

Integral means for starlike and convex functions with negative coefficients

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Abstract

Let $\mathcal{T}$ be the class of functions $f(z)$ with negative coefficients which are analytic and univalent in the open unit disk $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$. The classes $\mathcal{T}^*$ and $\mathcal{C}$ are defined as the subclasses of $\mathcal{T}$ which are starlike and convex in $\mathbb{U}$, respectively. In view of the interesting results for integral means given by H. Silverman (*Houston J. Math.* **23**(1977)), some generalization theorems are discussed in this paper.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be starlike with respect to the origin in $\mathbb{U}$ if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ with respect to the origin in $\mathbb{U}$. Further, a function $f(z) \in \mathcal{A}$ is said to be convex in $\mathbb{U}$ if it satisfies

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

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We also denote by $\mathcal{K}$ the subclass of $\mathcal{S}$ consisting of $f(z)$ which are convex in $\mathbb{U}$. By the above definitions, we know that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$, and that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$.

The class $\mathcal{T}$ is defined as the subclass of $\mathcal{S}$ consisting of all functions $f(z)$ which are given by

$$(1.4) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

Further, we denote by $\mathcal{T}^* = \mathcal{S}^* \cap \mathcal{T}$ and $\mathcal{C} = \mathcal{K} \cap \mathcal{T}$. It is well-known by Silverman[6] that

Remark 1.1. A function $f(z) \in \mathcal{T}^*$ if and only if

$$(1.5) \quad \sum_{n=2}^{\infty} n a_n \leq 1.$$

A function $f(z) \in \mathcal{C}$ if and only if

$$(1.6) \quad \sum_{n=2}^{\infty} n^2 a_n \leq 1.$$

For $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = g(\omega(z))$. We denote this subordination by

$$(1.7) \quad f(z) \prec g(z). \quad (\text{cf. Duren}[1])$$

For subordinations, Littlewood [2] has given the following integral mean.

Theorem A. If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\lambda > 0$ and $|z| = r$ ($0 < r < 1$),

$$(1.8) \quad \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta.$$

Furthermore, Silverman [6] has shown that

Remark 1.2. $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n}$ ($n \geq 2$) are extreme points of the class $\mathcal{T}^*$ (or $\mathcal{T}$). $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n^2}$ ($n \geq 2$) are extreme points of the class $\mathcal{C}$.

Applying Theorem A with extreme points of $\mathcal{T}$, Silverman [7] has proved the following results.

Theorem B. Suppose that $f(z) \in \mathcal{T}^*$, $\lambda > 0$ and $f_2(z) = z - \frac{z^2}{2}$. Then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(1.9) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta.$$

Theorem C. If $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_2(z) = z - \frac{z^2}{2}$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(1.10) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_2(z)|^\lambda d\theta.$$

In the present paper, we consider the generalization properties for Theorem B and Theorem C with $f(z) \in \mathcal{T}^*$ and $f(z) \in \mathcal{C}$.

Remark 1.3. More recently, applying Theorem A by Littlewood [2], Sekine, Tsurumi and Srivastava [4] and Sekine. Tsurumi, Owa and Srivastava [5] have discussed some interesting properties of integral means inequalities for fractional derivatives of some general subclasses of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Further, Owa and Sekine [3] has considered the integral means with some coefficient inequalities for certain analytic functions $f(z)$ in $\mathbb{U}$.

2 Generalization properties

Our first result for the generalization properties is contained in

Theorem 2.1. Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies

$$(2.1) \quad \sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0$$

for $k \geq 3$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{k-1} = k \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right),$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.$$

Proof. For $f(z) \in \mathcal{T}^*$, we have to show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta.$$

By Theorem A, it suffices to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k}.$$

Let us define the function $\omega(z)$ by

$$(2.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k} (\omega(z))^{k-1}.$$

It follows from (2.3) that

$$|\omega(z)|^{k-1} = \left| k \sum_{n=2}^{\infty} a_n z^{n-1} \right| \leq |z| \left(\sum_{n=2}^{\infty} k a_n \right).$$

Thus, we only show that

$$\sum_{n=2}^{\infty} k a_n \leq \sum_{n=2}^{\infty} n a_n,$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right).$$

