

A POINCARÉ-BENDIXSON TYPE THEOREM FOR HOLOMORPHIC VECTOR FIELDS

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INTRODUCTION

Let Z_1 be a linear vector field on the two-dimensional complex space $\mathbb{C}^2$:

$$Z_1 = \sum_{j=1}^2 \lambda_j z_j \partial / \partial z_j, \quad \lambda_j \in \mathbb{C}, \quad \lambda_j \neq 0.$$

We have the following well-known

Fact ([1]). If λ_1/λ_2 does not belong to $\mathbb{R}_-$, the set of negative real numbers, then the three-dimensional unit sphere $S^3(1) = S^3(1:0)$ centered at the origin 0 in $\mathbb{C}^2$ is transverse to the foliation $\mathcal{F}(Z_1)$ defined by the solutions of Z_1 .

If λ_1/λ_2 belongs to $\mathbb{R}_-$, $S^3(1)$ is not transverse to $\mathcal{F}(Z_1)$.

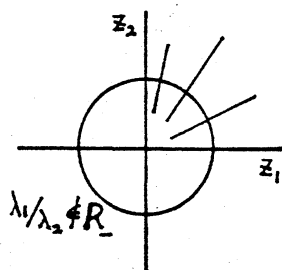


Fig. 1

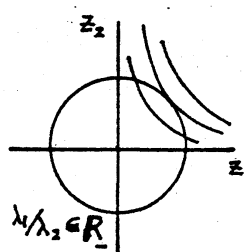


Fig. 2

We carry $S^3(1:0)$ to the sphere $S^3(1:(2,2))$ centered at the point $(2,2)$ in $\mathbb{C}^2$. Next we deform $S^3(1:(2,2))$ to $\tilde{S}^3(1:(2,2))$ as shown in Figures 5 and 6.

Intuitively it appears that $S^3(1:(2,2))$ and $\tilde{S}^3(1:(2,2))$ are not transverse to $\mathcal{F}(Z_1)$. The above figures suggest to us a topological property of the transversality between spheres and holomorphic vector fields. This observation leads us to the following Poincaré–Hopf type theorem for holomorphic vector fields.

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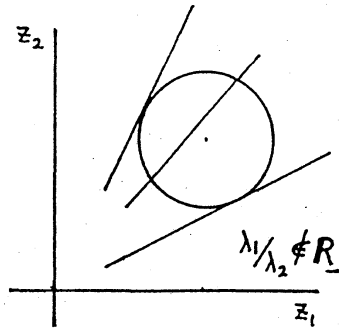


Fig. 3

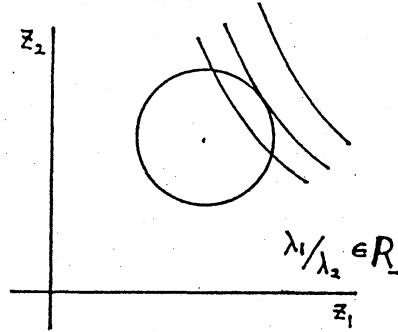


Fig. 4

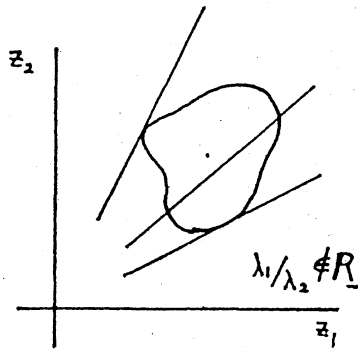


Fig. 5

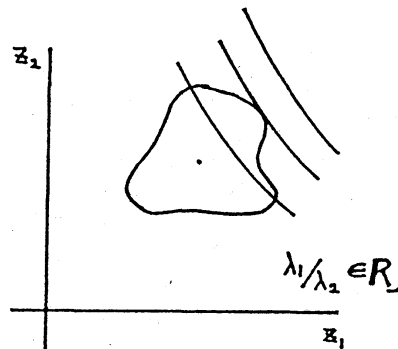


Fig. 6

Theorem 1. Let M be a subset of $\mathbb{C}^n$, diffeomorphic to the $2n$ -dimensional closed disk $\bar{D}^{2n}(1)$ consisting of all z in $\mathbb{C}^n$ with $\|z\| \leq 1$. We write $\mathcal{F}(Z)$ for the foliation defined by solutions of a holomorphic vector field Z in some neighborhood of M . If the boundary of M is transverse to $\mathcal{F}(Z)$, then Z has only one singular point, say p , in M . Furthermore, the index of Z at p is equal to one.

