

A definition of a field for Euclid's Elements without any set theories

法政大学 村上 雅彦 (Masahiko MURAKAMI)
Hosei University

We introduce a definition of a field for Euclid's Elements [E] without any set theories. Precisely, we define the non-negative part of a ordered field.

Since we never use any notion of set theory, we never say the language is $\{+, \cdot, 0, 1\}$, but we say the symbols of binary operations “+” and “.” and the symbols of constants “0” and “1”.

Since we never use any notion of set theory, we never say *infinitely* many variable symbols $v_1, v_2, \dots$, but we say variable symbols $a, b, c, \dots$ etc. as many as we need. because the words “infinite” and “finite” are notions of set theories. We introduce a unary predicate $N(\bullet)$, and we say n is a natural number if $N(n)$.

The symbol of equality is “=”, and the logical connections are “ $\wedge$ ”, “ $\vee$ ”, “ $\Rightarrow$ ” and “ $\neg$ ” and the quantifiers are “ $\forall$ ” and “ $\exists$ ”.

By usual way of BNF, we define *terms*, *equations*, *formulas*. They are not sets but they are on a paper, in our brain, in storages of computers, or etc.. We never say a *set* of formulas against model theory.

By usual way we adopt the axiom of equality — replacing the terms which are connected by “=”.

Finally, we work on the classical predicate logic. any proofs or Any deductions are never set.

Here are the definition. Every free variable is bound by universal quantifier.

A0 $N(0) \wedge N(1)$.

$$\mathbf{A1} \quad (a + b) + c = a + (b + c).$$

$$\mathbf{A3} \quad a + b = b = 1$$

$$\mathbf{A4} \quad \left[P(0, a, b, c, d, e) \wedge \forall n [\mathbb{N}(n) \wedge P(n, a, b, c, d, e)] \Rightarrow P(n + 1, a, b, c, d, e) \right] \\ \Rightarrow \forall n [\mathbb{N}(n) \Rightarrow P(n, a, b, c, d, e)], \text{ where } P(x, u, v, w, y, z) \text{ is a formula on a paper, in our brain, in storages of computers, or etc.,} \\ \text{and } x, u, v, w, y, z \text{ are meta-symbols to replace terms.}$$

$$\mathbf{A5} \quad a + c = b + c \Rightarrow a = b.$$

$$\mathbf{A6} \quad \neg[a + 1 = 0].$$

$$\mathbf{A7} \quad a \cdot 1 = a.$$

$$\mathbf{A8} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

$$\mathbf{A9} \quad a \cdot b = b \cdot a.$$

$$\mathbf{A10} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

$$\mathbf{A11} \quad \exists c [a = 0 \vee a \cdot c = b].$$

$$\mathbf{A12} \quad \exists b a = b \cdot b.$$

$$\mathbf{A13} \quad \exists n \exists b [\mathbb{N}(n) \wedge a + b = n].$$

We denote $\exists c a + c = b$ by $a \leq b$ and denote $\exists n [\mathbb{N}(n) \wedge a \cdot n = b]$ by $a \mid b$.

REFERENCE

- [E] *Ευκλειδου Στοιχεια*, translation to Japanese by K.Nakamura H.Terasaka S.Ito M.Ikeda, Kyoritsu Shuppan, 1971.
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