

MULTIPLICITY OF FILTERED RINGS*

筑波大学 数学系 泊 昌孝 (MASATAKA TOMARI)

§1 Introduction and results.

(1.1) Let (V, p) be a germ of a projective variety at a closed point p . It is a fundamental problem to study the ring theoretic properties of the local ring $O_{V,p}$ by means of resolution of singularities ; $\psi : (\tilde{V}, E) \rightarrow (V, p)$. In the case $\dim V = 2$, Artin's fundamental cycle for the resolution ψ is important and gives many information of singularities. Let Z_0 be Artin's fundamental cycle for ψ . For example the degree $(Z_0)^2$ is independent of choice of the resolution ψ and we have the relation

$$-(Z_0)^2 \leq \text{the multiplicity of } (V, p)$$

(Ph. Wagreich cf. [Wagreich]). Unfortunately no higher dimensional analogue of this object are studied . In this note we will study the multiplicity of singularities by filtered blowing-ups. We prove an inequality (1.6) which gives a lower bound of multiplicity by the data of tangent cone of the filtration.

An application of our results to a purely elliptic singularity of special type ($[IW][Y][T1]$) will be given in another note for the talk of "COMMUTATIVE RING THEORY ; JAPAN NO.11".

(1.2) Throughout this note we will fix the following situation. Our singularity (V, p) or local ring $(A, m) = (O_{(V,p)}, m)$ is always assumed as the material coming from some scheme over a field k . Further we will assume (A, m) is analytically unramified after (1.6). In particular,

(A, m) : d -dimensional Noether local ring over a field k ,

$F = \{F^k\}_{k \geq 0}$: a filtration of ideals as follows ;

$$(F^0 = A \supset F^1 = m, F^k \supset F^{k+1}, F^k \cdot F^j \subset F^{k+j},$$

(*) This is a preliminary version. 内容の変化にとわい, 講演時 "Multiplicity of normal graded rings" より, 七行も上記のものに変わりました。

$R = \oplus_{l \geq 0} F^l \cdot T^l \subset A[T]$ is a finitely generated A -algebra, where T is an indeterminate.

There is an integer $N > 0$ with $(F^N)^m = F^{Nm}$ for $m \geq 0$.

$F^N : m$ -primary)

By these assumption, $G_+ = \oplus_{l \geq 1} F^l / F^{l+1}$ is the homogeneous maximal ideal of G .

Problem (1.2.1). Study the multiplicity $e(m, A)$ of (A, m) from the associated graded ring $gr_F A = G = \oplus_{k \geq 0} F^k / F^{k+1}$ and compare the integers $e(m, A)$ and $e(G_+, G)$.

(We hope that these are very near when G is a "good" ring.)

First we shall prove the following .

FACT (1.3). Let the situation be as above. Then

$$l(A/m^{l+1}) \leq l(G/(G_+)^{l+1}) \text{ for } l \geq 0.$$

In particular we obtain the relations $e(m, A) \leq e(G_+, G)$ and $\text{embdim } A \leq \text{embdim } G$.

Proof. The induced filtration on A/m^{l+1} by $F = \{F^k\}$ is given as follows:

$$\begin{array}{ccccccc} 0 & \rightarrow & m^{l+1} & \longrightarrow & A & \longrightarrow & A/m^{l+1} \rightarrow 0 \\ & & \cup \parallel & & \cup \parallel & & \cup \parallel \\ 0 & \rightarrow & m^{l+1} \cap F^k & \rightarrow & F^k & \rightarrow & F^k(A/m^{l+1}) \rightarrow 0 \end{array}$$

Hence we obtain $gr_F(A/m^{l+1}) = gr_F(A)/gr_F(m^{l+1})$. Here we see

$$\begin{aligned} gr_F(m^{l+1}) &= \oplus_{k \geq 0} m^{l+1} \cap F^k / m^{l+1} \cap F^{k+1} \\ &\cong \oplus_{k \geq 0} \frac{F^k \cap m^{l+1} + F^{k+1}}{F^{k+1}}, \\ (G_+)^{l+1} &= \oplus_{k \geq l+1} \frac{\sum_{m_1 + \dots + m_{l+1} = k, m_i \geq 1} F^{m_1} \dots F^{m_{l+1}} + F^{k+1}}{F^{k+1}}. \end{aligned}$$

Clearly we have

$$\sum_{m_1 + \dots + m_{l+1} = k, m_i \geq 1} F^{m_1} \dots F^{m_{l+1}} \subset F^k \cap m^{l+1}.$$

Hence

$$(G_+)^{l+1} \subset gr_F(m^{l+1}).$$

Therefore

$$l(A/m^{l+1}) \leq l(gr_F(A/m^{l+1})) = l(G/gr_F(m^{l+1})) \leq l(G/(G_+)^{l+1}).$$

Example(1.4). We shall introduce a filtration F on the regular local ring $A = k[[x, y, z]]$ with $m = (x, y, z)A$ by means of the associated order function ν as in the following (cf. [Rees]) :

$\nu(x) = \nu(y) = \nu(z) = 1$, and $\nu(x^2 + y^2 + z^2) = 3(> 2)$ with $F^k = \{\alpha \in A \mid \nu(\alpha) \geq k\} \subset A$.

