

Theory of Centrifugal Type Impeller

By

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Chapter 1 Theory of Centrifugal Type Impeller with Vanes of Arbitrary Form

This chapter deals with the theory of the impeller of centrifugal type hydraulic machinery with vanes of arbitrary form. The theory contains a method of conformal mapping based on a mapping function, which maps conformally an impeller with logarithmic spiral vanes into a unit circle. The calculated results were compared with the experimental ones of impellers with three types of vanes.

1. Introduction

As is well-known, Spannhake¹⁾, Sorensen²⁾, and Busemann³⁾ have treated the theory of centrifugal type impeller and have developed the theory of the impeller with vanes of logarithmic spiral forms under the assumption that the flow in the impeller is a two-dimensional potential one. The calculated results are the exact solutions and show that the theoretical head obtained by the impeller is affected by the vane angle, number of vanes and inner-outer radii ratio. They did not, however, deal with the flow in the impeller, especially the pressure distributions along the vane surfaces. Recently J. D. Stanitz⁴⁾, Chung-Hua Wu⁵⁾, J. A. Jhonson⁶⁾, A. Ginsburg⁶⁾, J. T. Harmick⁷⁾, W. H. Isay⁸⁾ and H. Krüger⁹⁾ have developed the two-dimensional and the three-dimensional theory from the standpoint of hydrodynamics, and A. J. Acosta¹⁰⁾ experimentally measured the pressure distribution along the vane surfaces of the impeller.

The authors have performed already the measurement¹¹⁾ of the pressure distributions along the vanes of logarithmic spiral forms and investigated the effects of the vane angle and number of vanes upon them. Our tests were performed on the impellers with constant breadth, in which the flow may be considered to be two-dimensional. The test results showed that the measured pressure distributions were nearly equal to the ones calculated by the theory¹²⁾. Therefore it is very significant that the theoretical calculation of the impellers with vanes of arbitrary forms is developed and that the flow in the impeller, especially the pressure distributions, can be theoretically calculated.

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$$\log r = \frac{2\nu}{N(\tau^2 + \nu^2)} \left[\nu \tanh^{-1} \frac{2\kappa \cos \theta}{1 + \kappa^2} + \tau \tanh^{-1} \frac{2\kappa \sin \theta}{1 - \kappa^2} \right] + \frac{\nu}{2} + a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta' + b_n \sin n\theta') \quad \dots\dots\dots (6)$$

$$\varphi = \frac{2\tau}{N(\tau^2 + \nu^2)} \left[\nu \tanh^{-1} \frac{2\kappa \cos \theta}{1 + \kappa^2} + \tau \tanh^{-1} \frac{2\kappa \sin \theta}{1 - \kappa^2} \right] + \frac{\tau}{2} + a'_0 + \sum_{n=1}^{\infty} (a'_n \cos n\theta' + b'_n \sin n\theta') \quad \dots\dots\dots (7)$$

where $\theta' = \theta - \theta_T$, $\theta_T = \tan^{-1} \left(\frac{1 - \kappa^2}{1 + \kappa^2} \frac{\tau}{\nu} \right)$ and the coefficients a_n, b_n, a'_n and b'_n are the small quantities which represent the camber and thickness of the vane.

Next, denote the angles between the radii of the corresponding points and the tangents to the above basic spiral and to the vane contour as γ and γ' respectively. The so-called vane angle β is a complementary angle to this angle. From the assumption that the vane is a thin airfoil, the difference $\delta = \gamma' - \gamma$ is a small quantity.

Hence
$$\tan \gamma' = \frac{r d\varphi}{dr} = \frac{d\varphi}{d\theta} \left| \frac{d \log r}{d\theta} \right|$$

so the following relation is obtained:

$$\delta = \frac{N(1 - 2\kappa^2 \cos 2\theta + \kappa^4)}{4 \kappa \sqrt{K} \sin \theta'} \left\{ \tau \sum_{n=1}^{\infty} n(a_n \sin n\theta' - b_n \cos n\theta') - \nu \sum_{n=1}^{\infty} n(a'_n \sin n\theta' - b'_n \cos n\theta') \right\} \quad \dots\dots\dots (8)$$

3. Boundary Condition in Source and Vortex Flow

As shown in Fig. 2 (a), the flow inside of the impeller can be expressed by a source Q and a vortex Γ_1 , which are put at the origin. The flow with $\Gamma_1 = 0$ shows the one without pre-rotation. Denote the normal and the tangential velocities due to the above source and vortex, as v_n and v_t respectively, so that they are given as follows;

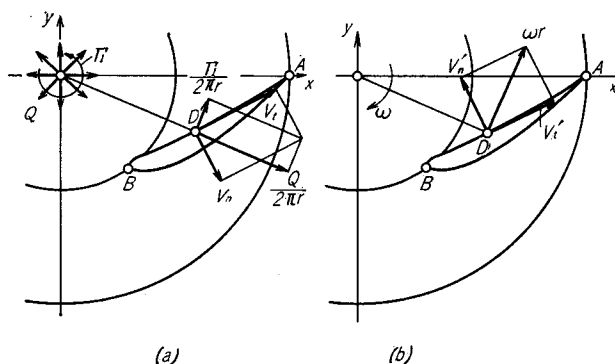


Fig. 2

As
$$\left. \begin{aligned} v_n &= \frac{Q}{2\pi r} \sin \gamma' - \frac{\Gamma_1}{2\pi r} \cos \gamma', \quad v_t = \frac{Q}{2\pi r} \cos \gamma' + \frac{\Gamma_1}{2\pi r} \sin \gamma' \\ \cos \gamma' &= \cos \gamma - \delta \sin \gamma, \quad \sin \gamma' = \sin \gamma + \delta \cos \gamma \end{aligned} \right\} \dots\dots\dots (9)$$

hence
$$\left. \begin{aligned} v_n &= \frac{Q}{2\pi r} \sin \gamma - \frac{\Gamma_1}{2\pi r} \cos \gamma + \delta \left(\frac{Q}{2\pi r} \cos \gamma + \frac{\Gamma_1}{2\pi r} \sin \gamma \right) \\ v_t &= \frac{Q}{2\pi r} \cos \gamma + \frac{\Gamma_1}{2\pi r} \sin \gamma - \delta \left(\frac{Q}{2\pi r} \sin \gamma - \frac{\Gamma_1}{2\pi r} \cos \gamma \right) \end{aligned} \right\} \dots\dots\dots (10)$$

Now, from the above assumption that the vane is a thin airfoil and its camber differs a little from the basic spiral, the mapping function, which represents conformally the given impeller vane into a unit circle on the ζ -plane, can be expressed as

$$\log z = \frac{e^{i \tan^{-1} \tau / \nu}}{N} \left[e^{-i \tan^{-1} \tau / \nu} \log \frac{\kappa \zeta + 1}{\kappa \zeta - 1} + e^{i \tan^{-1} \tau / \nu} \log \frac{\zeta + \kappa}{\zeta - \kappa} - \frac{N}{2} \sqrt{\tau^2 + \nu^2} \right] \dots\dots (11)$$

Therefore the radial and the circumferential velocity components v_r and v_θ at the corresponding point on the circle are given as follows;

$$v_r = v_n \left| \frac{dz}{d\zeta} \right|, \quad v_\theta = v_t \left| \frac{dz}{d\zeta} \right| \dots\dots\dots (12)$$

where

$$\left| \frac{dz}{d\zeta} \right| = -\frac{r}{N} \frac{4\kappa \sqrt{K_1} \sin \theta'}{1 - 2\kappa^2 \cos 2\theta + \kappa^4} \dots\dots\dots (13)$$

and

$$\begin{aligned} K_1 &= 1 + 2\kappa^2 \cos 2\gamma + \kappa^4 = K/(\tau^2 + \nu^2) \\ v_r &= \frac{2\kappa \sqrt{K_1}}{\pi N \sqrt{K}} (Q \sin \gamma - \Gamma_1 \cos \gamma) \left\{ \nu \sum_{n=1}^{\infty} \kappa^{2n} \sin (2n+1)\theta - \tau \sum_{n=1}^{\infty} \kappa^{2n} \cos (2n+1)\theta \right\} \\ &\quad - \frac{\nu}{\sqrt{\tau^2 + \nu^2}} (Q \cos \gamma + \Gamma_1 \sin \gamma) \left\{ \frac{\tau}{\nu} \sum_{n=1}^{\infty} n(a_n \sin n\theta' - b_n \cos n\theta') \right. \\ &\quad \left. - \sum_{n=1}^{\infty} n(a'_n \sin n\theta' - b'_n \cos n\theta') \right\} \dots\dots\dots (14) \end{aligned}$$

To satisfy the boundary condition at the vane surface, put the doublets of higher order at the center of the circle. The complex potential F_1 is expressed as

$$F_1 = \sum_{n=1}^{\infty} \frac{C_n}{\zeta^n} + \sum_{n=1}^{\infty} \frac{C'_n}{\zeta'^n} \dots\dots\dots (15)$$

where $C_n = A_n + iB_n$, $C'_n = A'_n + iB'_n$, n positive integers, such as 1, 2, 3, ... and $\zeta' = \zeta / e^{i\theta}$. The radial and the tangential components v_r^* and v_θ^* due to the doublets at the center of circle are obtained as follows;

$$v_r^* = -\sum_{n=1}^{\infty} nA_n \cos n\theta - \sum_{n=1}^{\infty} nB_n \sin n\theta - \sum_{n=1}^{\infty} nA'_n \cos n\theta' - \sum_{n=1}^{\infty} nB'_n \sin n\theta' \dots\dots\dots (16)$$

$$v_\theta^* = -\sum_{n=1}^{\infty} nA_n \sin n\theta + \sum_{n=1}^{\infty} nB_n \cos n\theta - \sum_{n=1}^{\infty} nA'_n \sin n\theta' + \sum_{n=1}^{\infty} nB'_n \cos n\theta' \dots\dots\dots (17)$$

From the boundary condition; $v_r + v_r^* = 0$ (18)

and $\tan \gamma = \frac{\tau}{\nu}$, $\cos \gamma = -\frac{\nu}{\sqrt{\tau^2 + \nu^2}}$, $\sin \gamma = -\frac{\tau}{\sqrt{\tau^2 + \nu^2}}$

the unknown coefficients A_n, A'_n, B_n and B'_n are determined as :

$$\left. \begin{aligned} A_{2n} &= 0, & B_{2n} &= 0 \\ A_{2n+1} &= \frac{2\tau\kappa^{2n+1}(\tau Q - \nu\Gamma_1)}{(2n+1)\pi N(\tau^2 + \nu^2)}, & B_{2n+1} &= -\frac{2\nu\kappa^{2n+1}(\tau Q - \nu\Gamma_1)}{(2n+1)\pi N(\tau^2 + \nu^2)} \\ A'_n &= -\frac{\nu Q + \tau\Gamma_1}{2\pi\nu} \frac{\nu^2}{\tau^2 + \nu^2} \left(\frac{\tau}{\nu} b_n - b'_n \right), & B'_n &= \frac{\nu Q + \tau\Gamma_1}{2\pi\nu} \frac{\nu^2}{\tau^2 + \nu^2} \left(\frac{\tau}{\nu} a_n - a'_n \right) \end{aligned} \right\} \dots (19)$$

