littlewood-Paley operators. Let Φ and φ be the functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying the following properties :

$$\begin{aligned} \operatorname{supp} \widehat{\Phi} &\subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq 5/6 \}, \quad \operatorname{supp} \widehat{\varphi} \subset \{ \xi \in \mathbb{R}^n \mid 3/5 \leq |\xi| \leq 5/3 \}, \\ \widehat{\Phi}(\xi) + \sum_{j=0}^{\infty} \widehat{\varphi}_j(\xi) &= 1, \quad \xi \in \mathbb{R}^n, \end{aligned}$$

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where $\varphi_j(x) = 2^{jn}\varphi(2^jx)$ and $\widehat{f}$ denotes the Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ on $\mathbb{R}^n$. Given $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote

$$\Delta_j f := \begin{cases} \Phi * f & j = -1, \\ \varphi_j * f & j \geq 0, \\ 0 & j \leq -2, \end{cases} \quad S_k f := \sum_{j \leq k} \Delta_j f, \quad k \in \mathbb{Z},$$

where $*$ denotes the convolution operator. Then, we define the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ by the following definition.

Definition 1.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the Besov space $B_{p,q}^s(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the following norm is finite :

$$\|f\|_{B_{p,q}^s} := \left\| \left\{ 2^{sj} \|\Delta_j f\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q}.$$

In the framework for decaying initial velocities, Kato [7] proved that for the given initial velocity field $u_0 \in H^m(\mathbb{R}^n)^n$ with $m > n/2 + 1$, there exist $T = T(\|u_0\|_{H^m})$ such that the Euler equation (E) possesses a unique solution u in the class $C([0, T]; H^m(\mathbb{R}^n)^n)$. Alinhac and Métivier [2] proved that Kato's solution is real analytic in $\mathbb{R}^n$ if the initial velocity is real analytic. See also Bardos, Benachour and Zerner [3], Le Bail [9] and Levermore and Oliver [10]. Kukavica and Vicol [8] considered the vorticity equations for (E) in $H^s(\mathbb{T}^3)^3$ with $s > 7/2$ and proved the propagation properties of the real analyticity. In particular, they improved the estimate for the size of the radius of the convergence of the Taylor expansion for the solution to the vorticity equations. On the other hand, in the framework for non-decaying initial velocities, Pak and Park [11] proved that for the given initial velocity $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = 0$, there exists a $T = T(\|u_0\|_{B_{\infty,1}^1}) > 0$ such that the Euler equation (E) possesses a unique solution u in the class $C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ with $\nabla p = \sum_{i,j=1}^n \nabla(-\Delta)^{-1} \partial_i u_j \partial_j u_i$. Note that the Besov space $B_{\infty,1}^1(\mathbb{R}^n)$ contains some non-decaying functions, for example, the trigonometric function $e^{ix \cdot a}$ with the wave vector $a \in \mathbb{R}^n$. In [12], we prove the propagation of the analyticity for the solution to (E) constructed by Pak and Park in [11]. In particular, we give an improvement for the estimate for the size of the radius of convergence of Taylor's expansion.

Before stating our result about the analyticity, we set some notation. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all positive integers. For $k \in \mathbb{N}_0$, put

$$m_k := c \frac{k!}{(k+1)^2},$$

where c is a positive constant such that one has

$$\begin{aligned} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} m_{|\beta|} m_{|\alpha-\beta|} &\leq m_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n, \\ \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} m_{|\beta|-1} m_{|\alpha-\beta|+1} &\leq |\alpha| m_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n \setminus \{0\}^n. \end{aligned}$$

For example, it suffices to take $c \leq 1/16$. For the detail, see Kahane [6] and Alinhac and Métivier [1].