Indeed, we see that

$$\begin{aligned} \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right) &= \left(1 - \frac{k-2}{k} \right) a_2 + \left(1 - \frac{k-3}{k} \right) a_3 + \cdots + \left(1 - \frac{2}{k} \right) a_{k-2} \\ &\quad + \left(1 - \frac{1}{k} \right) a_{k-1} + a_k + \left(1 + \frac{1}{k} \right) a_{k+1} + \left(1 + \frac{2}{k} \right) a_{k+2} \\ &\quad + \cdots + \left(1 + \frac{k+1}{k} \right) a_{2k+1} + \left(1 + \frac{k+2}{k} \right) a_{2k+2} + \cdots \\ &= \frac{k-2}{k} (a_{2k-2} - a_2) + \frac{k-3}{k} (a_{2k-3} - a_3) + \cdots + \frac{2}{k} (a_{k+2} - a_{k-2}) + \frac{1}{k} (a_{k+1} - a_{k-1}) \end{aligned}$$

$$+ \left(1 + \frac{k-1}{k}\right) a_{2k-1} + \left(1 + \frac{k}{k}\right) a_{2k} + \left(1 + \frac{k+1}{k}\right) a_{2k+1} + \cdots + \sum_{n=2}^{2k-2} a_n.$$

Nothing that

$$1 + \frac{k+j}{k} \geq 1 + \frac{2+j}{k} \quad (j = -1, 0, 1, \dots),$$

we obtain

$$\begin{aligned}
 (2.4) \quad & \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right) \geq \frac{k-2}{k} (a_{2k-2} - a_2) + \frac{k-3}{k} (a_{2k-3} - a_3) + \cdots \\
 & + \frac{2}{k} (a_{k+2} - a_{k-2}) + \frac{1}{k} (a_{k+1} - a_{k-1}) + \left(1 + \frac{1}{k}\right) a_{2k-1} + \left(1 + \frac{2}{k}\right) a_{2k} + \cdots \\
 & + \left(1 + \frac{k-3}{k}\right) a_{3k-5} + \left(1 + \frac{k-2}{k}\right) a_{3k-4} + \cdots + \sum_{n=2}^{2k-2} a_n \\
 & \geq \frac{1}{k} (a_{2k-1} + a_{k+1} - a_{k-1}) + \frac{2}{k} (a_{2k} + a_{k+2} - a_{k-2}) + \cdots \\
 & + \frac{k-2}{k} (a_{3k-4} + a_{2k-2} - a_2) + \sum_{n=2}^{\infty} a_n \\
 & = \sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) + \sum_{n=2}^{\infty} a_n \\
 & \geq \sum_{n=2}^{\infty} a_n
 \end{aligned}$$

with the following condition

$$\sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0.$$

Thus, we observe that the function $\omega(z)$ defined by (2.3) is analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$). This completes the proof of the theorem. $\square$

Remark 2.1. Taking $k = 2$ in Theorem 2.1, we have Theorem B by Silverman [7].

Example 2.1. Let us define

$$(2.5) \quad f(z) = z - \frac{37}{1200} z^2 - \frac{1}{18} z^3 - \frac{1}{48} z^4 - \frac{1}{100} z^5$$

and

$$(2.6) \quad f_3(z) = z - \frac{1}{3}z^3$$

with $k = 3$ in Theorem 2.1. Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n = \frac{217}{600} < 1,$$

we have $f(z) \in \mathcal{T}^*$. Furthermore, $f(z)$ satisfies,

$$\frac{1}{3}(a_5 + a_4 - a_2) = \frac{1}{3} \left(\frac{1}{100} + \frac{1}{48} - \frac{37}{1200} \right) = 0.$$

Thus, $f(z)$ satisfies the conditions in Theorem 2.1 with $k = 3$.

If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{9}r^4 \right) < \frac{20}{9}\pi = 6.9813 \dots$$

Corollary 2.1. Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies the conditions in Theorem 2.1, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.6) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^2}r^{2(k-1)} \right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k^2} \right)^{\frac{\lambda}{2}}.$$

Proof. It follows that

$$\int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z|^\lambda \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta.$$