From Theorem 1, we get an answer to the problem suggested by Figures 5 and 6.

Corollary 2. Consider a linear vector field in $\mathbb{C}^n$: $Z = \sum_{j=1}^n \lambda_j z_j \partial / \partial z_j$, $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$. If a smooth imbedding φ of $(2n-1)$ -sphere S^{2n-1} in $\mathbb{C}^n - \{0\}$ belongs to the zero element of the homotopy group $\pi_{2n-1}(\mathbb{C}^n - \{0\})$, then φ is not transverse to $\mathcal{F}(Z)$.

Since the distance function for solutions of a holomorphic vector field Z with respect to the origin 0 is subharmonic, each solution of Z is unbounded except the singular set of Z . Therefore we have formulated a Poincaré-Bendixson type theorem for holomorphic vector fields.

Theorem 3. Let M denote a subset of $\mathbb{C}^n$ holomorphic and diffeomorphic to the $2n$ -dimensional closed disk $\bar{D}^{2n}(1)$. Let Z be a holomorphic vector field in some neighborhood of M . If the boundary ∂M of M is transverse to the foliation $\mathcal{F}(Z)$, then each solution of Z which crosses ∂M tends to the unique singular point p of Z in M , that is, p is in the closure

of L . Further, the restriction $\mathcal{F}(Z)|_{M-\{p\}}$ of $\mathcal{F}(Z)$ to $M - \{p\}$ is C^ω -diffeomorphic to the foliation $\mathcal{F}(Z)|_{\partial M \times (0, 1]}$ of $M - \{p\}$, where $\mathcal{F}(Z)|_{\partial M}$ denotes the restriction of $\mathcal{F}(Z)$ to ∂M .

Adrien Douady proved Theorem 3 in the case $n = 2$.

From Theorem 3 we get an affirmative answer to a special case of the Seifert conjecture.

Corollary 4. *Let Z be a holomorphic vector field in some neighborhood of $\bar{D}^4(1) \subset \mathbb{C}^2$. If the boundary $\partial \bar{D}^4(1) = S^3(1)$ is transverse to $\mathcal{F}(Z)$, then the restriction $\mathcal{F}(Z)|_{S^3(1)}$ to S^3 has at least one compact leaf.*

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§1. DEFINITION OF TRANSVERSALITY BETWEEN MANIFOLDS AND HOLOMORPHIC VECTOR FIELDS

Let $Z = \sum_{j=1}^n f_j(z) \partial/\partial z_j$ be a holomorphic vector field in the complex space $\mathbb{C}^n$ of dimension n . We identify $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$ of dimension $2n$ by the natural correspondence. We have a real representation of Z :

$$\begin{aligned} Z &= \sum_{j=1}^n f_j(z) \partial/\partial z_j \\ &= \sum_{j=1}^n (g_j(x, y) + ih_j(x, y)) \frac{1}{2} (\partial/\partial x_j - i \partial/\partial y_j) \\ &= \frac{1}{2} \left\{ \left[\sum_{j=1}^n (g_j(x, y) \partial/\partial x_j + h_j(x, y) \partial/\partial y_j) \right] \right. \\ &\quad \left. - i \left[\sum_{j=1}^n (-h_j(x, y) \partial/\partial x_j + g_j(x, y) \partial/\partial y_j) \right] \right\} \\ &= \frac{1}{2} (X - iY), \end{aligned} \tag{1.1}$$

where we set

$$X = \sum_{j=1}^n (g_j(x, y) \partial/\partial x_j + h_j(x, y) \partial/\partial y_j) \tag{1.2}$$

and

$$Y = \sum_{j=1}^n (-h_j(x, y) \partial/\partial x_j + g_j(x, y) \partial/\partial y_j). \tag{1.3}$$

Let J be the natural almost complex structure of $\mathbb{C}^n$. The vector fields X and Y satisfy the following equations:

$$JX = Y, \quad JY = -X \quad \text{and} \quad [X, Y] = 0. \quad (1.4)$$

Let N be a smooth manifold of dimension $2n - 1$. We define below the transversality of a smooth map $\Phi : N \rightarrow \mathbb{C}^n$ to the foliation $\mathcal{F}(Z)$ determined by solutions of Z .