We can easily check that $G = gr_F(A) \cong k[x, y, z, w]/x^2 + y^2 + z^2$, $e(m, A) = 1$ and $e(G_+, G) = 2$.

In fact we have

$$F^0 = A$$

$$F^1 = m$$

$$F^2 = m^2$$

$$F^3 = m^3 + (x^2 + y^2 + z^2)A$$

$$F^4 = m^4 + (x^2 + y^2 + z^2)m$$

$$F^5 = m^5 + (x^2 + y^2 + z^2)m^2$$

$$F^6 = m^6 + (x^2 + y^2 + z^2)m^3 + (x^2 + y^2 + z^2)^2 A$$

..... ,

hence $x^2 + y^2 + z^2 \in F^3 - F^4$ can not be represented by $x, y, z \in F^1 - F^2$ in the ring G . To compute G as in the assertion, remark that if we regard A in the form $A \cong k[[x, y, z, w]/(w - x^2 - y^2 - z^2)]$, then F is the induced filtration from the filtration on $k[[x, y, z, w]]$ by the degree of monomials as $F^k = \{x^a y^b z^c w^d \in k[[x, y, z, w]] \mid a + b + c + 3d \geq k\}A$.

Example(1.5). We introduce the filtration F on the local ring $A (= k[[b, c, y, z]]/b^2 + (y^3 + z^7 + c^{21})c) \cong k[[a, b, c, y, z]]/(a + y^3 + z^7 + c^{21}, b^2 - ac)$ with $m = (a, b, c, y, z)A$ by the order function ν as :

$$\nu(a) = \nu(y^3 + z^7 + c^{21}) = 23, \nu(b) = 12, \nu(c) = 1, \nu(y) = 7 \text{ and } \nu(z) = 3.$$

Now $G = gr_F(A) \cong k[a, b, c, y, z]/(y^3 + z^7 + c^{21}, b^2 - ac)$, $e(m, A) = 2$ and $e(G_+, G) = 6$. Further one can see that G is a normal domain.

These examples say that the integers $e(m, A)$ and $e(G_+, G)$ are different, in general, even if we assume that G is a normal Gorenstein domain. The next is the main result of this note which gives a lower bound of $e(m, A)$ from the data of G .

THEOREM (1.6). *Let the situation be as in (1.2). Further we assume that A is analytically unramified and that k is an infinite field. Let a system of elements $x_1, \dots, x_s \in G_+$ be a minimal homogeneous generator system of G_+ with $\deg x_1 \leq \deg x_2 \leq \dots \leq \deg x_s$, with $s \geq d = \dim A = \dim G$. Then we have the following*

(1)

$$\begin{aligned} \left(\prod_{i=1}^d \deg x_i \right) \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda) \\ \leq_{(i)} e(m, A) \\ \leq e(G_+, G) \leq_{(ii)} (\deg x_s)^d \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda). \end{aligned}$$

where $P(G, \lambda) = \sum_{k \geq 0} l(G_k) \lambda^k \in \mathbb{Z}[[\lambda]]$.

(2) If the equality holds in (i), then $e(m, A) = e(G_+, G)$ and there is a parameter system $y_1, \dots, y_d$ of A whose initial form gives a homogeneous parameter system $in(y_1), \dots, in(y_d)$ of G such that $\deg in(y_i) = \deg x_i$ for $i = 1, \dots, d$.

(3) If the equality holds in (ii) and G is normal with $G.C.D.(\deg x_1, \dots, \deg x_s) = 1$, then $e(m, A) = e(G_+, G)$ and G is a homogeneous ring. That is $\deg x_i = 1$ holds for $i = 1, \dots, s$.

In general we have the following.

Remark (1.7) (1) Let $R = R(E, D)$ be a normal d -dimensional graded ring with Demazure's description.

$$D^{d-1} = \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(R, \lambda)$$

where $P(R, \lambda) = \sum_{k \geq 0} l(R_k) \lambda^k \in \mathbb{Z}[[\lambda]]$, with $d = \dim R$.