Applying Hölder inequality for $0 < \lambda < 2$, we obtain that

$$\begin{aligned} \int_0^{2\pi} |z|^\lambda \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta &\leq \left(\int_0^{2\pi} (|z|^\lambda)^{\frac{2}{2-\lambda}} d\theta \right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left(\left| 1 - \frac{z^{k-1}}{k} \right|^\lambda \right)^{\frac{2}{\lambda}} d\theta \right)^{\frac{\lambda}{2}} \\ &= \left(\int_0^{2\pi} |z|^{\frac{2\lambda}{2-\lambda}} d\theta \right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left| 1 - \frac{z^{k-1}}{k} \right|^2 d\theta \right)^{\frac{\lambda}{2}} \\ &= \left(2\pi r^{\frac{2\lambda}{2-\lambda}} \right)^{\frac{2-\lambda}{2}} \left(2\pi \left(1 + \frac{1}{k^2}r^{2(k-1)} \right) \right)^{\frac{\lambda}{2}} \\ &= 2\pi r^\lambda \left(1 + \frac{1}{k^2}r^{2(k-1)} \right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \frac{1}{k^2} \right)^{\frac{\lambda}{2}}. \end{aligned}$$

Further, it is clear for $\lambda = 2$. □

For the generalization of TheoremC by Silverman [7], we have

Theorem 2.2. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by*

$$(\omega(z))^{k-1} = \sum_{n=2}^{\infty} na_n z^{n-1},$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.7) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta.$$

Proof. For $f(z) \in \mathcal{T}^*$, it is sufficient to show that

$$(2.8) \quad 1 - \sum_{n=2}^{\infty} na_n z^{n-1} \prec 1 - z^{k-1}.$$

Let us define the function $\omega(z)$ by

$$(2.9) \quad 1 - \sum_{n=2}^{\infty} na_n z^{n-1} = 1 - \omega(z)^{k-1},$$

or, by

$$\omega(z)^{k-1} = \sum_{n=2}^{\infty} na_n z^{n-1}.$$

Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n \leq 1,$$

the function $\omega(z)$ is analytic in $\mathbb{U}$, $\omega(0) = 0$, and $|\omega(z)| < 1$ ($z \in \mathbb{U}$). □

Remark 2.2. If we take $k = 2$ in Theorem2.2, then we have TheoremC by Silverman [7].

Using Hölder inequality for Theorem2.2, we have

Corollary 2.2. *Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies the conditions in Theorem2.2, then, for $z = re^{i\theta}$ ($0 < r < 1$),*

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi(1 + r^{2(k-1)})^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}} \pi.$$

3 Integral means for functions in the class $\mathcal{C}$

In this section, we discuss the integral means for functions $f(z)$ in the class $\mathcal{C}$.

Theorem 3.1. *Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies*

$$(3.1) \quad \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) \geq 0$$

for $k \geq 3$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{k-1} = k^2 \sum_{n=2}^{\infty} a_n z^{n-1},$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.$$

Proof. For the proof, we need to show that

$$(3.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k^2}$$

by Theorem A. Define the function $\omega(z)$ by

$$(3.4) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k^2} \omega(z)^{k-1},$$

or by

$$(3.5) \quad (\omega(z))^{k-1} = k^2 \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right).$$

Therefore, we have to show that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right).$$

Using the same technique as in the proof of Theorem 2.1, we see that

$$\begin{aligned} \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right) &\geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n \\ &\geq \sum_{n=2}^{\infty} a_n. \end{aligned}$$

□

Example 3.1. Consider the functions

$$(3.6) \quad f(z) = z - \frac{1}{40}z^2 - \frac{1}{18}z^3 - \frac{1}{40}z^4$$

and

$$(3.7) \quad f_3(z) = z - \frac{1}{9}z^3$$

with $k = 3$ in Theorem 3.1. Then we have that

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{40} + \frac{9}{18} + \frac{16}{40} = 1$$

which implies $f(z) \in \mathcal{C}$, and that

$$\frac{5}{9}(a_4 - a_2) = 0.$$

Thus $f(z)$ satisfies the conditions of Theorem 3.1. If we make $\lambda = 2$, then we see that

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{81}r^4\right) < \frac{164}{81}\pi = 6.3607 \dots$$

Corollary 3.1. Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies the condition in Theorem 3.1, then, for $k \geq 3$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.8) \quad \begin{aligned} \int_0^{2\pi} |f(z)|^\lambda d\theta &\leq 2\pi r^\lambda \left(1 + \frac{1}{k^4}r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \frac{1}{k^4}\right)^{\frac{\lambda}{2}}. \end{aligned}$$