Definition 1.1. We say that the map Φ is transverse to the foliation $\mathcal{F}(Z)$ or the holomorphic vector field Z if the following equation is satisfied for each point $p \in N$:

$$\Phi_*(T_p N) + \{X, Y\}_{\Phi(p)} = T_{\Phi(p)} \mathbb{R}^{2n},$$

where $T_p N$ and $T_{\Phi(p)} \mathbb{R}^{2n}$ are the tangent space of N at p and the tangent space of $\mathbb{R}^{2n}$ at $\Phi(p)$ respectively, and $\{X, Y\}_{\Phi(p)}$ is the vector space generated by $X_{\Phi(p)}$ and $Y_{\Phi(p)}$. In particular, if N is a submanifold in $\mathbb{C}^n$, we say that N is transverse to $\mathcal{F}(Z)$.

For example consider the $(2n-1)$ -dimensional sphere $S^{2n-1}(r)$, consisting of all $z \in \mathbb{C}^n$ with $\|z\| = r$. $S^{2n-1}(r)$ is tangent to $\mathcal{F}(Z)$ at $p \in S^{2n-1}(r)$ if and only if the following equation is satisfied at p :

$$\sum_{j=1}^n f_j(z) \bar{z}_j = \langle X, N \rangle - i \langle Y, N \rangle = 0, \quad (1.6)$$

where we denote by $N = \sum_{j=1}^n (x_j \partial/\partial x_j + y_j \partial/\partial y_j)$ the usual normal vector field on $S^{2n-1}(r)$. We set $\Sigma = \{z \in \mathbb{C}^n \mid \sum_{j=1}^n f_j(z) \bar{z}_j = 0\}$ and say that Σ is the total contact set of spheres and $\mathcal{F}(Z)$. We denote by $R(z) = \sum_{j=1}^n |z_j|^2$ the distance function between $z \in \mathbb{C}^n$ and the origin 0 in $\mathbb{C}^n$. A critical point of the restriction $R|_L$ of R to a solution L of Z is a contact point of L and the sphere.

We will conclude this section by giving some examples of the contact set $\Sigma \cap S^{2n-1}(r)$ of $S^{2n-1}(r)$ and $\mathcal{F}(Z)$.

Example 1.2. Consider $Z = z_1(2 + z_1 + z_2) \partial/\partial z_1 + z_2(1 + z_1) \partial/\partial z_2$ defined in $\mathbb{C}^2$. The set $\text{Sing}(Z)$ of singular points of Z consists of three points: $(0, 0)$, $(-2, 0)$ and $(-1, -1)$. Now $\text{Sing}(Z) \cap \bar{D}^4(1)$ consists of $(0, 0)$ only, where $\bar{D}^4(1)$ is the four-dimensional closed disk centered at the origin in $\mathbb{C}^2$ with radius 1. For any r , $0 < r \leq 1$, the contact set $S^3(r) \cap \Sigma$ is empty; that is, $S^3(r)$ is transverse to $\mathcal{F}(Z)$. Therefore, each solution of Z which crosses $S^3(1)$ tends to the origin in $\mathbb{C}^2$.

Example 1.3. Let a be a complex number different from zero. Define Z on C^2 by $Z = (2z_1 + az_2^2) \partial/\partial z_1 + z_2 \partial/\partial z_2$. We mention here that one can find in [3] one of the normal forms of holomorphic vector fields in C^2 :

$$\tilde{Z} = (\lambda_1 z_1 + az_2^n) \partial/\partial z_1 + \lambda_2 z_2 \partial/\partial z_2, \quad \lambda_1 = n\lambda_2.$$

The singular set $\text{Sing}(Z)$ consists of a single point $(0, 0)$. There exists a number $r_0 > 0$ such that

- (i) if $0 < r < r_0$, $\Sigma \cap S^3(r)$ is empty;
- (ii) if $r = r_0$, $\Sigma \cap S^3(r_0)$ is diffeomorphic to the circle S^1 ;
- (iii) if $r_0 < r$, $\Sigma \cap S^3(r)$ is diffeomorphic to the disjoint union $S^1 \amalg S^1$ of two copies of the circle S^1 .