(2) For a graded complete intersection

$$R = k[x_1, \dots, x_{d+s}]/(f_1, \dots, f_s),$$

where $f_1, \dots, f_s$ is a homogeneous regular sequence of $k[x_1, \dots, x_{d+s}]$, we have

$$P(R, \lambda) = \frac{(1 - \lambda^{\deg f_1}) \dots (1 - \lambda^{\deg f_s})}{(1 - \lambda^{\deg x_1}) \dots (1 - \lambda^{\deg x_{d+s}})}.$$

Hence

$$\lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(R, \lambda) = \frac{(\deg f_1) \dots (\deg f_s)}{(\deg x_1) \dots (\deg x_{d+s})}.$$

(1.8) By using (1.7), we will observe (1.4) and (1.5).

(1.8.1) For G of (1.4), we have $\deg x = \deg y = \deg z = 1$ and $\deg w = 3$. Hence

$$1.1.1. \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda) = 1.1.1. \frac{2}{1.1.1.3} = \frac{2}{3} (\leq 1 = e(m, A)).$$

(1.8.2) For G of (1.5), we have $\deg a = 23$, $\deg b = 12$, $\deg c = 1$, $\deg y = 7$ and $\deg z = 3$. Hence

$$1.3.7. \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda) = 1.3.7. \frac{21.24}{1.3.7.12.23} = \frac{42}{23} (\leq 2 = e(m, A)).$$

COROLLARY (1.9). Let the situation be as in (1.6).

(1) If the condition

$$\text{the round up of the number } \left(\prod_{i=1}^d \deg x_i \right) \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda) = e(G_+, G)$$

holds, then the equality $e(m, A) = e(G_+, G)$ holds.

(2) If G is a hypersurface with the isolated singularity at G_+ , then $e(m, A) = e(G_+, G)$.

Proof. (1) is obvious from (1) of Theorem (1.6). (2) Let us represent G as $G = k[x_1, \dots, x_{d+1}]/f$ with $\deg f = h$ and $\deg x_i = q_i$. Let us represent f by a linear combination of monomials of the form $x^M = \prod_{i=1}^s x_i^{m_i}$ with $m_i \geq 0$ as

$$f = \sum_{M \in \left(\mathbb{Z}_{\geq 0} \right)^{d+1}} a_M x^M \quad \text{with } a_M \in k$$

We define the Newton support of f by

$$\text{Support}(f) = \{M \in \left(\mathbb{Z}_{\geq 0} \right)^{d+1} \mid a_M \neq 0\}$$

The condition $\sum_{i=1}^s q_i m_i = h$ implies $\frac{h}{q_{d+1}} \leq \sum_{i=1}^s m_i \leq \frac{h}{q_1}$. Hence we have $q_1 \dots q_d \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda) = \frac{h}{q_{d+1}} \leq \text{multiplicity of } f = \text{minimum of } \sum_{i=1}^{d+1} m_i$ for $x^M \in \text{Support}(f) \leq \frac{h}{q_1} = q_2 \dots q_{d+1} D^{d-1}$. Since $\{f = 0\}$ has only isolated singularity at o , a monomial of form $x_i^{m_i} x_{j(i)}$ with $j(i) \in \{1, \dots, d+1\}$ is contained in $\text{Support}(f)$ for each i (K. Saito [S1], V.I. Arnold, P. Orlik- Ph. Wagreich, and Fletcher[Fletcher]). In particular $x_{d+1}^{[h/q_{d+1}]} x_{j(d+1)} \in \text{Support}(f)$. Hence the multiplicity of f equals the round up of the rational number $\frac{h}{q_{d+1}}$.

Example (1.10). Let A be ("a normal graded complete intersection ") as follows : $A = k[[x, y, z, w, u]]/(f_1, f_2)$ with the filtration F on A naturally induced as $\deg x = \deg y = \deg z = \deg w = 1$, $\deg u = 2$ and $\deg f_1 = \deg f_2 = 3$. We have $G = k[x, y, z, w, u]/(f_1, f_2)$ with $\deg x = \deg y = \deg z = \deg w = 1$, $\deg u = 2$ and $\deg f_1 = \deg f_2 = 3$. By (1.6) we obtain

$$\frac{9}{2} \leq e(m, A) \leq e(G_+, G) \leq 36.$$

Since (A, m) is not a tangential complete intersection with respect to the maximal-ideal-adic filtration on A , the lower bound is the best. But the upper bound of this implication is very bad.

In the rest of this note we give a outline of proof of Theorem (1.6) and state some generalities on the rational number $\left(\prod_{i=1}^d \deg x_i \right) \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda)$ for the normal graded ring R in terms of Demazure's description of R .