Our result on the propagation of the analyticity now reads:

Theorem 1.2. *Let $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ be an initial velocity field satisfying $\operatorname{div} u_0 = 0$, and let $u \in C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ be the solution of (E). Suppose that $u_0 \in C^\omega(\mathbb{R}^n)^n$ in the following sense : there exist positive constants K_0 and ρ_0 such that*

$$\|\partial_x^\alpha u_0\|_{B_{\infty,1}^1} \leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^n$. Then, $u(\cdot, t) \in C^\omega(\mathbb{R}^n)^n$ for all $t \in [0, T]$ and satisfies the following estimate : there exist positive constants $K = K(n, K_0)$, $L = L(n, K_0)$ and $\lambda = \lambda(n)$ such that

$$\|\partial_x^\alpha u(\cdot, t)\|_{B_{\infty,1}^1} \leq K \left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|} (1+t)^{\max\{|\alpha|-1, 0\}} \exp \left\{ \lambda |\alpha| \int_0^t \|u(\cdot, \tau)\|_{B_{\infty,1}^1} d\tau \right\} \quad (1.1)$$

for all $\alpha \in \mathbb{N}_0^n$ and $t \in [0, T]$.

Remark 1.3. (i) Since K, L and λ do not depend on T , (1.1) gives a grow-rate estimate for large time behavior of the higher order derivatives of Pak-Park's solutions.

(ii) From (1.1), one can derive the estimate for the size of the uniform analyticity radius of the solutions as follows :

$$\liminf_{|\alpha| \rightarrow \infty} \left(\frac{\|\partial_x^\alpha u(t)\|_{L^\infty}}{\alpha!} \right)^{-\frac{1}{|\alpha|}} \geq \frac{\rho_0}{L} (1+t)^{-1} \exp \left\{ -\lambda \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau \right\}.$$

Moreover, since $B_{\infty,1}^1(\mathbb{R}^n)$ is continuously embedded in $C^1(\mathbb{R}^n)$ (see Triebel [14]), we have by (1.1) that

$$\liminf_{|\alpha| \rightarrow \infty} \left(\frac{\|\partial_x^\alpha \operatorname{rot} u(t)\|_{L^\infty}}{\alpha!} \right)^{-\frac{1}{|\alpha|}} \geq \frac{\rho_0}{L} (1+t)^{-1} \exp \left\{ -\lambda \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau \right\}.$$

Recently, Kukavica and Vicol [8] considered the vorticity equations of (E) in $H^s(\mathbb{T}^3)^3$ with $s > 7/2$, and obtained the following estimate for uniform analyticity radius :

$$\liminf_{|\alpha| \rightarrow \infty} \left(\frac{\|\partial_x^\alpha \operatorname{rot} u(t)\|_{L^\infty}}{\alpha!} \right)^{-\frac{1}{|\alpha|}} \geq \rho (1+t^2)^{-1} \exp \left\{ -\lambda \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\}$$

with some $\rho := \rho(r, \operatorname{rot} u_0)$ and $\lambda = \lambda(r)$. Hence our result is an improvement of the previous analyticity-rate in the sense that $(1+t^2)^{-1}$ is replaced by $(1+t)^{-1}$, and clarifies that $\rho = \rho_0/L$.

In [12], we also consider the propagation properties of the almost periodicity with respect to the spatial variables for the solution to the Euler equations (E). We recall the definition of the almost periodicity in the sense of Bohr.

Definition 1.4. Let f be a bounded continuous function on $\mathbb{R}^n$. Put

$$\Sigma_f := \{ \tau_\xi f \mid \xi \in \mathbb{R}^n \} (\subset L^\infty(\mathbb{R}^n)), \quad \tau_\xi f := f(\cdot + \xi).$$

Then, f is called almost periodic in $\mathbb{R}^n$ if Σ_f is relatively compact in $L^\infty(\mathbb{R}^n)$.

Our result on the propagation of the almost periodicity now reads:

Theorem 1.5. *Let $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ be an initial velocity field satisfying $\operatorname{div} u_0 = 0$, and let $u \in C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ be the solution of (E). Suppose that u_0 is almost periodic in $\mathbb{R}^n$, then the solution $u(\cdot, t)$ of (E) is almost periodic in $\mathbb{R}^n$ for all $t \in [0, T]$.*

The same assertion is known for the solutions to the Navier-Stokes equations by Giga, Mahalov and Nicolaenko [5]. Recently, Taniuchi, Tashiro and Yoneda [13] proved the almost periodicity of weak solutions to (E) in the whole plane $\mathbb{R}^2$ when $u_0 \in L^\infty(\mathbb{R}^2)^2$. On the other hand, in the Theorem 1.5, we treat the classical solutions and all space-dimensions $n \geq 2$. The proof of Theorem 1.5 is based on the argument given by [5]. The key of the proof is to use the estimate concerning the continuity with respect to the initial velocities. The details are given in [12].