Further, we may have

Theorem 3.2. Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies

$$(3.9) \quad \sum_{j=2}^{2k-2} j(k-j)a_j \leq 0,$$

and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{k-1} = k \sum_{n=2}^{\infty} n a_n z^{n-1},$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.10) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta.$$

Example 3.2. Take the functions

$$(3.11) \quad f(z) = z - \frac{1}{24}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4$$

and

$$(3.12) \quad f_3(z) = z - \frac{1}{9}z^3$$

with $k = 3$ in Theorem 3.2. Since

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{24} + \frac{9}{18} + \frac{16}{48} = \frac{5}{6} < 1$$

and

$$2(3-2)a_2 + 3(3-3)a_3 + 4(3-4)a_4 = \frac{1}{12} - \frac{1}{12} = 0,$$

$f(z)$ satisfies the conditions in Theorem 3.2. If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f'(z)|^2 d\theta \leq 2\pi \left(1 + \frac{1}{9}r^4\right) < \frac{20}{9}\pi.$$

Corollary 3.2. Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies the condition in Theorem 3.2, then, for $k \geq 2$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi \left(1 + \frac{1}{k}r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k}\right)^{\frac{\lambda}{2}}.$$

4 Applications for the integrated functions

For $f(z) \in \mathcal{T}$, we define

$$\begin{aligned} I_0 f(z) &= f(z) = z - \sum_{n=2}^{\infty} a_n z^n \\ I f(z) &= I_1 f(z) = \int_0^z f(t) dt = \frac{1}{2}z^2 - \sum_{n=2}^{\infty} \frac{a_n}{n+1} z^{n+1} \\ I_k f(z) &= I(I_{k-1} f(z)) = \frac{1}{(k+1)!} z^{k+1} - \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \quad (k = 1, 2, 3, \dots). \end{aligned}$$

Theorem 4.1. Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_j(z) = z - \frac{z^j}{j}$ ($j = 2, 3, 4, \dots$).
If $f(z)$ satisfies

$$(4.1) \quad \sum_{k=2}^{j^2+j-1} \frac{j^2+j-k}{j(j+1)} (a_{2j^2+2j-k} - a_k) \geq 0$$

for $j = 2, 3, 4, \dots$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{j-1} = j(j+1) \left(\sum_{n=2}^{\infty} \frac{1}{n+1} a_n z^{n-1} \right),$$

then

$$(4.2) \quad \int_0^{2\pi} |If(z)|^\lambda d\theta \leq \int_0^{2\pi} |If_j(z)|^\lambda d\theta.$$

Proof. We have to prove

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| 1 - \frac{2}{j(j+1)} z^{j-1} \right|^\lambda d\theta.$$

If

$$1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} \prec 1 - \frac{2}{j(j+1)} z^{j-1},$$

then the proof is completed by Theorem A.

Let us define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} = 1 - \frac{2}{j(j+1)} (\omega(z))^{j-1}.$$

Then

$$\begin{aligned} |\omega(z)|^{j-1} &= \left| j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n z^{n-1} \right| \\ &\leq |z| \left(j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} |a_n| \right). \end{aligned}$$

Thus, we only show that

$$j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} |a_n| \leq \sum_{n=2}^{\infty} n |a_n|$$

or

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n \left(\frac{1}{j(j+1)} + \frac{1}{n+1} \right) a_n.$$