§2 The openness of reduction property.

The purpose of this section is to prove (2.7) which we will use in §4.

(2.1) Let (V, p) be a singularity over a field k and $(O_{V,p}, m)$ be the associated local ring. We assume that p is a closed point of a projective variety $\bar{V}$ over the field k . Let I be an m -primary ideal of $O_{V,p}$. Let $\pi : (\tilde{V}, A) \rightarrow (V, p)$ be a projective morphism such that $I \cdot O_{\tilde{V}}$ is a locally principal $O_{\tilde{V}}$ -module. We will represent $I \cdot O_{\tilde{V}}$ as

$$I \cdot O_{\tilde{V}} = O_{\tilde{V}}(-D(I, \pi))$$

by a Cartier divisor on $\tilde{V}$.

THEOREM (2.2). *Let the situation below as above. Assume $d = \dim O_{\bar{V}}$. Then $e(I, O_{V,p}) = (-1)^{d+1} D(I, \pi)^d$.*

Proof. Let a projective variety $\bar{V}$ be a compactification of V and $\psi : \bar{V}_1 \rightarrow \bar{V}$ be the blowing-up of $\bar{V}$ with center I . There is a natural morphism $\tau : \bar{V} \rightarrow \bar{V}_1$ which satisfies the relation $\pi = \psi \cdot \tau$. We have $I^k = \psi_*(I^k O_{\bar{V}})$ and $R^i \psi_*(I^k O_{\bar{V}}) = 0$ ($i \geq 1$) for arbitrary large integer k (EGA III). We have $l(I^k/I^{k+1}) = \chi(\bar{V}, I^k/I^{k+1})$. By Leray's spectral sequence

$$E_2^{p,q} = H^p(\bar{V}, R^q \psi_*(I^k O_{\bar{V}_1})) \implies_p H^n(\bar{V}_1, I^k O_{\bar{V}_1}),$$

we have

$$\sum_{q \geq 0} (-1)^q \chi(\bar{V}, R^q \psi_*(I^k O_{\bar{V}_1})) = \chi(\bar{V}_1, I^k O_{\bar{V}_1}).$$

Hence for $k \gg 0$, we obtain

$$\begin{aligned} l(I^k/I^{k+1}) &= \chi(\bar{V}, I^k O_{\bar{V}}) - \chi(\bar{V}, I^{k+1} O_{\bar{V}}) \\ &= \chi(\bar{V}_1, I^k O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k+1} O_{\bar{V}_1}). \end{aligned}$$

Let $P \in \mathbb{Q}[t]$ be the Hilbert-Samuel polynomial defined as $P(k) = \chi(\bar{V}_1, I^k O_{\bar{V}_1}/I^{k+1} O_{\bar{V}_1})$ for $k \gg 0$ [Kl]. We have degree $P = d - 1$. Let us set of polynomials $\Delta^{(m)} P$ for $1 \leq m \leq d$ as ; $\Delta^{(1)} P(k) = P(k) - P(k-1)$, . . . , $\Delta^{(m)} P = \Delta(\Delta^{(m-1)} P)$, inductively. Here $\Delta^{(d-1)} P$ is the constant function $e(I, O_{V,p})$. Further we have

$$\begin{aligned} \Delta^{(1)} P(k) &= \chi(\bar{V}_1, I^k O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k+1} O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k-1} O_{\bar{V}_1}) + \chi(\bar{V}_1, I^k O_{\bar{V}_1}) \\ &= -\chi(\bar{V}_1, I^{k+1} O_{\bar{V}_1}) + 2 \cdot \chi(\bar{V}_1, I^k O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k-1} O_{\bar{V}_1}). \end{aligned}$$

An by similar calculations we obtain

$$\begin{aligned} \Delta^{(d-1)} P(k) &= - \sum_{i=0}^d \binom{d}{i} (-1)^i \chi(\bar{V}_1, I^{k+1-i} O_{\bar{V}_1}) \\ &= - \deg_{O_V(-D(I, \psi))}(O_{\text{bar } V_1}) \\ &= -(O_{V_1}(-D(I, \psi))^d \cdot O_{V_1})_{V_1} \text{ (the intersection symbol [Kl])} \\ &= (-1)^{d+1} D(I, \psi)^d. \end{aligned}$$

Since τ is birational, $(-1)^{d+1} D(I, \psi)^d = (-1)^{d+1} D(I, \pi)^d$.

Hence $e(I, O_{V,p}) = (-1)^{d+1} D(I, \pi)^d$. Q.E.D.

We will apply (2.2) to the following.