This note is organized as follows. In Section 2, we recall the key lemmas which play important roles in our proof. In Sections 3, we present the proof of Theorems 1.2.

2 Key Lemmas

In this section, we recall some key lemmas and prove a bilinear estimate in the Besov space $B_{\infty,1}^1(\mathbb{R}^n)$. We first prepare the commutator type estimates and the bilinear estimates in the Besov space $B_{\infty,1}^1(\mathbb{R}^n)$ for nonlinear terms of (E).

Lemma 2.1 (Pak-Park [11]). *There exists a positive constant $C = C(n)$ such that*

$$\sum_{j \in \mathbb{Z}} 2^j \|(S_{j-2}u \cdot \nabla) \Delta_j f - \Delta_j((u \cdot \nabla)f)\|_{L^\infty} \leq C \|u\|_{B_{\infty,1}^1} \|f\|_{B_{\infty,1}^1}$$

holds for all $(u, f) \in B_{\infty,1}^1(\mathbb{R}^n)^{n+1}$ with $\operatorname{div} u = 0$.

Lemma 2.2. *There exists a positive constant $C = C(n)$ such that*

$$\|fg\|_{B_{\infty,1}^1} \leq C(\|f\|_{L^\infty} \|g\|_{B_{\infty,1}^1} + \|g\|_{L^\infty} \|f\|_{B_{\infty,1}^1})$$

holds for all $f, g \in B_{\infty,1}^1(\mathbb{R}^n)$.

Proof. For the proof, we use the Bony paraproduct formula [4]. Let us decompose fg as

$$fg = \sum_{j=2}^{\infty} S_{j-3}f \Delta_j g + \sum_{j=2}^{\infty} S_{j-3}g \Delta_j f + \sum_{j=1}^{\infty} \sum_{k=j-2}^{j+2} \Delta_j f \Delta_k g.$$

Since $\operatorname{supp} \mathcal{F}[\varphi_j] \cap \operatorname{supp} \mathcal{F}[\varphi_{j'}] = \emptyset$ if $|j - j'| \geq 2$, we see that

$$\operatorname{supp} \mathcal{F}[S_{j-3}f \Delta_j g] \subset \{\xi \in \mathbb{R}^n \mid 2^{j-2} \leq |\xi| \leq 2^{j+2}\}$$

and

$$\operatorname{supp} \mathcal{F}[\Delta_j f \Delta_k g] \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq 2^{\max\{j,k\}+2}\},$$

which yield that

$$\Delta_j(fg) = \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} \Delta_j(S_{j'-3}f \Delta_{j'}g) + \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} \Delta_j(S_{j'-3}g \Delta_{j'}f)$$

$$\begin{aligned}
& + \sum_{\max\{j', j''\} \geq j-2} \sum_{|j''-j'| \leq 2} \Delta_j (\Delta_{j'} f \Delta_{j''} g) \\
& =: I_1 + I_2 + I_3.
\end{aligned} \tag{2.1}$$

By the Hausdorff-Young inequality and the Hölder inequality, we have that

$$\begin{aligned}
\|I_1\|_{L^\infty} & \leq C \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} \|S_{j'-3} f\|_{L^\infty} \|\Delta_{j'} g\|_{L^\infty} \\
& \leq C \|f\|_{L^\infty} \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} \|\Delta_{j'} g\|_{L^\infty}.
\end{aligned} \tag{2.2}$$

Similarly, it holds that

$$\|I_2\|_{L^\infty} \leq C \|g\|_{L^\infty} \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} \|\Delta_{j'} f\|_{L^\infty}. \tag{2.3}$$