Now the tangential velocity $v_\theta + v_\theta^*$ on the circle is not always zero at the point which corresponds to the trailing point A. So putting an unknown circulation Γ_Q around the circle, the complex potential F_2 is expressed as :

$$F_2 = \frac{i\Gamma_Q}{2\pi} \log \frac{\zeta - \kappa}{\zeta - 1/\kappa} \dots (20)$$

Hence the tangential component of velocity due to the circulation Γ_Q on the circle is given as follows ;

$$v_{\theta Q} = -\frac{\Gamma_Q}{2\pi} \frac{1 - \kappa^2}{1 - 2\kappa \cos \theta + \kappa^2} \dots (21)$$

From the Joukowski's hypothesis ;

$$(v_\theta + v_\theta^* + v_{\theta Q})_{\theta=\theta_T} = 0 \dots (22)$$

the unknown quantity of circulation Γ_Q can be determined as follows :

$$\begin{aligned} \frac{\Gamma_Q}{Q} &= \frac{4\kappa(\tau - \nu\frac{\Gamma_1}{Q})}{N(\sqrt{K} - 2\kappa\nu)} + \frac{\nu + \tau\frac{\Gamma_1}{Q}}{\nu} \frac{1 + \kappa^2}{1 - \kappa^2} \frac{\sqrt{K} + 2\kappa\nu}{\sqrt{K}} \frac{\nu^2}{\tau^2 + \nu^2} \left\{ \frac{\tau}{\nu} \sum_{n=1}^{\infty} n a_n - \sum_{n=1}^{\infty} n a'_n \right\} \\ &\quad - \frac{\tau - \nu\frac{\Gamma_1}{Q}}{\nu} \frac{1 + \kappa^2}{1 - \kappa^2} \frac{\sqrt{K} + 2\kappa\nu}{\sqrt{K}} \frac{\nu^2}{\tau^2 + \nu^2} \left\{ \frac{\tau}{\nu} \sum_{n=1}^{\infty} n b_n - \sum_{n=1}^{\infty} n b'_n \right\} \end{aligned} \dots (23)$$

The velocity v_Q on the circle is expressed as

$$\begin{aligned} v_Q &= v_\theta + v_\theta^* + v_{\theta Q} \\ &= \frac{\nu Q + \tau\Gamma_1}{2\pi N(\tau^2 + \nu^2)} - \frac{4\kappa\sqrt{K} \sin \theta'}{1 - 2\kappa^2 \cos 2\theta + \kappa^4} + \frac{2\kappa(\tau Q - \nu\Gamma_1)}{\pi N(\tau^2 + \nu^2)} \frac{(1 - \kappa^4)(\tau^2 + \nu^2) \cos \theta' + 4\kappa^2 \tau \nu \sin \theta'}{\sqrt{K}(1 + 2\kappa^2 \cos 2\theta + \kappa^4)} \\ &\quad - \frac{\nu Q + \tau\Gamma_1}{2\pi\nu} \frac{\nu^2}{\tau^2 + \nu^2} \left\{ \sum_{n=1}^{\infty} n \left(\frac{\tau}{\nu} a_n - a'_n \right) \cos n\theta' + \sum_{n=1}^{\infty} n \left(\frac{\tau}{\nu} b_n - b'_n \right) \sin n\theta' \right\} \\ &\quad - \frac{\tau Q - \nu\Gamma_1}{2\pi\nu} \frac{\nu^2}{\tau^2 + \nu^2} \left\{ \sum_{n=1}^{\infty} n \left(\frac{\tau}{\nu} a_n - a'_n \right) \sin n\theta' - \sum_{n=1}^{\infty} n \left(\frac{\tau}{\nu} b_n - b'_n \right) \cos n\theta' \right\} \\ &\quad - \frac{\Gamma_Q}{2\pi} \frac{1 - \kappa^2}{1 - 2\kappa \cos \theta + \kappa^2} \end{aligned} \dots (24)$$

4. Boundary Condition in Displacement Flow

Assuming that the impeller is at rest and that the fluid rotates at the angular velocity ω , as shown in Fig. 2 (b), the normal and the tangential velocity components v'_n and v'_t on the vane surface are given respectively as:

$$v'_n = -\omega r \cos \gamma + \delta \omega r \sin \gamma, \quad v'_t = \omega r \sin \gamma + \delta \omega r \cos \gamma \quad \dots\dots\dots(25)$$

Therefore the radial and the circumferential components v'_r and v'_θ , which correspond respectively to v'_n and v'_t , are given as follows;

$$v'_r = v'_n \left| \frac{dz}{d\zeta} \right|, \quad v'_\theta = v'_t \left| \frac{dz}{d\zeta} \right| \quad \dots\dots\dots(26)$$

The former is expressed as

$$v'_r = \frac{\omega r^2}{N} \cos \gamma \frac{4\kappa\sqrt{K} \sin \theta'}{1 - 2\kappa^2 \cos 2\theta' + \kappa^4} - \omega r^2 \sin \gamma \left\{ \tau \sum_{n=1}^{\infty} n(a_n \sin n\theta' - b_n \cos n\theta') - \nu \sum_{n=1}^{\infty} n(a'_n \sin n\theta' - b'_n \cos n\theta') \dots\dots\dots(27) \right.$$

Developing the following terms in the Eq. (27) into the Fourier's series as follows;

$$\begin{aligned} \frac{r^2 \sin \theta'}{1 - 2\kappa^2 \cos 2\theta' + \kappa^4} &= \sum_{n=0}^{\infty} C_n \cos n\theta' + \sum_{n=1}^{\infty} d_n \sin n\theta' \quad \dots\dots\dots(28) \\ r^2 \left\{ \tau \sum_{n=1}^{\infty} n(a_n \sin n\theta' - b_n \cos n\theta') - \nu \sum_{n=1}^{\infty} n(a'_n \sin n\theta' - b'_n \cos n\theta') \right\} \\ &= \sum_{n=0}^{\infty} e_n \cos n\theta' + \sum_{n=1}^{\infty} f_n \sin n\theta' \quad \dots\dots\dots(29) \end{aligned}$$

the radial velocity component v'_r of the Eq. (27) can be expressed as

$$v'_r = \frac{\omega \cos \gamma}{N} 4\kappa\sqrt{K} \left\{ \sum_{n=0}^{\infty} c_n \cos n\theta' + \sum_{n=1}^{\infty} d_n \sin n\theta' \right\} - \omega \sin \gamma \left\{ \sum_{n=0}^{\infty} e_n \cos n\theta' + \sum_{n=1}^{\infty} f_n \sin n\theta' \right\} \quad \dots\dots\dots(30)$$

Putting the doublets of higher order at the center of the circle to cancel the above component v'_r , the following complex potential F_3 is expressed as

$$F_3 = \sum_{n=1}^{\infty} \frac{G_n}{\zeta'^n}, \quad G_n = D_n + iE_n \quad \dots\dots\dots(31)$$

Then the velocity components $v_r'^*$ and $v_\theta'^*$ due to the doublets are as follows;

$$\left. \begin{aligned} v_r'^* &= - \sum_{n=1}^{\infty} nD_n \cos n\theta' - \sum_{n=1}^{\infty} nE_n \sin n\theta' \\ v_\theta'^* &= - \sum_{n=1}^{\infty} nD_n \sin n\theta' + \sum_{n=1}^{\infty} nE_n \cos n\theta' \end{aligned} \right\} \quad \dots\dots\dots(32)$$

$$\text{From the boundary condition; } v'_r + v_r'^* = 0 \quad \dots\dots\dots(33)$$

the coefficients D_n and E_n are determined as

$$\left. \begin{aligned} D_n &= \frac{c_n 4\kappa\sqrt{K}\omega \cos \gamma}{nN} - \frac{e_n \omega \sin \gamma}{n} \\ E_n &= \frac{d_n 4\kappa\sqrt{K}\omega \cos \gamma}{nN} - \frac{f_n \omega \sin \gamma}{n} \end{aligned} \right\} \dots\dots\dots(34)$$

The velocity $v'_\theta + v'^{*}_\theta$ at the point corresponding to the trailing edge is not always zero.

So add an unknown circulation Γ_ω around the circle, then the complex potential function F_4 is expressed as follows;

$$F_4 = \frac{i\Gamma_\omega}{2\pi} \log \frac{\zeta - \kappa}{\zeta - 1/\kappa} \dots\dots\dots(35)$$

then the tangential velocity component v_{Γ_ω} due to the circulation is given as

$$v_{\Gamma_\omega} = -\frac{\Gamma_\omega}{2\pi} \frac{1 - \kappa^2}{1 - 2\kappa \cos \theta + \kappa^2} \dots\dots\dots(36)$$

From the Joukowski's hypothesis;

$$(v'_\theta + v'^{*}_\theta + v_{\Gamma_\omega})_{\theta=\theta_T} = 0 \dots\dots\dots(37)$$

the unknown magnitude of circulation Γ_ω is determined as follows:

$$\frac{\Gamma_\omega}{\omega} = 2\pi \frac{4\kappa\sqrt{K}\omega \cos \gamma \cdot K'}{(1 - \kappa^2)N} \sum_1^\infty d_n + 2\pi \frac{\cos \gamma \cdot K'}{1 - \kappa^2} \left\{ \tau \sum_{n=1}^\infty b_n - \nu \sum_{n=1}^\infty b'_n \right\} + 2\pi \frac{\sin \gamma \cdot K'}{1 - \kappa^2} \sum_{n=1}^\infty f_n \dots\dots(38)$$

where

$$K' = 1 - 2\kappa \cos \theta_T + \kappa^2.$$

Hence the velocity v_ω on the circle is expressed as

$$\begin{aligned} v_\omega &= v'_\theta + v'^{*}_\theta + v_{\Gamma_\omega} \\ &= -\frac{4\kappa\sqrt{K}\omega \sin \gamma}{N} \frac{r^2 \sin \theta'}{1 - 2\kappa^2 \cos 2\theta + \kappa^4} - \omega \nu \cos \gamma \left\{ \sum_{n=1}^\infty e_n \cos n\theta' + \sum_{n=1}^\infty f_n \sin n\theta' \right\} \\ &\quad + \frac{4\kappa\sqrt{K}\omega \cos \gamma}{N} \left\{ \sum_{n=1}^\infty d_n \cos n\theta' - \sum_{n=1}^\infty c_n \sin n\theta' \right\} \\ &\quad + \omega \nu \sin \gamma \left\{ \sum_{n=1}^\infty e_n \sin n\theta' - \sum_{n=1}^\infty f_n \cos n\theta' \right\} - \frac{\Gamma_\omega}{2\pi} \frac{1 - \kappa^2}{1 - 2\kappa \cos \theta + \kappa^2} \dots\dots\dots(39) \end{aligned}$$

5. Hydrodynamic Characteristics of Impeller

As shown in the preceding paragraphs, the circulations Γ_Q and Γ_ω can be calculated by the Eqs. (23) and (38) respectively. Hence an impeller vane has the circulation $\Gamma = \Gamma_Q + \Gamma_\omega$. So the theoretical head H_{th} of the impeller is calculated, as is well-known, by the following relation:

$$H_{th} = \frac{N\Gamma\omega}{2\pi g} \dots\dots\dots(40)$$

Then the head coefficient ψ_{th} is expressed as

$$\psi_{th} = H_{th} \left/ \frac{u_2^2}{g} \right. = N \left\{ \left(\frac{\Gamma_Q}{Q} \right) \varphi + \frac{1}{2\pi} \left(\frac{\Gamma_\omega}{\omega} \right) \right\} \dots\dots\dots(41)$$

where $\varphi(=c_{m_2}/u_2)$ is the flow rate coefficient, and the head versus flow quantity curve of impeller can be plotted in dimensionless form.