Let $J \subset I$ be m -primary ideals of $O_{V,p}$. Recall J is a reduction of I if there is an integer $r > 0$ such that $I^r J = I^{r+1}$ (Northcott-Rees [NR]).

THEOREM (2.3) (D. REES [HIO][REES], SEE ALSO J. LIPMAN [LIPMAN]). Assume that (A, m) is analytically unramified. Then J is a reduction of I if and only if the equality $e(J, O_{V,p}) = e(I, O_{V,p})$ holds.

COROLLARY (2.4). For m -primary ideals $J \subset I$, the following three conditions are equivalent each other.

- (1) The equality $e(J, O_{V,p}) = e(I, O_{V,p})$ holds.
- (2) J is a reduction of I .
- (3) There exists a birational morphism $\psi : \tilde{V} \rightarrow V$ such that the relation $J.O_{\tilde{V}} = I.O_{\tilde{V}}$ holds.

Proof. The equivalence of (1) and (2) are due to (2.3). Assume the condition (3) holds. There exists a birational morphism $\tau : V' \rightarrow \tilde{V}$ such that $I.O_{V'} = J.O_{V'}$ is local principal. By (2.2) we have $e(J, O_{V,p}) = e(I, O_{V,p})$. Next we assume there is an integer $r > 0$ such that $I^r J = I^{r+1}$. Let $\varphi : V' \rightarrow V$ be a birational morphism such that $I.O_{V'}$ is local principal. Then we have $J.I^r.O_{V'} = I^{r+1}.O_{V'}$ and have $J.O_{V'} = I.O_{V'}$.

By this we can see the reduction property of ideals are open condition as in the following sense.

DEFINITION (2.5). Let J be an ideal of $O_{V,p}$. A deformation of ideal J $\varrho : \tilde{J} \rightarrow Y \ni o$ over a scheme Y with a reference point o is an ideal $\tilde{J}$ of $O_{V \times Y}$ at $p \times Y$ such that $\varrho^{-1}(o) = J$.

PROPOSITION (2.6). Let $J \subset I$ be m -primary ideals of $O_{V,p}$ and $\varrho : \tilde{J} \rightarrow Y \ni o$ be a deformation of ideal of J . Suppose J is a reduction of I . Then there is a Zariski open neighborhood U of o in Y where $\tilde{J}_y = \varrho^{-1}(y)$ is a reduction of I for any point y of U .

Proof. There is an integer $r > 0$ such that $I^r J = I^{r+1}$ and $\varphi : V' \rightarrow V$ be a birational morphism such that $I.O_{V'}$ is local principal and V' is normal. Then we have $J.O_{V'} = I.O_{V'}$. Consider the morphism $\tilde{\varphi} : V' \times Y \rightarrow V \times Y$ with $I.O_{V' \times Y} \supset \tilde{J}$. Here $I.O_{V' \times Y}$ is defined as an invertible $O_{V' \times Y}$ in a trivial extension of $I.O_{V'}$. Now we have

the relation $\tilde{J}O_{V' \times Y} + \varphi^{-1}(m_o)I.O_{V' \times Y} = I.O_{V' \times Y}$. Hence $I.O_{V' \times Y}$ and $\tilde{J}.O_{V' \times Y}$ are equal at each generic points of $V' \times Y$ which contains a point of $\tilde{\varphi}^{-1}(o) = V' \times o$. In particular the reflexive hull $(\tilde{J}.O_{V' \times Y})^{**}$ equals $I.O_{V' \times Y}$. Let $S \subset V' \times Y$ be the non-reflexive locus of $\tilde{J}.O_{V' \times Y}$. Then S does not intersects $V' \times o$ and $\varphi(S)$ does not contain the point o . By Corollary (2.4) at any point $y \in Y - \varphi(S)$, $\tilde{J}_y$ is a reduction of I . Q.E.D.

COROLLARY (2.7). Let I be an m -primary ideal of $O_{V,p}$ generated as $I = (f_1, \dots, f_s)$. Suppose that $O_{V,p}$ contains a field k and that there is a reduction J of I written as

$J = (y_1, \dots, y_m)O_{V,p}$ where

$$y_i = \sum_{j=1}^s a_{i,j} x_j, \quad \text{with } a_{i,j} \in k \quad 1 \leq i \leq m, \quad 1 \leq j \leq s.$$

Then there is a Zariski open neighborhood U of $(a_{i,j})$ in k'^m such that $J_b = (z_1, \dots, z_m)O_{V,p}$ is a reduction of I for $z_i = \sum_{j=1}^s b_{i,j} x_j$, with $(b_{i,j}) \in U$

Proof. Define the deformation of J by $\tilde{J} = \coprod_{b \in k'^m} J_b$ over k'^m . The the assertion follows from Proposition 3. Q.E.D.