Moreover, we see that

$$\begin{aligned}
\|I_3\|_{L^\infty} & \leq C \sum_{\max\{j', j''\} \geq j-2} \sum_{|j''-j'| \leq 2} \|\Delta_{j'} f\|_{L^\infty} \|\Delta_{j''} g\|_{L^\infty} \\
& \leq C \|g\|_{L^\infty} \sum_{j' \geq j-4} \|\Delta_{j'} f\|_{L^\infty}.
\end{aligned} \tag{2.4}$$

Hence it follows from (2.1), (2.2), (2.3) and (2.4) that

$$\begin{aligned}
\|fg\|_{B_{\infty,1}^1} & = \sum_{j \in \mathbb{Z}} 2^j \|\Delta_j(fg)\|_{L^\infty} \\
& \leq C \|f\|_{L^\infty} \sum_{j=-1}^{\infty} \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} 2^j \|\Delta_{j'} g\|_{L^\infty} + C \|g\|_{L^\infty} \sum_{j=-1}^{\infty} \sum_{\substack{j' \geq 2 \\ |j'-j| \leq 3}} 2^j \|\Delta_{j'} f\|_{L^\infty} \\
& \quad + C \|g\|_{L^\infty} \sum_{j=-1}^{\infty} \sum_{j' \geq j-4} 2^j \|\Delta_{j'} f\|_{L^\infty} \\
& =: J_1 + J_2 + J_3.
\end{aligned} \tag{2.5}$$

For the estimate of J_1 , we have that

$$\begin{aligned}
I_1 & \leq C \|f\|_{L^\infty} \sum_{|k| \leq 3} 2^{-k} \sum_{j=-1}^{\infty} 2^{j+k} \|\Delta_{j+k} g\|_{L^\infty} \\
& \leq C \|f\|_{L^\infty} \|g\|_{B_{\infty,1}^1}.
\end{aligned} \tag{2.6}$$

Similarly, we have for I_2 that

$$I_2 \leq C \|g\|_{L^\infty} \|f\|_{B_{\infty,1}^1}. \tag{2.7}$$

Concerning the estimate of I_3 , we have

$$I_3 \leq C \|g\|_{L^\infty} \sum_{k \geq -4} 2^{-k} \sum_{j=-1}^{\infty} 2^{j+k} \|\Delta_{j+k} f\|_{L^\infty}$$

$$\leq C \|g\|_{L^\infty} \|f\|_{B_{\infty,1}^1}. \quad (2.8)$$

Substituting (2.6), (2.7) and (2.8) into (2.5), we obtain that

$$\|fg\|_{B_{\infty,1}^1} \leq C (\|f\|_{L^\infty} \|g\|_{B_{\infty,1}^1} + \|g\|_{L^\infty} \|f\|_{B_{\infty,1}^1}).$$

This completes the proof of Lemma 2.2. $\square$

Next, we give the estimate for the gradient of pressure $\pi = \nabla p$.

Lemma 2.3 (Pak-Park [11]). *There exists a positive constant $C = C(n)$ such that*

$$\|\pi(u, v)\|_{B_{\infty,1}^1} \leq C \|u\|_{B_{\infty,1}^1} \|v\|_{B_{\infty,1}^1}$$

holds for all $u, v \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u = \operatorname{div} v = 0$, where

$$\pi(u, v) = \sum_{j,k=1}^n \nabla(-\Delta)^{-1} \partial_{x_j} u^k \partial_{x_k} v^j = \nabla(-\Delta)^{-1} \operatorname{div} \{(u \cdot \nabla) v\}.$$

Finally, we recall the Gronwall inequality.

Lemma 2.4 (The Gronwall inequality). *Let $A \geq 0$, and let f, g and h be non-negative, continuous functions on $[0, T]$ satisfying*

$$f(t) \leq A + \int_0^t g(s) ds + \int_0^t h(s) f(s) ds$$

for all $t \in [0, T]$. Then it holds that

$$f(t) \leq A e^{\int_0^t h(\tau) d\tau} + \int_0^t e^{\int_s^t h(\tau) d\tau} g(s) ds$$

for all $t \in [0, T]$.