The relative velocity w along the vane surfaces can be calculated by the tangential velocities v_Q and v_ω , as given by the Eqs. (24) and (39), follows:

$$w = w_Q + w_\omega = (v_Q + v_\omega) \left| \frac{d\zeta}{dz} \right| \quad \dots\dots\dots (42)$$

From the shockless entrance condition at the leading edge:

$$(v_Q + v_\omega)_{\theta=\theta_T+\pi} = 0 \quad \dots\dots\dots (43)$$

the flow rate coefficient φ_e at the shockless entrance condition can be determined. At last the pressure distribution along the vane surfaces can be expressed as

$$\frac{\Delta H}{u_2^2/g} = \frac{p-p_T}{\rho u_2^2} = \frac{1}{2} \left\{ \left(\frac{r}{r_2} \right)^2 - \left(\frac{w}{u_2} \right)^2 \right\} \quad \dots\dots\dots (44)$$

6. Test Apparatus and Experimental Method

Fig. 3 shows a photograph of the test apparatus and Fig. 4 a diagrammatic sketch. The main parts of the apparatus are made of transparent Acryl Resin.

In Fig. 4, *A* indicates the rotating manometers for measuring the pressure distribution along the vane surfaces, which consist of glass tubes and rotate with

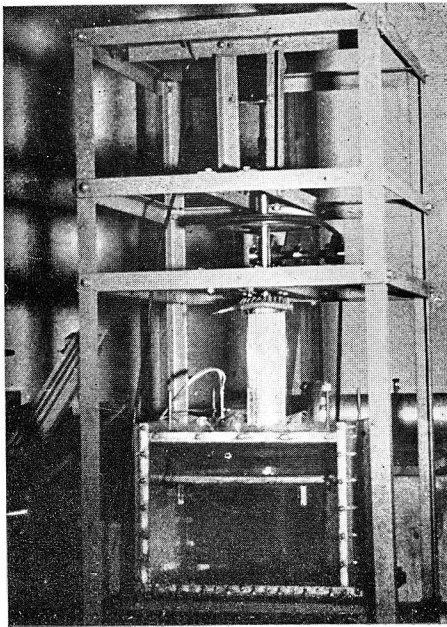


Fig. 3

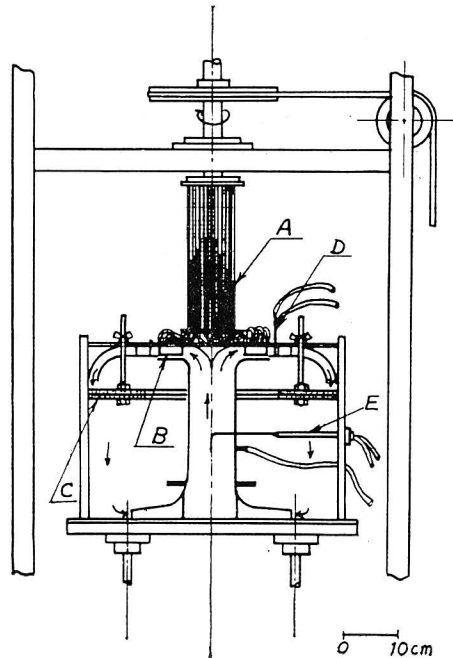


Fig. 4

the impeller. *B* is a test impeller, made of Acryl Resin, and it has an outer diameter of 200 mm, and an inner diameter of 120 mm.

The holes for measuring the pressure have diameters of 0.5 mm and are drilled at the midsection of the impeller breadth. The vane has 11 holes on the front surface and 12 holes on the back surface.

Hence one piece of the manometers *A* shows the total pressure at the inlet of the impeller and the other 23 pieces indicate the pressure distribution along the vane surfaces. Fig. 5 shows a photograph of joints between the pressure taps and manometers.

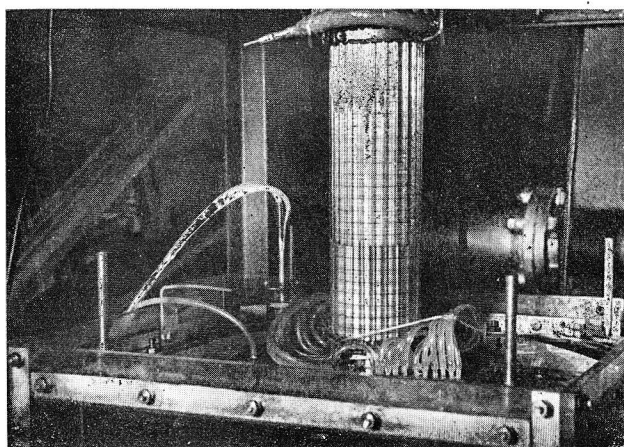


Fig. 5

The impeller rotates in the tank at 200 revolutions per minute, the tank having a plate *C* for regulating the flow quantity. The flow quantity *Q* is controlled by shifting of the regulating plate *C*, the fluid circulating in the tank in the direction of arrow as shown in Fig. 4. The direction and total pressure of flow at the inlet of the impeller can be measured by the cylindrical Pitot-tube *D*, and the flow quantity *Q* can be measured by the traversing of Pitot-tube *E* in the suction pipe.

The experiments were performed at various discharges under a definite speed of revolution. The total pressures of flow at the inlet and exit of the impeller were measured with the Pitot-tube, and the pressure distribution indicated on rotating manometers was observed by a synchronized strobo-light.

7. Comparison between the Theoretical and Experimental Results

The calculations were performed on the following three impellers;

- (i) impeller with logarithmic spiral vanes; $N=6$, $\beta=30^\circ$ and $r_1/r_2=0.6$
- (ii) impeller with circular arc vanes, based upon the logarithmic spiral of the impeller (i); $N=6$, $r_1/r_2=0.6$, $\beta_1=21^\circ$ and $\beta_2=35^\circ$

- (iii) impeller with airfoil vanes of modified NACA 6409 airfoils based upon the impeller (i) ; $N=6$, $r_1/r_2=0.6$

The test models are shown in Fig. 6. The impellers were tested on the test bed, as shown in Figs. 3 and 4, and the characteristics and the pressure distributions along the vane surfaces were measured, as shown in Figs. 7, 8, 9 and 10 respectively. The theoretical curve, calculated by the Eq. (41) and the shockless flow rate coefficient φ_s by the Eq. (43) are plotted in Fig. 7, especially the latter being indicated by the mark $\odot$. The efficiency curve is also shown in Fig. 7.

The pressure distributions along the vane surfaces are shown with respect to the radius ratio r/r_2 according to the Eq. (44), in Figs. 8, 9 and 10, and the distributions along the front and back surfaces are indicated by the full and broken lines, respectively. Especially the pressure distributions at the flow condition of maximum efficiency are shown by the thick lines. As the impellers with the vanes of circular arc and airfoil are based upon the other impeller (i) with vanes of logarithmic spiral, the pressure distributions along them differ a little from the