We state the following which is a higher dimensional analogue of a theorem of Laufer (cf. [L1]):

THEOREM (2.8). Let (W, w) be a normal d -dimensional singularity and $(x_1, \dots, x_d)$ a parameter system of $O_{W,w}$. Let $\psi : X \rightarrow W$ be a projective modification with normal X and $E = \psi^{-1}(w)$. We write $\text{div}_X(x_i O_X)$ by

$$\text{div}_X(x_i O_X) = D(x_i O_W, \psi) + W_{x_i, \psi} \quad i = 1, \dots, d,$$

where $W_{x_i, \psi}$ is the strict transform of $\{x_i = 0\}$ and $D(x_i O_W, \psi)$ is the part of E . We assume that the divisor $W_{x_i, \psi}$ is $\mathbb{Q}$ -Cartier for $i = 1, \dots, d$.

If $W_{x_1, \psi} \cap \dots \cap W_{x_d, \psi}$ is empty, we have the relation

$$e((x_1, \dots, x_d), O_{W,w}) = (-1)^{d+1} D(x_1 O_W, \psi) \dots D(x_d O_W, \psi)$$

We omit the proof.

§3 On Demazure's description of normal graded rings.

(3.1) The purpose of this section is to collect the generalities of Demazure's description $R(E, D)$ of the normal graded ring R in the connection with the number

$$\left(\prod_{i=1}^d \deg x_i \right) \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda).$$

As there are many good references on this subjects [D], [W 敬 1], [W 敬 2], we will review a computation method for Demazure's divisor D by a tentative way as follows (cf. [T1]): Let $R = \bigoplus_{k \geq 0} R_k$ be a normal d -dimensional graded ring with $R_0 = k$, R_+ the homogeneous maximal ideal with a generator consisting in homogeneous elements $x_1, \dots, x_s$ as $R_+ = (x_1, \dots, x_s)R$ with $x_i \in R_{q_i}$ for $i = 1, \dots, s$. We assume the condition $G.C.D.(q_1, \dots, q_s) = 1$. There are integers $u_1, \dots, u_s$ such that $\sum_{i=1}^s u_i q_i = 1$. We choose a homogeneous element T of the quotient field of R as $T = \prod_{i=1}^s (x_i)^{u_i}$. We represent

$$x_i R = \bigcap_{Q \in HP(R)} Q^{(a_i Q)}, \quad i = 1, \dots, s.$$

Here $HP(R)$ is the set of homogeneous prime ideals of height 1. By Demazure's fundamental works we can represent D as follows:

THEOREM (3.2)(DEMAZURE [D]). *In the above situation, we define the divisor D associated to T as*

$$D = \sum_{i=1}^s \left(\sum_{Q \in HP(R)} \frac{u_i a_i Q}{N(Q)} \cdot V(Q) \right) \in \text{Div}(E) \otimes \mathbb{Q}$$

where $V(Q)$ is the integral Weil divisor on $E = \text{Proj}(R)$ defined by Q and $N(Q)$ is the integer defined as ;

$$N(Q) = G.C.D.\{n \in \mathbb{Z} \mid n > 0 \text{ and } (R/Q)_n \neq 0\} \text{ (cf. (5.9.1) of [TW1]).}$$

Then we obtain the equality

$$R = \bigoplus_{k \geq 0} H^0(E, \mathcal{O}_E(kD)) \cdot T^k \text{ in } k(E)[T].$$

Example (3.3). Let R is a normal d -dimensional ($d \geq 2$) graded ring of the Brieskorn type as follows : $R = \mathbb{C}[x_1, \dots, x_{d+1}] / \{(x_1)^{a_1} + \dots + (x_{d+1})^{a_{d+1}}\}$ where

$a_1, \dots, a_{d+1}$ are integers ≥ 2 . Introduce the weight of each monomials of $\mathbb{C}[x_1, \dots, x_{d+1}]$ as the degree of $x_i = L.C.M.(a_1, \dots, a_{d+1})/a_i$. We simply denote it as q_i , for $i = 1, \dots, d+1$.

Then the Demazure divisor D associated to T is written as

$$D = \sum_{i=1}^{d+1} \frac{u_i}{G.C.D.(q_1, \dots, \hat{i}, \dots, q_{d+1})} . D_i \in \text{Div}(E) \otimes \mathbb{Q}$$

where D_i is the integral Weil divisor on $E = \text{Proj}(R)$ defined by the canonical morphism $D_i = \text{Proj}(R/x_i R) \rightarrow \text{Proj}(R) = E$ for $i = 1, \dots, d+1$ (see (1.9) of [T1] for a proof).