3 Proof of Theorem 1.2

Proof of Theorem 1.2. Let u_0 satisfy the assumption of Theorem 1.2. We first remark that $u \in C([0, T]; B_{\infty,1}^s(\mathbb{R}^n)^n)$ for all $s \geq 1$ if $u_0 \in B_{\infty,1}^s(\mathbb{R}^n)^n$ for all $s \geq 1$. Hence $u(\cdot, t) \in C^\infty(\mathbb{R}^n)^n$ for all $t \in [0, T]$ by our assumption on the initial velocity u_0 and the embedding theorem. Moreover, the time-interval in which the solution exists does not depend on s . Indeed, we can choose T such that $T \geq C/\|u_0\|_{B_{\infty,1}^1}$ with some positive constant C depending only on n by the blow-up criterion, and the solution u satisfies

$$\sup_{t \in [0, T]} \|u(t)\|_{B_{\infty,1}^1} \leq C_0 \|u_0\|_{B_{\infty,1}^1} \quad (3.1)$$

with some positive constant C_0 depending only on n .

Now we discuss with the induction argument. In the case $\alpha = 0$, (1.1) follows from (3.1) with $K = C_0 K_0$. Next, we consider the case $|\alpha| \geq 1$. We first introduce some notation. For $l \in \mathbb{N}$ and $\lambda, L > 0$, we put

$$X_l(t) := \max_{|\alpha|=l} \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1}, \quad t \in [0, T],$$

$$Y_l = Y_l^{\lambda,L} := \max_{1 \leq k \leq l} \sup_{t \in [0, T]} \left\{ \frac{M_k(t)}{m_k} X_k(t) \right\},$$

where

$$M_k(t) = M_k^{\lambda,L}(t) := \rho_0^k L^{-(k-1)} (1+t)^{-(k-1)} e^{-\lambda k \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau}.$$

The similar notation were used in [1] and [2]. In what follows, we shall show that $Y_{|\alpha|} \leq 2K_0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$ when λ and L are sufficiently large. We now consider the case $|\alpha| = 1$. Let k be an integer with $1 \leq k \leq n$. Taking the differential operation ∂_{x_k} to the first equation of (E), we have

$$\partial_t(\partial_{x_k} u) + (\partial_{x_k} u \cdot \nabla)u + (u \cdot \nabla)\partial_{x_k} u + \partial_{x_k} \pi(u, u) = 0, \quad (3.2)$$

where

$$\nabla p = \pi(u, u) = \sum_{j,k=1}^n \nabla(-\Delta)^{-1} \partial_{x_j} u^k \partial_{x_k} u^j = \nabla(-\Delta)^{-1} \operatorname{div} \{(u \cdot \nabla)u\}.$$

Applying the Littlewood-Paley operator Δ_j and adding the term $(S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u)$ to the both sides of (3.2), we have

$$\begin{aligned} & \partial_t \Delta_j(\partial_{x_k} u) + (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u) \\ &= (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u) - \Delta_j((u \cdot \nabla)\partial_{x_k} u) - \Delta_j((\partial_{x_k} u \cdot \nabla)u) - \Delta_j(\partial_{x_k} \pi(u, u)). \end{aligned} \quad (3.3)$$

Here we consider the family of trajectory flows $\{Z_j(y, t)\}$ defined by the solution of the ordinary differential equations

$$\begin{cases} \frac{\partial}{\partial t} Z_j(y, t) = S_{j-2}u(Z_j(y, t), t), \\ Z_j(y, 0) = y. \end{cases} \quad (3.4)$$

Note that $Z_j \in C^1(\mathbb{R}^n \times [0, T])^n$, and $\operatorname{div} S_{j-2}u = 0$ implies that each $y \mapsto Z_j(y, t)$ is a volume preserving mapping from $\mathbb{R}^n$ onto itself. From (3.3) and (3.4), we see that

$$\partial_t \Delta_j(\partial_{x_k} u) + (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u) \Big|_{(x,t)=(Z_j(y,t),t)} = \frac{\partial}{\partial t} \{ \Delta_j(\partial_{x_k} u)(Z_j(y, t), t) \},$$

which yields that

$$\begin{aligned} \Delta_j(\partial_{x_k} u)(Z_j(y, t), t) &= \Delta_j(\partial_{x_k} u_0)(y) - \int_0^t \Delta_j((\partial_{x_k} u \cdot \nabla)u)(Z_j(y, s), s) ds \\ &\quad + \int_0^t \{ (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u) - \Delta_j((u \cdot \nabla)\partial_{x_k} u) \} (Z_j(y, s), s) ds \\ &\quad - \int_0^t \Delta_j(\partial_{x_k} \pi(u, u))(Z_j(y, s), s) ds. \end{aligned} \quad (3.5)$$