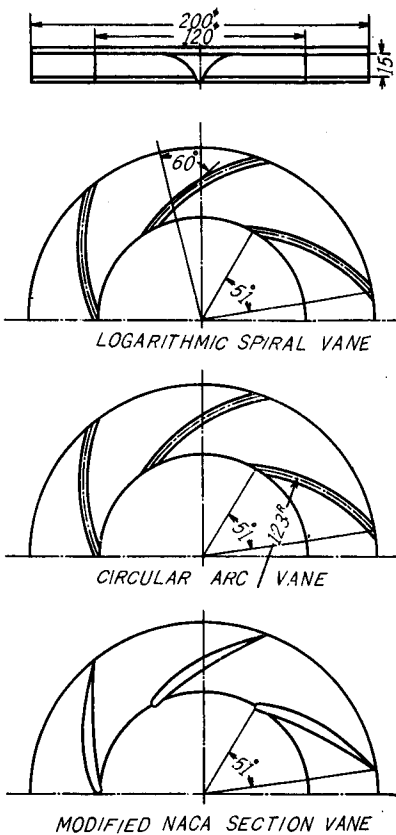


Fig. 6

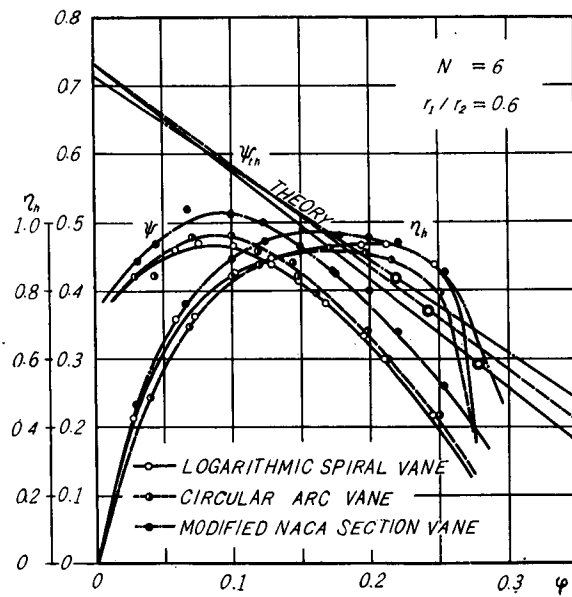


Fig. 7

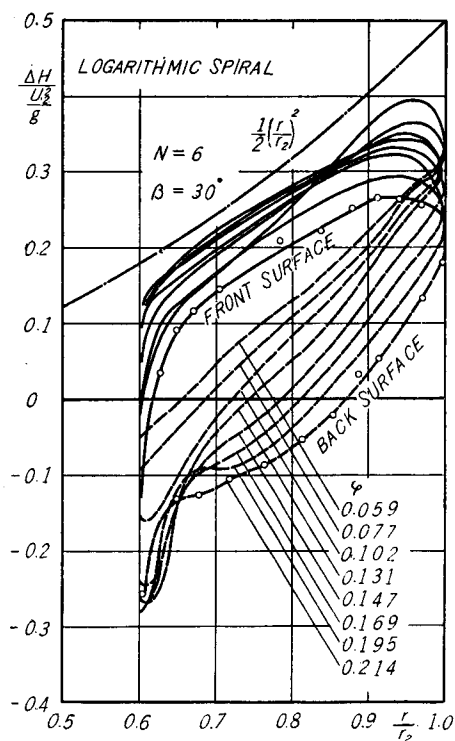


Fig. 8

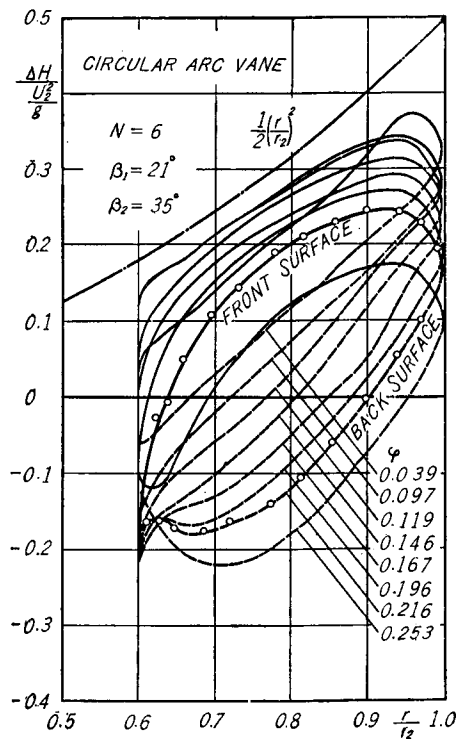


Fig. 9

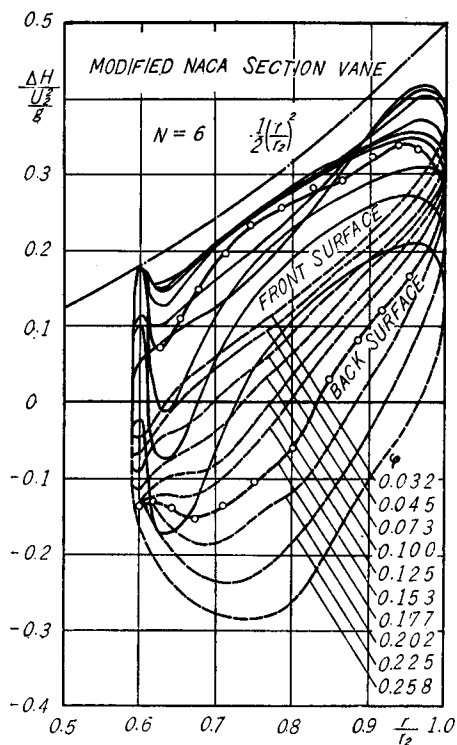


Fig. 10

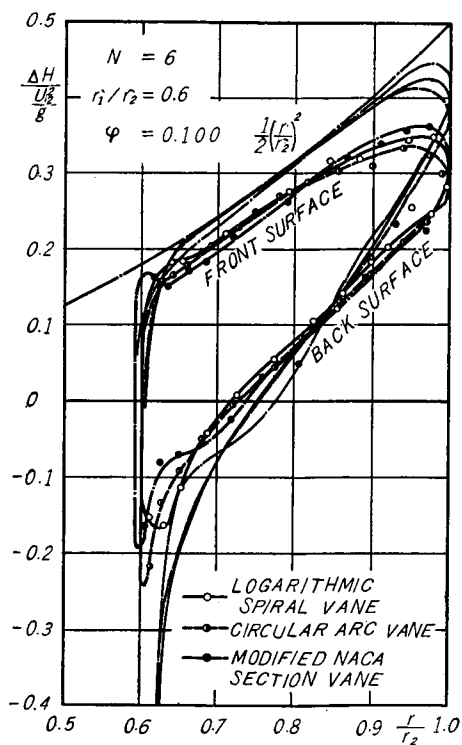


Fig. 11

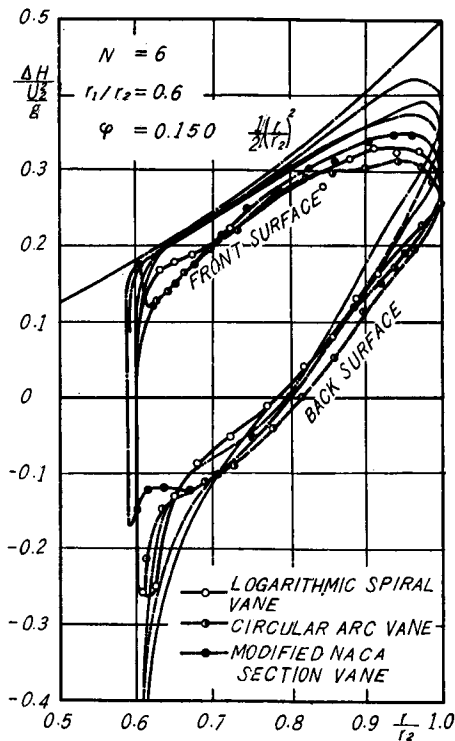


Fig. 12

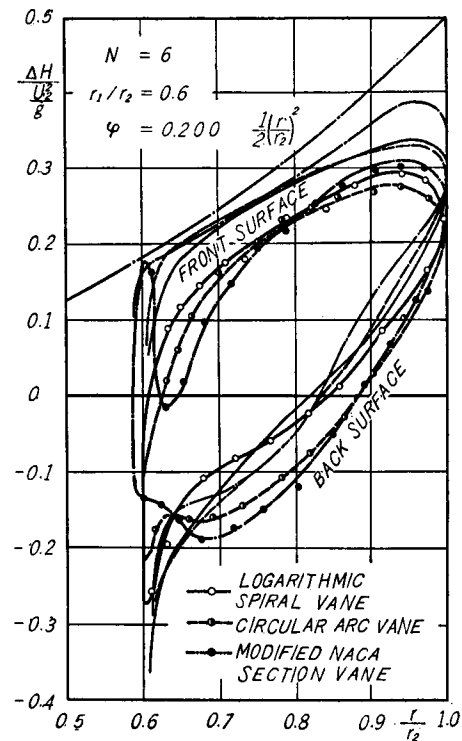


Fig. 13

other. But the distributions around the airfoil vanes show considerably lower pressure near the leading edge, especially on the front surface, because this impeller vane has the thick leading edge near the inner radius of impeller. Moreover the pressure at the holes, located near the leading edge, often shows the stagnation pressure because of the bluntness of the leading edge.

On the other hand, Figs. 11, 12, and 13 show the theoretical pressure distributions, calculated by the above obtained theory, at the flow rate coefficients $\varphi=0.100$, 0.150 and 0.200 . The distributions are shown by the fine lines and compared with the above test results. As shown in another report of the authors¹⁴⁾, the measured pressure distributions show lower values than the theoretical ones owing to the viscosity of the fluid.

8. Conclusion

A hydrodynamic theory of the centrifugal type impeller with the vanes of arbitrary form has been developed by using conformal mapping under the assumption that the flow is a two-dimensional potential one. The calculated results by this theory agree well with the test results.

Chapter 2 Theory of Centrifugal Type Impeller by Singularity Method

This chapter deals with the two-dimensional theory of the impeller of centrifugal type hydraulic machinery, which is based on the singularity method. This theory has such a merit that the vortex, velocity and pressure distributions can be calculated, even if the impeller has a great number of vanes, without difficulty. The calculations were performed with several impellers and their results were compared respectively with the experimental ones or the calculated ones by the conformal mapping method.

1. Introduction

As expressed in the former chapter, the two-dimensional theory of the impeller of centrifugal type hydraulic machinery has been investigated by many people and various methods. Though a method of conformal mapping is the most exact one of them, it can not produce desirable results in the case of a great number of vanes, small vane angle and small inner-outer radii ratio of the impeller.

Therefore the theory of the two-dimensional impeller, explained here, deals with the singularity method which is resolved by distributing the vortices on the vane surface.

As is well known, this method has been developed by Birnbaum¹⁵⁾ on the single airfoil and Schlichting¹⁶⁾ on the straight wing lattice. Recently W. H. Isay⁸⁾ has reported a theory of rotating circular wing lattices with a method similar to ours, but our method has the point of advantage that it can be applied to the theory of impeller with splitter vanes, without difficulty.

In this paper, the theories of three kinds of impellers with vanes of logarithmic spiral forms, arbitrary ones and with splitter vanes are developed under the assumption that the flow in the impeller is a two-dimensional potential one. And the theoretical vortex, velocity and pressure distributions along the vane surfaces of the impeller are calculated for several impellers.

2. Theory of Impeller with Vanes of Logarithmic Spiral Forms

As shown in Fig. 14, first consider one of the vanes with logarithmic spiral forms and vane angle β . Take the two points A and B , leading and trailing edges, and denote r_1 and r_2 , inner and outer radius, respectively, and τ , the angle taken

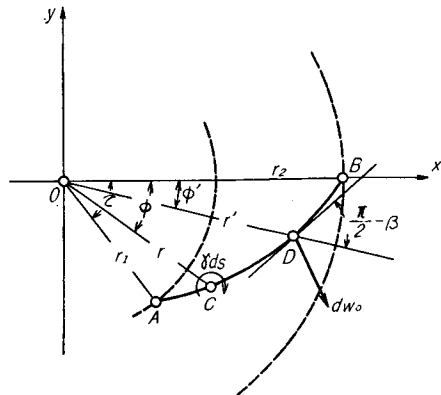


Fig. 14

for the leading and trailing edges to the center of the impeller.