LEMMA (3.4). *Let $R = R(E, D)$ be a normal d -dimensional graded ring with Demazure's description . Let us consider the singularity of $\text{Spec}(R)$ at $V(R_+)$. Let*

$$\psi : C = C(E, D) = \text{Spec}_E(\oplus_{k \geq 0} O_E(kD)) \rightarrow \text{Spec}(R)$$

be the partial resolution by the filtered blowing-up of $\text{Spec}(R)$ with respect to the filtration induced by grading of $\text{Spec}(R)$. Let $x_1, \dots, x_d \in R$ be a parameter system at R_{R_+} . Suppose $x_1, \dots, x_r$ with $r \leq d$ be homogeneous elements. Then we have

$$\dim W_{x_1, \psi} \cap \dots \cap W_{x_d, \psi} \leq d - r - 1 \text{ in } C(E, D).$$

Hence in the case $r = d$, $W_{x_1, \psi} \cap \dots \cap W_{x_d, \psi}$ is empty . In this case $e((x_1, \dots, x_d), R_{R_+})$ is computed by Theorem (2.8).

By (2.8) and (3.4) we obtain the following.

COROLLARY (3.5). *Let $R = R(E, D)$ be a normal d -dimensional graded ring with Demazure's description and $x_1, \dots, x_d \in R$ be a homogeneous parameter system of R .*

$$e((x_1, \dots, x_d), R) = (-1)^{d+1} \left(\prod_{i=1}^d \deg x_i \right) . E^d.$$

Here E^d is the intersection multiplicity in $C = C(E, D)$.

LEMMA (3.6). *Let $R = R(E, D)$ be a normal d -dimensional graded ring with Demazure's description and*

$$\psi : C = C(E, D) = \text{Spec}_E(\oplus_{k \geq 0} O_E(kD)) \rightarrow \text{Spec}(R)$$

be the filtered blowing-up of $\text{Spec}(R)$ with respect to the filtration induced by grading of $\text{Spec}(R)$.

Then we have the relation

$$D^{d-1} = (-1)^{d+1} E^d.$$

By using (1.7) we obtain the following.

COROLLARY (3.7). *In the situation (3.6), assume $x_1, \dots, x_d \in R$ be a homogeneous parameter system.*

$$\begin{aligned} e((x_1, \dots, x_d), R) &= \left(\prod_{i=1}^d \deg x_i \right) \cdot D^{d-1} \\ &= \left(\prod_{i=1}^d \deg x_i \right) \cdot \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(R, \lambda). \end{aligned}$$

§4 Proof of Theorem (1.6).

(4.1) *The inequality (i) of (1).* Let a system of elements $x_1, \dots, x_s$ of the maximal ideal m of A whose initial forms with respect to the filtration F give the minimal homogeneous generator of G_+ as follows ; $x_i \in F^{q_i} - F^{q_i+1}$ and the initial forms $\text{in}_F(x_i) = \bar{x}_i \in G_{q_i}$ satisfies the relations $G_+ = (\bar{x}_1, \dots, \bar{x}_s)G$ and $q_1 \leq \dots \leq q_s$. We can easily see the relations $m = F^n + (x_1, \dots, x_s)A$ for any positive integer n . There is an integer n such that $F^n \subset m^2$. Hence $m = (x_1, \dots, x_s)$ by NAK.

There is a system of parameter $y_1, \dots, y_d$ which is a minimal reduction of m and given as linear combination of $x_1, \dots, x_s$ as follows :

$$y_i = \sum_{j=1}^s a_{i,j} x_j, \quad \text{where } a_{i,j} \in k.$$

with $1 \leq i \leq d$, $1 \leq j \leq s$. By the openness of reduction property (Corollary (2.7)), we may assume $A = (a_{i,j})_{1 \leq i, j \leq d}$ is regular. So we can choose y_i in the following form:

$$y_i = x_i + \sum_{j=d+1}^s a_{i,j} x_j, \quad \text{where } a_{i,j} \in k$$

for $1 \leq i \leq d$ from the beginning .