In the case (ii), the circle $\Sigma \cap S^3(r_0)$ consists of degenerate critical points. If L_p is the solution of Z passing through $p \in \Sigma \cap S^3(r_0)$, then $L_p \cap \Sigma$ is a singleton set $\{p\}$.

In the case (iii), one circle of $\Sigma \cap S^3(r)$ consists of minimal points and the other consists of saddle points. In particular, for $p \in \Sigma \cap S^3(r)$ the set $L_p \cap \Sigma$ consists of two points p and q , $p \neq q$. More precisely, one of these two points is a saddle point of $R|_{L_p}$ and the other a minimal point of $R|_{L_p}$.

Example 1.4. One finds in [4] the following example of a one-form ω on C^2 : $\omega = z_2(1 - i - z_1 z_2) dz_1 - z_1(1 + i - z_1 z_2) dz_2$. We consider here $Z = z_1(1 + i - z_1 z_2) \partial/\partial z_1 + z_2(1 - i - z_1 z_2) \partial/\partial z_2$ on C^2 . The singular set $\text{Sing}(Z)$ consists of a single point, namely $(0, 0)$. If $0 < r < \sqrt{2}$, $\Sigma \cap S^3(r)$ is empty. If $r = \sqrt{2}$, $\Sigma \cap S^3(\sqrt{2})$ is diffeomorphic to the circle S^1 . Indeed $\Sigma \cap S^3(\sqrt{2})$ belongs to the solution $z_1 z_2 = 1$ of Z . If $r > \sqrt{2}$, $\Sigma \cap S^3(r)$ is diffeomorphic to the disjoint union $S^1 \amalg S^1$ of two copies of the circle S^1 , and consists of saddle points.

§2. PROOF OF THEOREM 1

In this section we shall use the same notation as in the previous sections.

First, we note that the following property of analytic sets in C^n : the set of singular points of Z in M consists of isolated finite points. Since the boundary ∂M of M is transverse to $\mathcal{F}(Z)$, there exists a smooth vector field ξ in some neighborhood of ∂M such that

- (i) ξ is represented by $aX + bY \neq 0$, where a and b are smooth functions defined in some neighborhood of ∂M ;
- (ii) ξ is required to point outward at each point of ∂M .

We obtain a smooth map (a, b) of some neighborhood of ∂M to $\mathbb{R}^2 - \{0\}$. When $n \geq 2$ using obstruction theory (see [9]), we can extend the map (a, b) to a smooth map (α, β) of some neighborhood of M to $\mathbb{R}^2 - \{0\}$ such that the restriction of (α, β) to some neighborhood of ∂M is the map (a, b) .

There should be no confusion if we use ξ for the extended smooth vector field $\xi = \alpha X + \beta Y$. By the definition of ξ on a neighborhood of M , the set $\text{Sing}(Z)$ of the singular points of Z coincides with that of ξ .

In order to calculate the index of ξ at $p \in \text{Sing}(Z)$, we may think of the vector field ξ as a map $\xi : M \rightarrow \mathbb{R}^{2n}$. Similarly we may think of the holomorphic vector field Z as a map $Z : M \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ or as a map $Z : M \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. We say that the vector field Z is non-degenerate at $p \in \text{Sing}(Z)$ if the Jacobian $\det(D(Z)(p))$ of Z at p is different from zero. By a direct calculation we obtain the following:

$$\begin{aligned} \det(D(\xi)(p)) &= \det \begin{pmatrix} \alpha(p)I_n & -\beta(p)I_n \\ \beta(p)I_n & \alpha(p)I_n \end{pmatrix} \det(D(Z)(p)) \\ &= |\det((\alpha(p) + i\beta(p))I_n)|^2 \left| \det \left(\frac{\partial g_j}{\partial x_k}(p) + i \frac{\partial g_j}{\partial y_k}(p) \right) \right|^2, \end{aligned} \quad (2.1)$$

where $\det A$ denotes the determinant of a matrix A and I_n is the identity matrix of $\text{GL}(n, \mathbb{R})$. In particular, since $\det(D(Z)(p))$ is positive at a non-degenerate singular point $p \in \text{Sing}(Z)$, the index of ξ at p is one (see [6]).