Now, assuming that the vortices are distributed along the vane surfaces, the induced velocity dw_0 at a point $D(r', \phi')$ on the vane surface due to the vortex $\gamma(\phi)ds$ at an arbitrary point $C(r, \phi)$ on the vane surface is expressed from Biot-Savart's law, as follows.

$$dw_0 = \frac{1}{2\pi} \frac{\gamma(\phi)ds}{\sqrt{r^2 + r'^2 - 2rr' \cos(\phi - \phi')}} \quad \dots\dots\dots(45)$$

where $r = e^{\phi \tan \beta}$, $\log r = \phi \tan \beta$, $d \log r / d\phi = \tan \beta$

therefore the infinitesimal vane length ds is;

$$\begin{aligned} ds &= \sqrt{(dr)^2 + (r d\phi)^2} = r d\phi \sqrt{1 + \left(\frac{d \log r}{d\phi}\right)^2} \\ &= \sqrt{1 + \tan^2 \beta} r d\phi = \frac{1}{\cos \beta} r d\phi \end{aligned} \quad \dots\dots\dots(46)$$

so Eq. (45) is expressed as follows;

$$dw_0 = \frac{1}{2\pi \cos \beta} \frac{\gamma(\phi) d\phi}{\sqrt{1 - 2\left(\frac{r'}{r}\right) \cos(\phi - \phi') + \left(\frac{r'}{r}\right)^2}} \quad \dots\dots\dots(47)$$

Next, denote the normal and tangential velocity components of dw_0 on the vane surface, as dv_{n0} and dv_{t0} respectively, so that they are given by Fig. 15 as follows:

$$\begin{aligned} dv_{n0} &= dw_0 \cos\left(\frac{\pi}{2} - \beta - \alpha\right) \\ &= dw_0(\sin \beta \cdot \cos \alpha + \cos \beta \cdot \sin \alpha) \quad \dots(48) \end{aligned}$$

$$\begin{aligned} dv_{t0} &= dw_0 \sin\left(\frac{\pi}{2} - \beta - \alpha\right) \\ &= dw_0(\cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta) \quad \dots(49) \end{aligned}$$

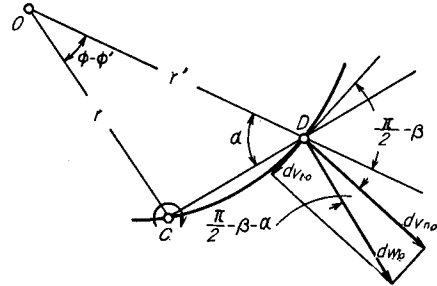


Fig. 15

Then put the Eq. (47) into the Eqs. (48) and (49), so dv_{n0} and dv_{t0} are expressed respectively as follows:

$$dv_{n0} = \frac{1}{2\pi \cos \beta} \frac{\sin \beta \cdot \cos \alpha + \cos \beta \cdot \sin \alpha}{\sqrt{1 - 2\left(\frac{r'}{r}\right) \cos(\phi - \phi') + \left(\frac{r'}{r}\right)^2}} \gamma(\phi) d\phi \quad \dots\dots\dots(50)$$

$$dv_{t0} = \frac{1}{2\pi \cos \beta} \frac{\cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta}{\sqrt{1 - 2\left(\frac{r'}{r}\right) \cos(\phi - \phi') + \left(\frac{r'}{r}\right)^2}} \gamma(\phi) d\phi \quad \dots\dots\dots(51)$$

hence $\sin \alpha$ and $\cos \alpha$ are expressed as follows:

$$\left. \begin{aligned} \sin \alpha &= \frac{\sin (\phi - \phi')}{\sqrt{1 - 2\left(\frac{r'}{r}\right) \cos (\phi - \phi') + \left(\frac{r'}{r}\right)^2}} \\ \cos \alpha &= \frac{\cos (\phi - \phi') - \frac{r'}{r}}{\sqrt{1 - 2\left(\frac{r'}{r}\right) \cos (\phi - \phi') + \left(\frac{r'}{r}\right)^2}} \end{aligned} \right\} \dots\dots\dots(52)$$

Therefore dv_{n0} and dv_{t0} can be expressed ultimately as follows:

$$dv_{n0} = \frac{1}{2\pi} \frac{\tan \beta \left\{ \cos (\phi - \phi') - \frac{r'}{r} \right\} + \sin (\phi - \phi')}{1 - 2\left(\frac{r'}{r}\right) \cos (\phi - \phi') + \left(\frac{r'}{r}\right)^2} r(\phi) d\phi \dots\dots\dots(53)$$

$$dv_{t0} = \frac{1}{2\pi} \frac{\cos (\phi - \phi') - \frac{r'}{r} - \tan \beta \cdot \sin (\phi - \phi')}{1 - 2\left(\frac{r'}{r}\right) \cos (\phi - \phi') + \left(\frac{r'}{r}\right)^2} r(\phi) d\phi \dots\dots\dots(54)$$

So, denote the induced velocity at a point D by the distributed vortices on a vane surface, as w_0 , and the normal and tangential velocity components of it, as v_{n0} and v_{t0} respectively.

Their velocity components are expressed as follows:

$$v_{n0} = \frac{1}{2\pi} \int_0^\tau \frac{\tan \beta \left\{ \cos (\phi - \phi') - \frac{r'}{r} \right\} + \sin (\phi - \phi')}{1 - 2\left(\frac{r'}{r}\right) \cos (\phi - \phi') + \left(\frac{r'}{r}\right)^2} r(\phi) d\phi \dots\dots\dots(55)$$

$$v_{t0} = \frac{1}{2\pi} \int_0^\tau \frac{\cos (\phi - \phi') - \frac{r'}{r} - \tan \beta \cdot \sin (\phi - \phi')}{1 - 2\left(\frac{r'}{r}\right) \cos (\phi - \phi') + \left(\frac{r'}{r}\right)^2} r(\phi) d\phi \dots\dots\dots(56)$$

Next, denote the number of vanes as N , then the velocity components v_n and v_t , which is induced at a point D on the No. 0 vane surface by the distributed vortices on the N sheets of vanes, are expressed respectively as follows:

$$v_n = \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\tan \beta \left\{ \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) - \frac{r'}{r} \right\} + \sin \left(\phi - \frac{2n\pi}{N} - \phi' \right)}{1 - 2\left(\frac{r'}{r}\right) \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r}\right)^2} r(\phi) d\phi \dots\dots(57)$$

$$v_t = \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) - \frac{r'}{r} - \tan \beta \cdot \sin \left(\phi - \frac{2n\pi}{N} - \phi' \right)}{1 - 2\left(\frac{r'}{r}\right) \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r}\right)^2} r(\phi) d\phi \dots\dots(58)$$

where n is the vane number; such as 0, 1, 2, ... and $N-1$.

Now, the fluid can not flow through the vane surface, so the boundary condition must be satisfied between v_n and the normal velocity components v'_n upon the

vane surface, which is induced by displacement and source flow in the impeller, as follows:

$$v_n + v'_n = 0 \quad \dots\dots\dots(59)$$

From Fig. 16 v'_n is expressed as

$$\begin{aligned} v'_n &= \frac{Q}{2\pi r'} \cos \beta - r' \omega \sin \beta \\ &= \frac{Q}{2\pi} \cos \beta \cdot e^{-\phi' \tan \beta} - \omega \sin \beta \cdot e^{\phi' \tan \beta} \quad \dots\dots(60) \end{aligned}$$

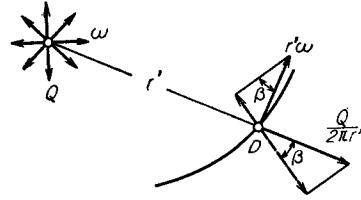


Fig. 16

Put the Eqs. (57) and (60) into the Eq. (59), and it can be expressed in dimensionless form as follows:

$$\begin{aligned} &\sin \beta \cdot e^{\phi' \tan \beta} - \varphi \cos \beta \cdot e^{-\phi' \tan \beta} \\ &= \frac{1}{2\pi} \int_0^{\tau} \sum_{n=0}^{N-1} \frac{\tan \beta \left\{ \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) - \frac{r'}{r} \right\} + \sin \left(\phi - \frac{2n\pi}{N} - \phi' \right) \gamma(\phi)}{1 - 2 \left(\frac{r'}{r} \right) \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r} \right)^2} \frac{\gamma(\phi)}{u_2} d\phi \quad \dots\dots(61) \end{aligned}$$

where u_2 is the circumferential velocity at the outer radius of impeller, and φ is the flow rate coefficient.

The Eq. (61) is a kind of integral equation in which the vortex $\gamma(\phi)/u_2$ is unknown, and it can be calculated by numerical solution.

Hence, from Joukowski's hypothesis $\gamma(\phi)/u_2$, should be zero at the trailing edge of the vane.

Next, putting the value $\gamma(\phi)/u_2$ determined by the Eq. (61) into the Eq. (58), and integrate it numerically, the velocity component v_t/u_2 along the vane surface is determined.

And the velocity component v'_t/u_2 along the vane surface, induced by the displacement and source flow in the impeller, is expressed as follows:

$$\frac{v'_t}{u_2} = \varphi \sin \beta \cdot e^{-\phi' \tan \beta} + \cos \beta \cdot e^{\phi' \tan \beta} \quad \dots\dots\dots(62)$$

Therefore the required velocity w/u_2 along the vane surface is determined ultimately from the following equation:

$$\frac{w}{u_2} = \frac{v_t}{u_2} + \frac{v'_t}{u_2} \pm \frac{\gamma(\phi)}{2u_2} \quad \dots\dots\dots(63)$$

where the positive and negative signs in the third term indicate the back and front vane surfaces respectively.

Next, the pressure coefficient, which expresses the pressure distribution along the vane surface, can be defined by Bernoulli's theorem in the flow of a rotating tube as follows;

$$\frac{\Delta H}{u_2^2/g} = \frac{1}{2} \left\{ \left(\frac{r}{r_2} \right)^2 - \left(\frac{w}{u_2} \right)^2 \right\} \quad \dots\dots\dots(64)$$

Moreover, as an impeller vane has the circulation Γ which is determined by the vortex distribution, the theoretical head H_{th} of the impeller is calculated, as is well-known, by the following relation :

$$H_{th} = \frac{N\Gamma\omega}{2\pi g} \quad \dots\dots\dots(65)$$

3. Theory of Impeller with Vanes of Arbitrary Form.

Next, when the impeller vane has the form which can be expressed by the polar coordinate (r, ϕ) , the vane angle β is not constant generally.