Since the map $y \mapsto Z_j(y, t)$ is bijective and volume-preserving for all $t \in [0, T]$, by taking the L^∞ -norm with respect to y to both sides of (3.5), we have

$$\begin{aligned} \|\Delta_j(\partial_{x_k} u)(t)\|_{L^\infty} &\leq \|\Delta_j(\partial_{x_k} u_0)\|_{L^\infty} + \int_0^t \|\Delta_j((\partial_{x_k} u \cdot \nabla)u)(s)\|_{L^\infty} ds \\ &\quad + \int_0^t \|\{(S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u) - \Delta_j((u \cdot \nabla)\partial_{x_k} u)\}(s)\|_{L^\infty} ds \\ &\quad + \int_0^t \|\Delta_j(\partial_{x_k} \pi(u, u))(s)\|_{L^\infty} ds. \end{aligned} \quad (3.6)$$

Multiplying both sides of (3.6) by 2^j and then taking the ℓ^1 -norm in j , we obtain that

$$\begin{aligned} \|\partial_{x_k} u(t)\|_{B_{\infty,1}^1} &\leq \|\partial_{x_k} u_0\|_{B_{\infty,1}^1} + \int_0^t \|(\partial_{x_k} u \cdot \nabla)u(s)\|_{B_{\infty,1}^1} ds + \int_0^t \|\partial_{x_k} \pi(u, u)(s)\|_{B_{\infty,1}^1} ds \\ &\quad + \int_0^t \sum_{j \in \mathbb{Z}} 2^j \|\{(S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k} u) - \Delta_j((u \cdot \nabla)\partial_{x_k} u)\}(s)\|_{L^\infty} ds \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (3.7)$$

It follows from the assumption on u_0 that

$$I_1 \leq K_0 \rho_0^{-1} m_1. \quad (3.8)$$

From Lemma 2.2, we see that

$$\begin{aligned} I_2 &\leq C \int_0^t \|\nabla u(s)\|_{L^\infty} \|\nabla u(s)\|_{B_{\infty,1}^1} ds \\ &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds, \end{aligned} \quad (3.9)$$

where we used the continuous embedding $B_{\infty,1}^1(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n)$. For the pressure term I_3 , it follows from Lemma 2.3 that

$$\begin{aligned} I_3 &\leq 2 \int_0^t \|\pi(\partial_{x_k} u, u)(s)\|_{B_{\infty,1}^1} ds \\ &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds. \end{aligned} \quad (3.10)$$

For the estimate of I_4 , we have from Lemma 2.1 that

$$\begin{aligned} I_4 &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} \|\partial_{x_k} u(s)\|_{B_{\infty,1}^1} ds \\ &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds. \end{aligned} \quad (3.11)$$

Substituting (3.8), (3.9), (3.10) and (3.11) into (3.7), we have

$$\|\partial_{x_k} u(t)\|_{B_{\infty,1}^1} \leq K_0 \rho_0^{-1} m_1 + C_1 \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds \quad (3.12)$$

with some positive constant C_1 depending only on n . Since $k \in \{1, \dots, n\}$ is arbitrary, it follows from (3.12) that

$$X_1(t) \leq K_0 \rho_0^{-1} m_1 + C_1 \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds,$$

which implies by Lemma 2.4 that

$$X_1(t) \leq K_0 \rho_0^{-1} m_1 e^{C_1 \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau}. \quad (3.13)$$

By choosing $\lambda \geq C_1$, we obtain from (3.13) that

$$\frac{M_1(t)}{m_1} X_1(t) \leq K_0 e^{(C_1 - \lambda) \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} \leq K_0,$$

which yields that

$$Y_1 \leq K_0. \quad (3.14)$$

Next, we consider the case $|\alpha| \geq 2$. Let α be a multi-index with $|\alpha| \geq 2$. Taking the differential operation ∂_x^α to the first equation of (E), we have

$$\partial_t(\partial_x^\alpha u) + \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u + \partial_x^\alpha \pi(u, u) = 0. \quad (3.15)$$