Indeed,

$$\begin{aligned} \sum_{n=2}^{\infty} n \left(\frac{1}{j(j+1)} + \frac{1}{n+1} \right) a_n &= 2 \left(\frac{1}{j(j+1)} + \frac{1}{3} \right) a_2 + 3 \left(\frac{1}{j(j+1)} + \frac{1}{4} \right) a_3 + \cdots \\ &+ (j-1) \left(\frac{1}{j(j+1)} + \frac{1}{j} \right) a_{j-1} + j \left(\frac{1}{j(j+1)} + \frac{1}{j+1} \right) a_j + (j+1) \left(\frac{1}{j(j+1)} + \frac{1}{j+2} \right) a_{j+1} \\ &+ \cdots + (2j^2 + 2j - 3) \left(\frac{1}{j(j+1)} + \frac{1}{2j^2 + 2j - 2} \right) a_{2j^2 + 2j - 3} \\ &+ (2j^2 + 2j - 2) \left(\frac{1}{j(j+1)} + \frac{1}{2j^2 + 2j - 1} \right) a_{2j^2 + 2j - 1} + \cdots \\ &\geq \left(1 - \frac{j(j+1) - 2}{j(j+1)} \right) a_2 + \left(1 - \frac{j(j+1) - 3}{j(j+1)} \right) a_3 + \cdots + \left(1 - \frac{j(j+1) - (j-1)}{j(j+1)} \right) a_{j-1} \\ &+ \left(1 - \frac{j(j+1) - j}{j(j+1)} \right) a_j + \left(1 - \frac{j(j+1) - (j+1)}{j(j+1)} \right) a_{j+1} + \cdots \\ &+ \left(1 - \frac{j(j+1) - (2j^2 + 2j - 3)}{j(j+1)} \right) a_{2j^2 + 2j - 3} + \left(1 - \frac{j(j+1) - (2j^2 + 2j - 2)}{j(j+1)} \right) a_{2j^2 + 2j - 2} + \cdots \\ &= \frac{j^2 + j - 2}{j(j+1)} (a_{2j^2 + 2j - 2} - a_2) + \frac{j^2 + j - 3}{j(j+1)} (a_{2j^2 + 2j - 3} - a_3) + \cdots + \frac{j^2 + 1}{j(j+1)} (a_{2j^2 + j + 1} - a_{j-1}) \\ &+ \frac{j^2}{j(j+1)} (a_{2j^2 + j} - a_j) + \frac{j^2 - 1}{j(j+1)} (a_{2j^2 + j - 1} - a_{j+1}) + \cdots + a_2 + a_3 + \cdots + a_{2j^2 + 2j - 2} + \cdots \\ &= \sum_{k=2}^{j^2 + j - 1} \frac{j^2 + j - k}{j(j+1)} (a_{2j^2 + 2j - k} - a_k) + \sum_{n=2}^{\infty} a_n \\ &\geq \sum_{n=2}^{\infty} a_n \end{aligned}$$

for

$$\sum_{k=2}^{j^2 + j - 1} \frac{j^2 + j - k}{j(j+1)} (a_{2j^2 + 2j - k} - a_k) \geq 0.$$

This completes the proof of Theorem 4.1. □

Finally, we derive

Theorem 4.2. Let $f(z) \in \mathcal{T}^*, \lambda > 0$, and $f_j(z) = z - \frac{z^j}{j}$ ($j = 2, 3, 4, \dots$). If $f(z)$ satisfies

$$(4.3) \quad \sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \left(1 - \frac{2n(j-1)!}{(j+k)!}\right) (a_n - a_{\frac{(j+k)!}{(j-1)!}-n})$$

for $k = 2, 3, 4, \dots$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{j-1} = \frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n-1},$$

then

$$(4.4) \quad \int_0^{2\pi} |I_k f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I_k f_j(z)|^\lambda d\theta.$$

Proof. We have to show that

$$1 - \sum_{n=2}^{\infty} \frac{n!(k+1)!}{(n+k)!} a_n z^{n-1} \prec 1 - \frac{(j-1)!(k+1)!}{(j+k)!} z^{j-1}.$$

Define $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} \frac{n!(k+1)!}{(n+k)!} a_n z^{n-1} = 1 - \frac{(j-1)!(k+1)!}{(j+k)!} (\omega(z))^{j-1}$$

or by

$$(\omega(z))^{j-1} = \frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n-1}.$$

Then we have to show that

$$\frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n \leq \sum_{n=2}^{\infty} n a_n,$$

that is, that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n \leq \frac{(j-1)!}{(j+k)!} \sum_{n=2}^{\infty} n a_n.$$

Since

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n &= \sum_{n=2}^{\infty} \frac{1}{(n+1)(n+2) \cdots (n+k)} a_n \\
&= \sum_{n=2}^{\infty} \left\{ \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \left(\frac{1}{n+3} - \frac{1}{n+4} \right) \cdots \right\} a_n \\
&\leq \sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)^{\left[\frac{k}{2} \right]} a_n \\
&\leq \sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) a_n,
\end{aligned}$$

We obtain

$$\sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) a_n \leq \frac{(j-1)!}{(j+k)!} \sum_{n=2}^{\infty} n a_n$$