In order to calculate the index of ξ at a degenerate singular point $p \in \text{Sing}(Z)$, we recall the following

Proper mapping theorem ([5]). Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic map such that $F(0)$ is equal to 0. Assume that 0 is an isolated point in $F^{-1}(0)$ and $\det(D(F)(0))$ is 0. Then there exists a number $\epsilon > 0$ together with a neighborhood W of 0 such that $F|_W : W \rightarrow \Delta(0 : \epsilon) = \{z \in \mathbb{C}^n \mid \|z\| < \epsilon\}$ is surjective.

Using the proper mapping theorem we find a sufficiently small number $\epsilon > 0$ and a neighborhood W of $p \in \text{Sing}(Z)$ such that $W \cap \text{Sing}(Z)$ is a singleton set. Since there exist regular values y of Z in $\Delta(0 : \epsilon)$, by (2.1), we may select a regular value y of ξ in $\Delta(0 : \epsilon_1) = \{y \in \mathbb{R}^{2n} \mid \|y\| < \epsilon_1\}$, $0 < \epsilon_1 < \epsilon$. The set $N_1 = \xi^{-1}(\bar{\Delta}(0 : \epsilon_1)) \cap W$ is compact. We then choose a compact set N with $W \supset N \supset N_1$ and a smooth function λ which takes on the value one at $x \in N_1$ and zero at $x \notin N$. Define $\tilde{\xi}$ by $\tilde{\xi}(x) = \xi(x) - \lambda(x)y$. Then $\tilde{\xi}$ is different from zero at each point $x \in N - N_1$; hence $\tilde{\xi}^{-1}(0) \cap W$ is compact and each point $\tilde{p} \in \tilde{\xi}^{-1}(0) \cap W$ is non-degenerate. Now we are ready to calculate the index of the vector field ξ at a degenerate point $p \in \text{Sing}(Z)$:

$$\begin{aligned} \text{index}_p \xi &= \sum_{\tilde{p} \in \tilde{\xi}^{-1}(0) \cap W} \text{index}_{\tilde{p}} \tilde{\xi} \\ &= \text{the number of elements of } \tilde{\xi}^{-1}(0) \cap W \geq 1, \end{aligned} \quad (2.2)$$

where $\text{index}_p \xi$ denotes the index of ξ at p .

On the other hand, by the Poincaré-Hopf theorem we have the following:

$$1 = \chi(M) = \sum_{p \in \text{Sing}(Z) \cap M} \text{index}_p \xi, \quad (2.3)$$

where $\chi(M)$ denotes the Euler number of M . From (2.2) and (2.3) we conclude that the number of elements of $\text{Sing}(Z)$ in M is one. This completes the proof of Theorem 1.

§3. PROOF OF THEOREM 3

We continue to use the same notation.

Since M is holomorphic, diffeomorphic to the $2n$ -dimensional closed disk $\bar{D}^{2n}(1)$, we give a proof of Theorem 3 for $\bar{D}^{2n}(1)$. Using a Möbius transformation, we can assume that the sole singular point of Z in $\bar{D}^{2n}(1)$ is the origin 0 . We define a function F in some neighborhood of $\bar{D}^{2n}$ minus the origin 0 by

$$F(z) = \frac{\sum_{j=1}^n f_j(z) \bar{z}_j}{\sum_{j=1}^n |z_j|^2}.$$

Since the boundary $S^{2n-1}(1)$ of $\bar{D}^{2n}(1)$ is transverse to $\mathcal{F}(Z)$, the restriction $F|_{S^{2n-1}(1)}$ of F to $S^{2n-1}(1)$ takes on the values in $\mathbb{C} - \{0\}$. Consider a complex line l_z through a point $z \in S^{2n-1}(1)$: $l_z = \{tz \in \mathbb{C}^n | t \in \mathbb{C}\}$. We define a holomorphic function $\tilde{F}(t : z)$ in some neighborhood of $\bar{D}^2(1 : 0) = \{t \in \mathbb{C} | |t| \leq 1\}$ by

$$\tilde{F}(t : z) = \begin{cases} \frac{\sum_{j=1}^n f_j(tz) \bar{t} \bar{z}_j}{t \bar{t}}, & \text{if } t \neq 0 \\ \sum_{j,k=1}^n \frac{\partial f_j}{\partial z_k}(0) z_k \bar{z}_j, & \text{if } t = 0. \end{cases}$$