Applying the Littlewood-Paley operator Δ_j and adding the term $(S_{j-2}u \cdot \nabla) \Delta_j(\partial_x^\alpha u)$ to the both sides of (3.15), we have

$$\begin{aligned} & \partial_t \Delta_j(\partial_x^\alpha u) + (S_{j-2}u \cdot \nabla) \Delta_j(\partial_x^\alpha u) \\ &= (S_{j-2}u \cdot \nabla) \Delta_j(\partial_x^\alpha u) - \Delta_j((u \cdot \nabla) \partial_x^\alpha u) \\ & \quad - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \Delta_j((\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u) - \Delta_j(\partial_x^\alpha \pi(u, u)) \end{aligned} \quad (3.16)$$

Similarly to the case of $|\alpha| = 1$, we have from (3.16) that

$$\begin{aligned} \|\Delta_j(\partial_x^\alpha u)(t)\|_{L^\infty} &\leq \|\Delta_j(\partial_x^\alpha u_0)\|_{L^\infty} \\ & \quad + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|\Delta_j((\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u)(s)\|_{L^\infty} ds \\ & \quad + \int_0^t \|\Delta_j(\partial_x^\alpha \pi(u, u))(s)\|_{L^\infty} ds \\ & \quad + \int_0^t \|\{(S_{j-2}u \cdot \nabla) \Delta_j(\partial_x^\alpha u) - \Delta_j((u \cdot \nabla) \partial_x^\alpha u)\}(s)\|_{L^\infty} ds. \end{aligned} \quad (3.17)$$

Multiplying both sides of (3.17) by 2^j and then taking the ℓ^1 -norm in j , we obtain that

$$\|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} \leq \|\partial_x^\alpha u_0\|_{B_{\infty,1}^1}$$

$$\begin{aligned}
& + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|(\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} ds \\
& + \int_0^t \|\partial_x^\alpha \pi(u, u)(s)\|_{B_{\infty,1}^1} ds \\
& + \int_0^t \sum_{j \in \mathbb{Z}} 2^j \|\{(S_{j-2} u \cdot \nabla) \Delta_j(\partial_x^\alpha u) - \Delta_j((u \cdot \nabla) \partial_x^\alpha u)\}(s)\|_{L^\infty} ds \\
& =: J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.18}$$

It follows from the assumption on u_0 that

$$J_1 \leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|}. \tag{3.19}$$

For the estimate of J_2 , we have from Lemma 2.2 and the continuous embedding that

$$\begin{aligned}
J_2 & \leq C \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \left(\|\partial_x^\beta u(s)\|_{L^\infty} \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} + \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{L^\infty} \|\partial_x^\beta u(s)\|_{B_{\infty,1}^1} \right) ds \\
& = C \sum_{j=1}^n \binom{\alpha}{e_j} \int_0^t \|\partial_{x_j} u(s)\|_{L^\infty} \|\nabla \partial_x^{\alpha-e_j} u(s)\|_{B_{\infty,1}^1} ds \\
& \quad + C \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t \|\partial_x^\beta u(s)\|_{L^\infty} \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} ds \\
& \quad + C \int_0^t \|\nabla u(s)\|_{L^\infty} \|\partial_x^\alpha u(s)\|_{B_{\infty,1}^1} ds \\
& \quad + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{L^\infty} \|\partial_x^\beta u(s)\|_{B_{\infty,1}^1} ds \\
& \leq C |\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds + C \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t X_{|\beta|-1}(s) X_{|\alpha-\beta|+1}(s) ds \\
& \quad + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds.
\end{aligned} \tag{3.20}$$

For the pressure term J_3 , from Lemma 2.3, we have

$$\begin{aligned}
J_3 & \leq \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|\pi(\partial_x^\beta u, \partial_x^{\alpha-\beta} u)(s)\|_{B_{\infty,1}^1} ds \\
& \leq C \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|\partial_x^\beta u(s)\|_{B_{\infty,1}^1} \|\partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} ds \\
& \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds.
\end{aligned} \tag{3.21}$$