Thereupon the vane angle β at a point (r', ϕ') on the vane surface is expressed as,

$$\beta = \tan^{-1} \left(\frac{1}{r'} \frac{dr'}{d\phi'} \right) \quad \dots\dots\dots(66)$$

By a similar procedure to the one applied to the impeller with the vanes of logarithmic spiral forms, the integral equation in order to determine the vortex distribution $\gamma(\phi)/u_2$ is expressed as follows :

$$\begin{aligned} & \frac{dr'}{d\phi'} - \frac{\varphi}{r'} \\ &= \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\frac{dr'}{r'd\phi'} \left\{ \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) - \frac{r'}{r} \right\} + \sin \left(\phi - \frac{2n\pi}{N} - \phi' \right) \sqrt{\left(\frac{dr}{r d\phi} \right)^2 + 1}}{1 - 2 \left(\frac{r'}{r} \right) \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r} \right)^2} \sqrt{\left(\frac{dr}{r d\phi} \right)^2 + 1} \frac{\gamma(\phi)}{u_2} d\phi \end{aligned} \quad \dots\dots\dots(67)$$

and the velocity component v_t/u_2 along the vane surface, which is induced at a point of the vane surface by the vortex distributions, is expressed as,

$$\frac{v_t}{u_2} = \frac{1}{2\pi} \int_0^\tau \sqrt{\left(\frac{dr}{r d\phi} \right)^2 + 1} \sum_{n=0}^{N-1} \frac{\cos \left\{ \phi - \frac{2n\pi}{N} - \phi' \right\} - \frac{r'}{r} \frac{dr'}{r'd\phi'} \sin \left(\phi - \frac{2n\pi}{N} - \phi' \right)}{1 - 2 \left(\frac{r'}{r} \right) \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r} \right)^2} \frac{\gamma(\phi)}{u_2} d\phi \quad \dots\dots\dots(68)$$

and v'_t/u_2 is expressed as,

$$v'_t/u_2 = \frac{1}{\sqrt{\left(\frac{dr'}{r'd\phi'} \right)^2 + 1}} \left(\frac{\varphi}{r'} \frac{dr'}{d\phi'} + r' \right) \quad \dots\dots\dots(69)$$

Therefore the velocity w/u_2 , the pressure coefficient $\Delta H/(u_2^2/g)$ along the vane surface and theoretical head can be calculated by the Eqs. (63), (64) and (65) respectively.

4. Theory of Impeller with Splitter Vanes

At first, we must deal with one of both the principal vanes and the splitter ones of the impeller, in the same way as before.

Now, assume that both the vane forms consist of logarithmic spirals, and the splitter vanes are put one by one in the middle part between two principal vanes. Sufficing m for the splitter vane, denote $A, B, A_m, B_m, r_1, r_2, r_{1m}, r_{2m}, \tau$ and τ_m as shown in Fig. 17.

Then the induced velocity dw_{00} at a point $D(r', \phi')$ on the principal vane surface due to the vortex $\gamma(\phi)ds$ at an arbitrary point $C(r, \phi)$ on the principal vane surface is the same as Eq. (47).

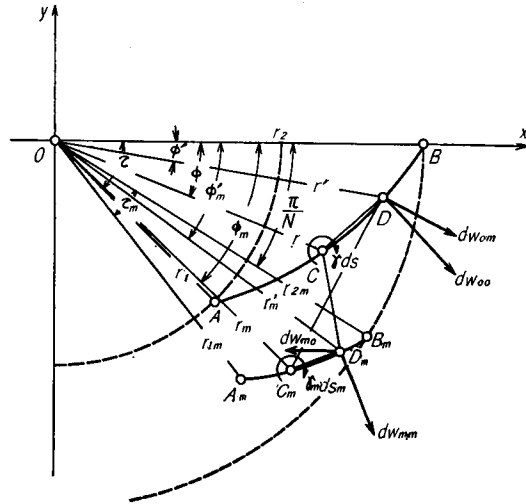


Fig. 17

Next, the induced velocity dw_{0m} at a point $D(r', \phi')$ on the principal vane surface due to the vortex $\gamma_m(\phi_m)ds_m$ at an arbitrary point $C_m(r_m, \phi_m)$ on the splitter vane surface is expressed as follows:

$$dw_{0m} = \frac{1}{2\pi} \frac{\gamma_m(\phi_m)ds_m}{\sqrt{r_m^2 + r'^2 - 2r_mr' \cos(\phi_m - \phi')}} \quad \dots\dots\dots(70)$$

where $r_m = e^{(\phi_m + \frac{\pi}{N}) \tan \beta}$, $\log r_m = (\phi_m + \frac{\pi}{N}) \tan \beta$, $\frac{d \log r_m}{d\phi_m} = \tan \beta$

therefore the infinitesimal length ds_m of the splitter vane is,

$$\begin{aligned} ds_m &= \sqrt{(dr_m)^2 + (r_m d\phi_m)^2} = r_m d\phi_m \sqrt{1 + \left(\frac{d \log r_m}{d\phi_m}\right)^2} \\ &= \sqrt{1 + \tan^2 \beta} r_m d\phi_m = \frac{1}{\cos \beta} r_m d\phi_m \quad \dots\dots\dots(71) \end{aligned}$$

So, Eq. (70) is expressed as follows;

$$dw_{0m} = \frac{1}{2\pi \cos \beta} \frac{\gamma_m(\phi_m)d\phi_m}{\sqrt{1 - 2\left(\frac{r'}{r_m}\right) \cos(\phi_m - \phi') + \left(\frac{r'}{r_m}\right)^2}} \quad \dots\dots\dots(72)$$

Next, denote the normal and tangential velocity components of dw_{00} and dw_{0m} , as dv_{n00} , dv_{n0m} , dv_{t00} and dv_{t0m} respectively, so that they are given as follows:

$$dv_{n00} = \frac{1}{2\pi} \frac{\tan \beta \left\{ \cos(\phi - \phi') - \frac{r'}{r} \right\} + \sin(\phi - \phi')}{1 - 2\left(\frac{r'}{r}\right) \cos(\phi - \phi') + \left(\frac{r'}{r}\right)^2} \gamma(\phi)d\phi \quad \dots\dots\dots(73)$$

$$dv_{n0m} = \frac{1}{2\pi} \frac{\tan \beta \left\{ \cos (\phi_m - \phi') - \frac{r'}{r_m} \right\} + \sin (\phi_m - \phi')}{1 - 2 \left(\frac{r'}{r_m} \right) \cos (\phi_m - \phi') + \left(\frac{r'}{r_m} \right)^2} \gamma_m(\phi_m) d\phi_m \quad \dots\dots\dots (74)$$

$$dv_{t00} = \frac{1}{2\pi} \frac{\cos (\phi - \phi') - \frac{r'}{r} - \tan \beta \cdot \sin (\phi - \phi')}{1 - 2 \left(\frac{r'}{r} \right) \cos (\phi - \phi') + \left(\frac{r'}{r} \right)^2} \gamma(\phi) d\phi \quad \dots\dots\dots (75)$$

$$dv_{t0m} = \frac{1}{2\pi} \frac{\cos (\phi_m - \phi') - \frac{r'}{r_m} - \tan \beta \cdot \sin (\phi_m - \phi')}{1 - 2 \left(\frac{r'}{r_m} \right) \cos (\phi_m - \phi') + \left(\frac{r'}{r_m} \right)^2} \gamma_m(\phi_m) d\phi_m \quad \dots\dots\dots (76)$$

Now denote the induced velocity at a point D due to the distributed vortices on a principal vane and a splitter vane, as w_0 , and the normal and tangential velocity components of it, as v_{n0} and v_{t0} respectively. Their velocity components are expressed as follows:

$$v_{n0} = \frac{1}{2\pi} \int_0^\tau \frac{\tan \beta \left\{ \cos (\phi - \phi') - \frac{r'}{r} \right\} + \sin (\phi - \phi')}{1 - 2 \left(\frac{r'}{r} \right) \cos (\phi - \phi') + \left(\frac{r'}{r} \right)^2} \gamma(\phi) d\phi \\ + \frac{1}{2\pi} \int_{-\frac{\pi}{N}}^{\tau_m - \frac{\pi}{N}} \frac{\tan \beta \left\{ \cos (\phi_m - \phi') - \frac{r'}{r_m} \right\} + \sin (\phi_m - \phi')}{1 - 2 \left(\frac{r'}{r_m} \right) \cos (\phi_m - \phi') + \left(\frac{r'}{r_m} \right)^2} \gamma_m(\phi_m) d\phi_m \quad \dots\dots\dots (77)$$

$$v_{t0} = \frac{1}{2\pi} \int_0^\tau \frac{\cos (\phi - \phi') - \frac{r'}{r} - \tan \beta \cdot \sin (\phi - \phi')}{1 - 2 \left(\frac{r'}{r} \right) \cos (\phi - \phi') + \left(\frac{r'}{r} \right)^2} \gamma(\phi) d\phi \\ + \frac{1}{2\pi} \int_{-\frac{\pi}{N}}^{\tau_m - \frac{\pi}{N}} \frac{\cos (\phi_m - \phi') - \frac{r'}{r_m} - \tan \beta \cdot \sin (\phi_m - \phi')}{1 - 2 \left(\frac{r'}{r_m} \right) \cos (\phi_m - \phi') + \left(\frac{r'}{r_m} \right)^2} \gamma_m(\phi_m) d\phi_m \quad \dots\dots\dots (78)$$

Next, denote both the number of principal and splitter vanes as N . Then the velocity components v_n and v_t , which are induced at a point D on the No. 0 principal vane surface by the distributed vortices on all the principal and the splitter vanes, are expressed respectively as follows:

$$v_n = \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\tan \beta \left\{ \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) - \frac{r'}{r} \right\} + \sin \left(\phi - \frac{2n\pi}{N} - \phi' \right)}{1 - 2 \left(\frac{r'}{r} \right) \cos \left(\phi - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r} \right)^2} \gamma(\phi) d\phi \\ + \frac{1}{2\pi} \int_{-\frac{\pi}{N}}^{\tau_m - \frac{\pi}{N}} \sum_{n=0}^{N-1} \frac{\tan \beta \left\{ \cos \left(\phi_m - \frac{2n\pi}{N} - \phi' \right) - \frac{r'}{r_m} \right\} + \sin \left(\phi_m - \frac{2n\pi}{N} - \phi' \right)}{1 - 2 \left(\frac{r'}{r_m} \right) \cos \left(\phi_m - \frac{2n\pi}{N} - \phi' \right) + \left(\frac{r'}{r_m} \right)^2} \gamma_m(\phi_m) d\phi_m \\ \dots\dots\dots (79)$$

$$\begin{aligned}
 v_t = & \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\cos\left(\phi - \frac{2n\pi}{N} - \phi'\right) - \frac{r'}{r} - \tan\beta \cdot \sin\left(\phi - \frac{2n\pi}{N} - \phi'\right)}{1 - 2\left(\frac{r'}{r}\right) \cos\left(\phi - \frac{2n\pi}{N} - \phi'\right) + \left(\frac{r'}{r}\right)^2} \gamma(\phi) d\phi \\
 & + \frac{1}{2\pi} \int_{-\frac{\pi}{N}}^{\tau_m - \frac{\pi}{N}} \sum_{n=0}^{N-1} \frac{\cos\left(\phi_m - \frac{2n\pi}{N} - \phi'\right) - \frac{r'}{r_m} - \tan\beta \cdot \sin\left(\phi_m - \frac{2n\pi}{N} - \phi'\right)}{1 - 2\left(\frac{r'}{r_m}\right) \cos\left(\phi_m - \frac{2n\pi}{N} - \phi'\right) + \left(\frac{r'}{r_m}\right)^2} \gamma_m(\phi_m) d\phi_m \\
 & \dots\dots\dots(80)
 \end{aligned}$$

where $\tau \leq \phi' \leq 0$, $\tau \leq \phi \leq 0$, $\tau_m - \frac{\pi}{N} \leq \phi_m \leq -\frac{\pi}{N}$ and N is the vane number, such as 0, 1, 2, ... and $N-1$.