Furthermore, we have

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(\frac{2n(j-1)!}{(j+k)!} + \frac{2n}{n+1} - \frac{n}{n+2} \right) a_n.$$

Let the function $h(n)$ be given by

$$h(n) = \frac{2n}{n+1} - \frac{n}{n+2} = 1 - \frac{2}{n^2 + 3n + 2}.$$

Since $h(n)$ is increasing for $n \geq 2$,

$$h(n) \geq \frac{5}{6}.$$

Thus, we only show that

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(\frac{11}{6} - \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \right) a_n.$$

In fact,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left(\frac{11}{6} - \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \right) a_n \\
&= \left(\frac{11}{6} - \frac{(j+k)! - 4(j-1)!}{(j+k)!} \right) a_2 + \left(\frac{11}{6} - \frac{(j+k)! - 6(j-1)!}{(j+k)!} \right) a_3 + \cdots
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{11}{6} - \frac{4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}-2} + \left(\frac{11}{6} - \frac{2(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}-1} + \left(\frac{11}{6} - 0 \right) a_{\frac{(j+k)!}{2(j-1)!}} \\
& + \left(\frac{11}{6} - \frac{2(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}+1} + \left(\frac{11}{6} + \frac{4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}+2} + \cdots \\
& + \left(\frac{11}{6} + \frac{(j+k)!-6(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{(j-1)!}-3} + \left(\frac{11}{6} + \frac{(j+k)!-4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{(j-1)!}-2} + \cdots \\
& \geq \frac{11}{6} \sum_{n=2}^{\infty} a_n + \frac{(j+k)!-4(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{(j-1)!}-2} - a_2) + \frac{(j+k)!-6(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{(j-1)!}-3} - a_3) \\
& + \frac{4(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{2(j-1)!}+2} - a_{\frac{(j+k)!}{2(j-1)!}-2}) + \frac{2(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{2(j-1)!}+1} - a_{\frac{(j+k)!}{2(j-1)!}-1}) \\
& = \sum_{n=2}^{\infty} a_n + \frac{5}{6} \sum_{n=2}^{\infty} a_n + \frac{(j+k)!-4(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{2(j-1)!}-2} - a_2) \\
& + \frac{(j+k)!-6(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{2(j-1)!}-3} - a_3) + \cdots \\
& + \frac{(j+k)!-\{(j+k)!-4(j-1)!\}}{(j+k)!} (a_{\frac{(j+k)!}{2(j-1)!}+2} - a_{\frac{(j+k)!}{2(j-1)!}-2}) \\
& + \frac{(j+k)!-\{(j+k)!-2(j-1)!\}}{(j+k)!} (a_{\frac{(j+k)!}{2(j-1)!}+1} - a_{\frac{(j+k)!}{2(j-1)!}-1}) \\
& = \sum_{n=2}^{\infty} a_n + \frac{5}{6} \sum_{n=2}^{\infty} a_n + \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \frac{(j+k)!-2n(j-1)!}{(j+k)!} (a_{\frac{(j+k)!}{(j-1)!}-n} - a_n) \\
& \geq \sum_{n=2}^{\infty} a_n
\end{aligned}$$

for

$$\sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \left(1 - \frac{2n(j-1)!}{(j+k)!} \right) (a_n - a_{\frac{(j+k)!}{(j-1)!}-n}).$$

This completes the proof of Theorem 4.2. □

Remark 4.1. Letting $k = 2$, if $f(z)$ satisfies,

$$(4.5) \quad \sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{j(j+1)(j+2)}{2}-1} \left(1 - \frac{2n}{j(j+1)(j+2)} \right) (a_n - a_{j(j+1)(j+2)-n})$$

for $j = 2, 3, 4, \dots$, then

$$(4.6) \quad \int_0^{2\pi} |I_2 f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I_2 f_j(z)|^\lambda d\theta.$$

Remark 4.2. Letting $k = 3$, if $f(z)$ satisfies,

$$(4.7) \quad \sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{j(j+1)(j+2)(j+3)}{2}-1} \left(1 - \frac{2n}{j(j+1)(j+2)(j+3)}\right) (a_n - a_{j(j+1)(j+2)(j+3)-n})$$

for $j = 2, 3, 4, \dots$, then

$$(4.8) \quad \int_0^{2\pi} |I_3 f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I_3 f_j(z)|^\lambda d\theta.$$

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