For the estimate of J_4 , it follows from Lemma 2.1 that

$$\begin{aligned} J_4 &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} \|\partial_x^\alpha u(s)\|_{B_{\infty,1}^1} ds \\ &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds. \end{aligned} \quad (3.22)$$

Substituting (3.19), (3.20), (3.21) and (3.22) to (3.18), we have

$$\begin{aligned} \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} + C|\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds \\ &\quad + C \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t X_{|\beta|-1}(s) X_{|\alpha-\beta|+1}(s) ds \\ &\quad + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds. \end{aligned} \quad (3.23)$$

Furthermore, for the third term of the right-hand side of (3.23), we see that

$$\begin{aligned} &\sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t X_{|\beta|-1}(s) X_{|\alpha-\beta|+1}(s) ds \\ &= \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t \frac{M_{|\beta|-1}(s)}{m_{|\beta|-1}} X_{|\beta|-1}(s) \frac{M_{|\alpha-\beta|+1}(s)}{m_{|\alpha-\beta|+1}} X_{|\alpha-\beta|+1}(s) \frac{m_{|\beta|-1}}{M_{|\beta|-1}(s)} \frac{m_{|\alpha-\beta|+1}}{M_{|\alpha-\beta|+1}(s)} ds \\ &\leq \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} m_{|\beta|-1} m_{|\alpha-\beta|+1} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds \\ &\leq |\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds. \end{aligned} \quad (3.24)$$

Similarly, for the fourth term of the right hand side of (3.23), we have

$$\begin{aligned} &\sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds \\ &\leq m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds. \end{aligned} \quad (3.25)$$

Substituting (3.24) and (3.25) to (3.23), we have

$$\begin{aligned} \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} + C|\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds \\ &\quad + C|\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds, \end{aligned}$$

which implies that

$$\begin{aligned} X_{|\alpha|}(t) &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} + C|\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds \\ &\quad + C|\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds. \end{aligned} \quad (3.26)$$

By Lemma 2.4, we obtain from (3.26) that

$$\begin{aligned} X_{|\alpha|}(t) &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} e^{C_2|\alpha| \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} + C_2|\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \\ &\quad \times \int_0^t (1+s)^{|\alpha|-2} e^{C_2|\alpha| \int_s^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau + \lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds \end{aligned}$$

with some positive constant C_2 depending only on n . By choosing $\lambda \geq C_2$ and $L \geq 1$, we thus have

$$\begin{aligned} \frac{M_{|\alpha|}(t)}{m_{|\alpha|}} X_{|\alpha|}(t) &\leq K_0 L^{-(|\alpha|-1)} (1+t)^{-(|\alpha|-1)} e^{(C_2-\lambda)|\alpha| \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} \\ &\quad + C_2|\alpha| L^{-1} (1+t)^{-(|\alpha|-1)} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{(C_2-\lambda)|\alpha| \int_s^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds \\ &\leq K_0 + C_2|\alpha| L^{-1} (1+t)^{-(|\alpha|-1)} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} ds \\ &\leq K_0 + \frac{2C_2}{L} (Y_{|\alpha|-1})^2. \end{aligned}$$

The above estimate with (3.14) implies that

$$Y_{|\alpha|} \leq K_0 + \frac{2C_2}{L} (Y_{|\alpha|-1})^2 \quad (3.27)$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 2$. From (3.14) and (3.27), we obtain by the standard inductive argument that

$$Y_{|\alpha|} \leq 2K_0 \quad (3.28)$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$, provided $\lambda \geq \max\{C_1, C_2\}$ and $L \geq \max\{1, 8C_2K_0\}$. Therefore, it follows from (3.28) that

$$\|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} \leq \frac{2K_0}{L} \left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|} (1+t)^{|\alpha|-1} e^{\lambda|\alpha| \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} \quad (3.29)$$

for all $t \in [0, T]$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$. From (3.1) and (3.29) with $K = K_0 \max\{C_0, 2/L\}$, we complete the proof of Theorem 1.2. $\square$

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