In the same way, the velocity components v_{nm} and v_{tm} , which are induced, at a point D_m on the splitter vane, by the distributed vortices on all the vanes, are expressed respectively as follows:

$$\begin{aligned}
 v_{nm} = & \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\tan\beta \left\{ \cos\left(\phi - \frac{2n\pi}{N} - \phi'_m\right) - \frac{r'_m}{r} \right\} + \sin\left(\phi - \frac{2n\pi}{N} - \phi'_m\right)}{1 - 2\left(\frac{r'_m}{r}\right) \cos\left(\phi - \frac{2n\pi}{N} - \phi'_m\right) + \left(\frac{r'_m}{r}\right)^2} \gamma(\phi) d\phi \\
 & + \frac{1}{2\pi} \int_{-\frac{\pi}{N}}^{\tau_m - \frac{\pi}{N}} \sum_{n=0}^{N-1} \frac{\tan\beta \left\{ \cos\left(\phi_m - \frac{2n\pi}{N} - \phi'_m\right) - \frac{r'_m}{r_m} \right\} + \sin\left(\phi_m - \frac{2n\pi}{N} - \phi'_m\right)}{1 - 2\left(\frac{r'_m}{r_m}\right) \cos\left(\phi_m - \frac{2n\pi}{N} - \phi'_m\right) + \left(\frac{r'_m}{r_m}\right)^2} \gamma_m(\phi_m) d\phi_m \\
 & \dots\dots\dots(81)
 \end{aligned}$$

$$\begin{aligned}
 v_{tm} = & \frac{1}{2\pi} \int_0^\tau \sum_{n=0}^{N-1} \frac{\cos\left(\phi - \frac{2n\pi}{N} - \phi'_m\right) - \frac{r'_m}{r} - \tan\beta \cdot \sin\left(\phi - \frac{2n\pi}{N} - \phi'_m\right)}{1 - 2\left(\frac{r'_m}{r}\right) \cos\left(\phi - \frac{2n\pi}{N} - \phi'_m\right) + \left(\frac{r'_m}{r}\right)^2} \gamma(\phi) d\phi \\
 & + \frac{1}{2\pi} \int_{-\frac{\pi}{N}}^{\tau_m - \frac{\pi}{N}} \sum_{n=0}^{N-1} \frac{\cos\left(\phi_m - \frac{2n\pi}{N} - \phi'_m\right) - \frac{r'_m}{r_m} - \tan\beta \cdot \sin\left(\phi_m - \frac{2n\pi}{N} - \phi'_m\right)}{1 - 2\left(\frac{r'_m}{r_m}\right) \cos\left(\phi_m - \frac{2n\pi}{N} - \phi'_m\right) + \left(\frac{r'_m}{r_m}\right)^2} \gamma_m(\phi_m) d\phi_m \\
 & \dots\dots\dots(82)
 \end{aligned}$$

where $\tau_m - \frac{\pi}{N} \leq \phi'_m \leq -\frac{\pi}{N}$

Now, the following boundary conditions must be satisfied in the same way as the former sections 2.

$$v_n + v'_n = 0, \quad v_{nm} + v'_{nm} = 0 \quad \dots\dots\dots(83)$$

Then v'_n and v'_{nm} are expressed respectively as follows:

$$\left. \begin{aligned}
 v'_n &= \frac{Q}{2\pi} \cos\beta \cdot e^{-\phi' \tan\beta} - \omega \sin\beta \cdot e^{\phi' \tan\beta} \\
 v'_{nm} &= \frac{Q}{2\pi} \cos\beta \cdot e^{-\left(\phi'_m + \frac{\pi}{N}\right) \tan\beta} - \omega \sin\beta \cdot e^{\left(\phi'_m + \frac{\pi}{N}\right) \tan\beta}
 \end{aligned} \right\} \quad \dots\dots\dots(84)$$

Then, put the Eq. (84) into the Eq. (83), and it can be expressed in dimensionless form, as follows;

$$\left. \begin{aligned} \frac{v_n}{u_2} + \varphi \cos \beta \cdot e^{-\phi' \tan \beta} - \sin \beta \cdot e^{\phi' \tan \beta} &= 0 \\ \frac{v_{nm}}{u_2} + \varphi \cos \beta \cdot e^{-\left(\phi'_m + \frac{\pi}{N}\right) \tan \beta} - \sin \beta \cdot e^{\left(\phi'_m + \frac{\pi}{N}\right) \tan \beta} &= 0 \end{aligned} \right\} \dots\dots\dots (85)$$

The Eqs. (85) are simultaneous integral equations in which the vortices $\gamma(\phi)/u_2$ and $\gamma_m(\phi_m)/u_2$ are unknown, and they can be calculated by numerical solution.

Hence, from Joukowski's hypothesis $\gamma(\phi)/u_2$ and $\gamma_m(\phi_m)/u_2$ should both be zero at the trailing edges of the principal and splitter vanes.

Next putting the values $\gamma(\phi)/u_2$ and $\gamma_m(\phi_m)/u_2$ determined by the Eq. (85) into the Eqs. (80) and (82), and integrating them numerically, the velocity components v_t/u_2 and v_{tm}/u_2 along both the vane surfaces may be determined. The tangential velocity components v'_t/u_2 and v'_{tm}/u_2 are expressed respectively as follows:

$$\left. \begin{aligned} \frac{v'_t}{u_2} &= \varphi \sin \beta \cdot e^{-\phi' \tan \beta} + \cos \beta \cdot e^{\phi' \tan \beta} \\ \frac{v'_{tm}}{u_2} &= \varphi \sin \beta \cdot e^{-\left(\phi'_m + \frac{\pi}{N}\right) \tan \beta} + \cos \beta \cdot e^{\left(\phi'_m + \frac{\pi}{N}\right) \tan \beta} \end{aligned} \right\} \dots\dots\dots (86)$$

Therefore ultimately the required velocities w/u_2 and w_m/u_2 along the principal and the splitter vane surfaces are determined respectively from the following equation:

$$\left. \begin{aligned} \frac{w}{u_2} &= \frac{v_t}{u_2} + \frac{v'_t}{u_2} \pm \frac{\gamma(\phi)}{2u_2} \\ \frac{w_m}{u_2} &= \frac{v_{tm}}{u_2} + \frac{v'_{tm}}{u_2} \pm \frac{\gamma_m(\phi_m)}{2u_2} \end{aligned} \right\} \dots\dots\dots (87)$$

where the positive and the negative signs in the third term indicate the back and front vane surfaces respectively.

Then the pressure coefficients $\Delta H/g$ and $\Delta H_m/g$ can be calculated respectively as follows:

$$\left. \begin{aligned} \Delta H/g &= \frac{1}{2} \left\{ \left(\frac{r'}{r_2} \right)^2 - \left(\frac{w}{u_2} \right)^2 \right\} \\ \Delta H_m/g &= \frac{1}{2} \left\{ \left(\frac{r'_m}{r_2} \right)^2 - \left(\frac{w_m}{u_2} \right)^2 \right\} \end{aligned} \right\} \dots\dots\dots (88)$$

Moreover, as the circulations Γ and Γ_m of each of the principal and splitter vanes are determined by the vortices $\gamma(\phi)/u_2$ and $\gamma_m(\phi_m)/u_2$, the theoretical head

H_{th} of the impeller with splitter vanes is calculated by the following relation:

$$H_{th} = \frac{N(I' + I'_m)\omega}{2\pi g} \quad \dots\dots\dots (89)$$

5. Theoretical Results

Calculations were performed on the following three impellers:

- (i) impeller with logarithmic spiral vanes; $N=6$, $\beta=30^\circ$ and $r_1/r_2=0.6$
- (ii) impeller with splitter vanes of $2/3$ length of which the forms are logarithmic spirals, both $N=6$, $\beta=30^\circ$ and $r_1/r_2=0.6$
- (iii) impeller with logarithmic spiral vanes; $N=12$, $\beta=30^\circ$ and $r_1/r_2=0.6$

They are shown in Fig. 18.

The vortex, the velocity and the pressure distributions along the vane surfaces, calculated by the Eqs. (61), (63), (64), (86), (87) and (88), are shown in Figs. 19, 20, 21, 22, 23, 24, 25, 26, and 27, respectively, and the experimental pressure distribution of an impeller with $2/3$ splitter vanes is shown in Fig. 28.

Their vortex distributions were calculated by using the KDC-1 Digital Computer except for an impeller with splitter vanes.

In the vortex distributions, the abscissas are the dimensionless vane length s/r_2 , and in the velocity and pressure distributions, the ones are the radius ratio r/r_2 . The chain lines in these pressure distributions show the pressure distribution curve $1/2(r/r_2)^2$ in the case where the fluid rotates with the impeller.

The impeller with 6 vanes is compared by the conformal mapping method, shown by the broken lines, and the results agree well as shown in these figures. Also on the pressure distribution of the impeller with 12 vanes as shown in Fig. 26, the flow rate coefficient $\varphi=0.25$ is the nearly shockless one, so the theoretical curve is compared with the experimental results at flow conditions of maximum efficiency, indicated by the thick broken line.