Then the degree of $\tilde{F}|_{|t|=1}$ is zero, because $F|_{S^{2n-1}(1)}$ is homotopic to a constant map. Hence, for any $z \in S^{2n-1}(1)$, $\tilde{F}(t : z)$ is not zero; that is, the only element of $\Sigma \cap \bar{D}^{2n}(1)$ is the origin 0 in $\mathbb{C}^n$. In other words, $S^{2n-1}(r)$, $0 < r \leq 1$, are transverse to $\mathcal{F}(Z)$. Let $\tilde{N} \in T\mathcal{F}(Z)$ be the vector field of the projection of N to $T\mathcal{F}(Z)$. The set of singular points of $\tilde{N}$ in $\bar{D}^{2n}(1)$ is the singleton set $\{0\}$ in $\mathbb{C}^n$. Then each solution of Z which crosses $S^{2n-1}(1)$ tends to 0 along the orbit of $\tilde{N}$. Furthermore, the restricted foliation $\mathcal{F}(Z)|_{S^{2n-1}(r)}$ of $S^{2n-1}(r)$ is C^ω -diffeomorphic to the foliation $\mathcal{F}(Z)|_{S^{2n-1}(1)}$ of $S^{2n-1}(1)$ by the correspondence along orbits of $\tilde{N}$. This completes the proof of Theorem 3.

§4. A SPECIAL CASE OF SEIFERT CONJECTURE

The notation used in the Introduction, §1 and §3 carries over in the present section.

We first recall the Seifert conjecture. Consider the vector field $e = z_1 \partial/\partial z_1 + z_2 \partial/\partial z_2$ on C^2 . All leaves of the restricted foliation $\mathcal{F}(e)|_{S^3(1)}$ of $S^3(1)$ are fibres of the Hopf fibration $S^3 \rightarrow S^2$. On the other hand, consider the vector field $e_\epsilon = (z_1 + \epsilon z_2) \partial/\partial z_1 + z_2 \partial/\partial z_2$, where the number ϵ is sufficiently small. Then the restricted foliation $\mathcal{F}(e_\epsilon)|_{S^3(1)}$ of $S^3(1)$ has one closed orbit $|z_1| = 1$ but all other leaves are diffeomorphic to R^1 . In [8] H. Seifert proved the following

Theorem (H. Seifert). A continuous vector field on the three-sphere which differs sufficiently little from $\mathcal{F}(e)|_{S^3(1)}$ and which sends through every point exactly one integral curve, has at least one closed integral curve.

The Seifert conjecture says "every non-singular vector field on the three-dimensional sphere S^3 has a closed integral curve".

In [7] Paul Schweitzer constructed a counterexample to the Seifert conjecture: There exists a non-singular C^1 vector field on S^3 which has no closed integral curves.

In this section we investigate a certain property of a non-singular vector field on S^3 induced by a holomorphic vector field in some neighborhood of $\bar{D}^4(1)$ which is transverse to $S^3(1)$. This will prove Corollary 4.

Proof of Corollary 4. Using a Möbius transformation, we can assume that the only singular point of Z in $\bar{D}^4(1)$ is the origin. First, we note that the existence of a separatrix of Z at 0 was proved by C. Camacho and P. Sad [2]. Let L be a separatrix of Z at 0. There is a sufficiently small number $\epsilon > 0$ together with a holomorphic function f defined in $D^4(\epsilon)$ such that $D^4(\epsilon) \cap L = \{f = 0\}$. Then for each ϵ_1 , $0 < \epsilon_1 < \epsilon$, $S^3(\epsilon_1) \cap L$ is a circle. Since $\mathcal{F}(F)|_{S^3(\epsilon_1)}$ is C^ω -diffeomorphic to $\mathcal{F}(F)|_{S^3(1)}$, the latter has at least one compact leaf. This completes the proof of Corollary 4.

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