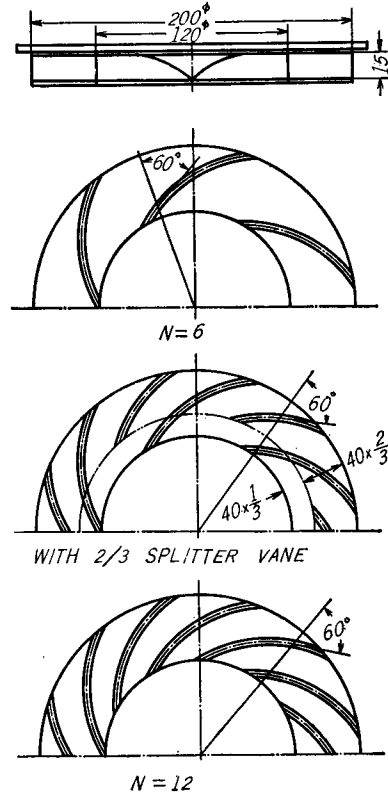


Fig. 18

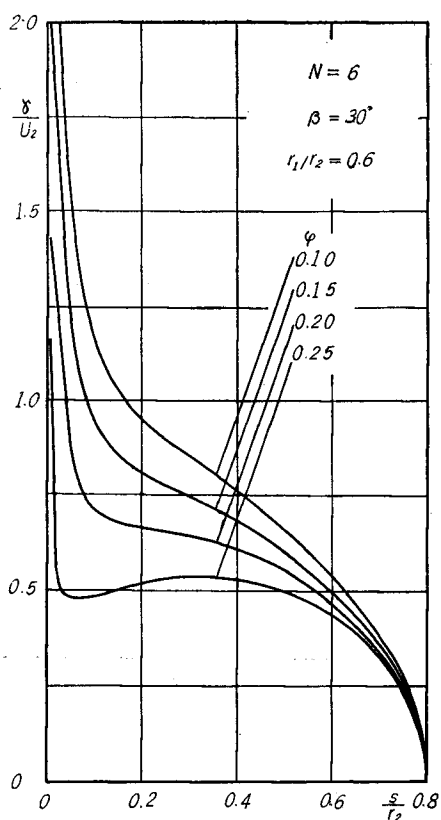


Fig. 19

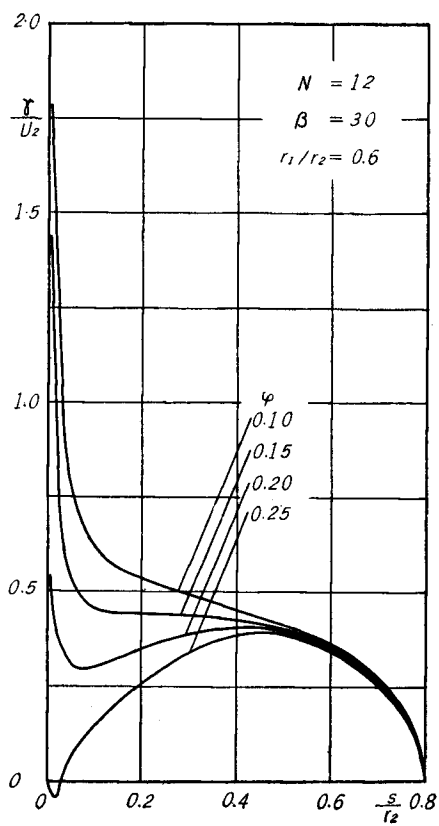


Fig. 20

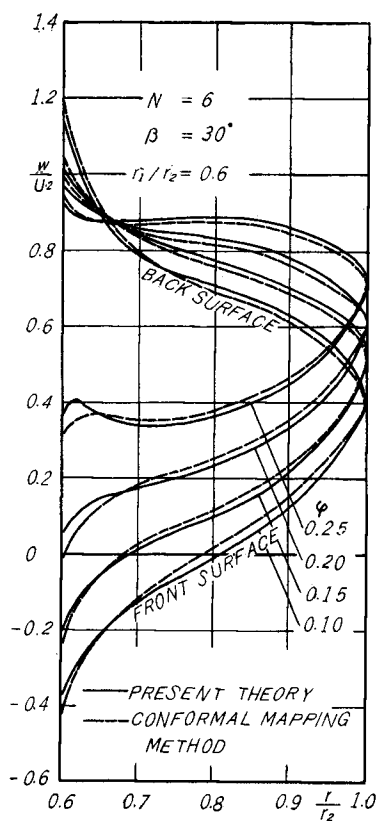


Fig. 21

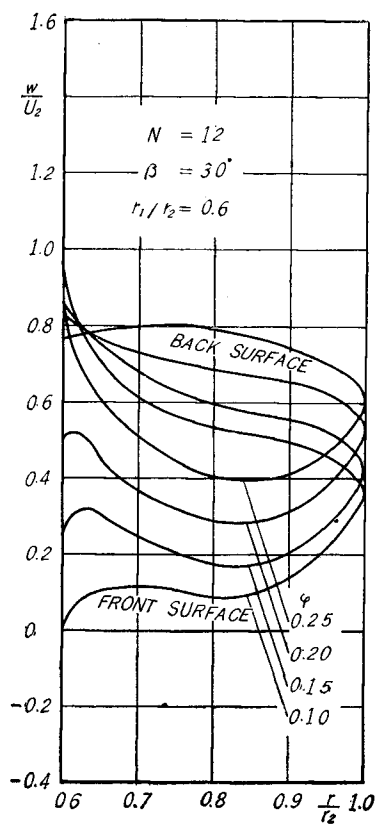


Fig. 22

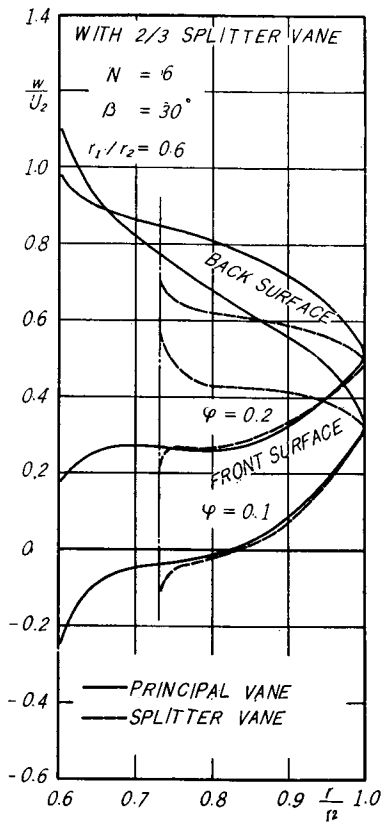


Fig. 23

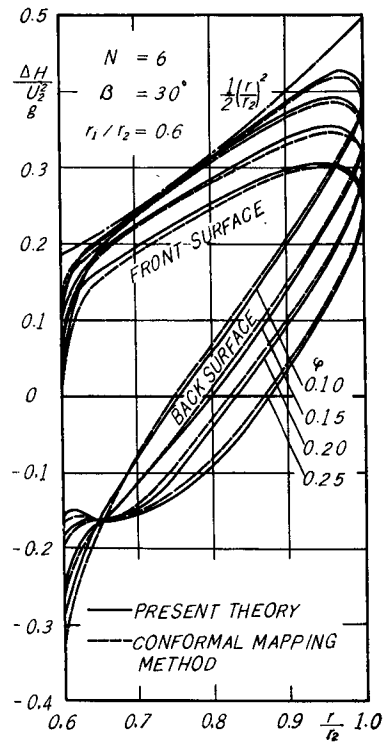


Fig. 24

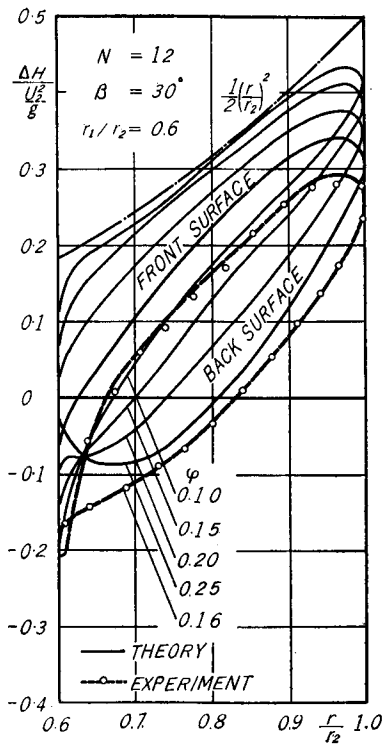


Fig. 25

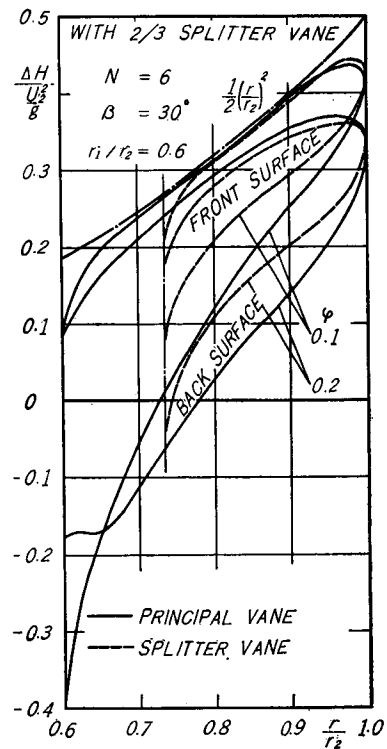


Fig. 26

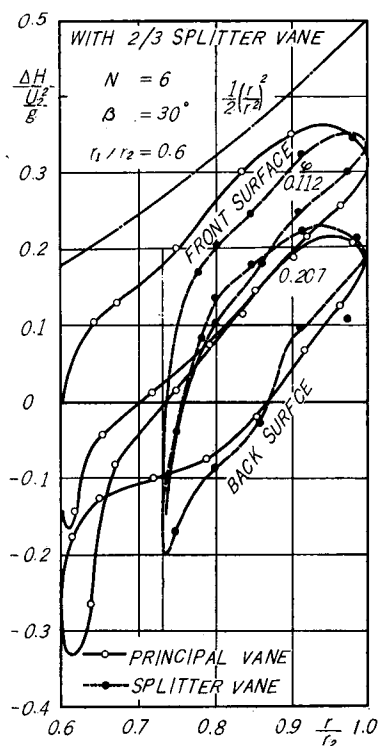


Fig. 27

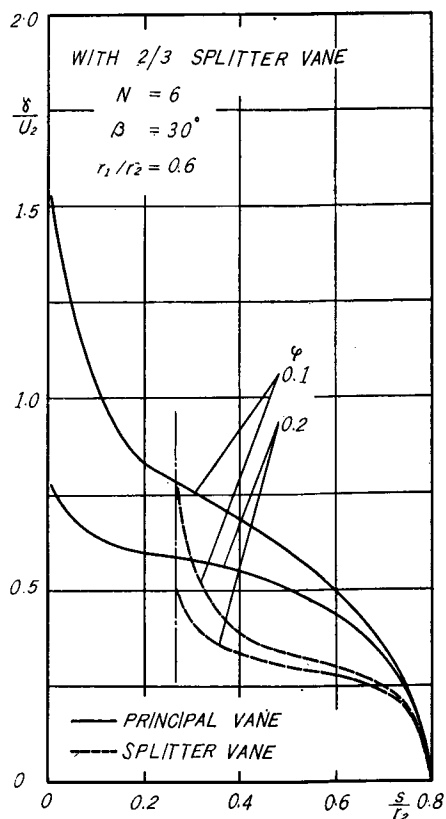


Fig. 28

Though the forms of their distributions are both similar, the measured one shows lower value than the theoretical one owing to the viscosity of the fluid.

And as shown in Figs. 27 and 28, the theoretical pressure distributions of the impeller with 2/3 splitter vanes differ from the experimental ones owing to the viscosity of fluid, as in the former case.

6. Conclusion

In this chapter, the two-dimensional theory of the impeller of centrifugal type hydraulic machinery has been developed by using the singularity method. Though the distributed vortices along the vane surfaces should be calculated by numerical solution, they can be determined exactly by using a digital computer, even if the impeller has a great number of vanes.

The calculated results from this theory agree well with those from the theory of the conformal mapping method or from experimental results.

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