Let L be a positive integer divided by $L.C.M.(deg x_1, \dots, deg x_d)$. By Leck's lemma

$$e((y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}), A) = \frac{L}{q_1} \cdot \dots \cdot \frac{L}{q_d} \cdot e((y_1, \dots, y_d), A).$$

Since $y_i^{\frac{L}{q_i}} \in F^L$ for $1 \leq i \leq d$, we have

$$e(F^L, A) \leq \frac{L}{q_1} \cdot \dots \cdot \frac{L}{q_d} \cdot e((y_1, \dots, y_d), A).$$

There is an integer L as above and satisfies the relation $F^{mL} = (F^L)^m$ for any positive integer m , that is $(\oplus_{k \geq 0} F^k \cdot T^k)^{(L)} = A[F^L \cdot T^L]$. To finish the proof, it is sufficient to show the following:

LEMMA (4.2). Let L be a positive integer such that the relation $F^{mL} = (F^L)^m$ holds for any positive integer m . Then

$$e(F^L, A) = L^d \cdot \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda).$$

Proof. By the assumption, $\oplus_{k \geq 0} F^{kL} / F^{(k+1)L}$ is generated by F^L / F^{2L} . Hence we obtain the equality (see §13 and §14 of [Matsumura]):

$$e(F^L, A) = \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(\oplus_{k \geq 0} F^{kL} / F^{(k+1)L}, \lambda).$$

Let us introduce the notation as $G^{(L)} = \oplus_{k \geq 0} F^{kL} / F^{(k+1)L}$ and

$G^{(L,l)} = \oplus_{k \geq 0} F^{kL+l} / F^{(k+1)L+l}$ for $l = 0, \dots, L-1$. Since there is an integer M such that $F^L \cdot F^b = F^{L+b}$ holds for any $b \geq M$, $G^{(L,l)}$ is a finite $G^{(L)}$ -module for $l = 0, \dots, L-1$.

As graded $G^{(L)}$ -modules, we calculate the Poincare series ; $P(G^{(L,l)}, \mu) \in \mathbb{Z}[[\mu]]$ for $l = 0, \dots, L-1$. For each i , $\lim_{\mu \rightarrow 1} (1 - \mu)^d P(G^{(L,l)}, \mu)$ is a finite number. We have the relations

$$\begin{aligned} (1 - \mu)^d P(G, \mu) &= \sum_{l=0}^{L-1} (1 - \mu)^d P(G^{(L,l)}, \mu^L) \cdot \mu^l \\ &= \sum_{l=0}^{L-1} (1 - \mu^L)^d P(G^{(L,l)}, \mu^L) \cdot \frac{(1 - \mu)^d}{(1 - \mu^L)^d} \\ &+ \sum_{l=0}^{L-1} (1 - \mu^L)^d P(G^{(L,l)}, \mu^L) \cdot \frac{(1 - \mu)^d}{(1 - \mu^L)^d} (\mu^l - 1). \end{aligned}$$

Hence

$$\begin{aligned}
\lim_{\mu \rightarrow 1} (1 - \mu)^d P(G, \mu) &= \lim_{\mu \rightarrow 1} (1 - \mu^L)^d \sum_{l=0}^{L-1} P(G^{(L,l)}, \mu^L) \cdot \frac{1}{L^d} \\
&= \lim_{\mu \rightarrow 1} (1 - \mu)^d P\left(\sum_{l=0}^{L-1} G^{(L,l)}, \mu\right) \cdot \frac{1}{L^d} \\
&= \lim_{\mu \rightarrow 1} (1 - \mu)^d P\left(\bigoplus_{k \geq 0} F^{kL} / F^{(k+1)L}, \mu\right) \cdot \frac{1}{L^d} \\
&= e(F^L, A) \frac{1}{L^d}.
\end{aligned}$$

(4.3) *Proof of (2).* Let $y_1, \dots, y_d$ be a parameter system of A as in the arguments of (4.1). By the assumption we have the equality

$$e((y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}), A) = e(F^L, A).$$

Hence $(y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}})$ is a reduction of F^L by a Theorem of Rees. There is an integer $r > 0$ such that

$$(F^L)^{r+1} = (F^L)^r (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) \text{ in } A.$$

Let $\psi : X = \text{Proj}(\bigoplus_{k \geq 0} F^k \cdot T^k) \rightarrow \text{Spec}(A)$ be the filtered blowing-up of $\text{Spec}(A)$ by F . We have

$$(F^L)^{r+1} O_X = (F^L)^r (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) O_X \text{ in } O_X.$$

Here $R^L O_X = O_X(L)$ is an invertible O_X -module sheaf, we obtain the relation

$$F^L O_X = O_X(L) = (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) O_X.$$

We represent the strict transform of the scheme $\text{Spec}(A/y_i)$ by ψ as $W_{y_i, \psi}$ for $i = 1, \dots, d$. Since $(y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) O_X$ is locally free, $W_{y_1, \psi} \cap \dots \cap W_{y_d, \psi} \cap E$ is empty. Here

$$W_{y_j, \psi} \cap E = \text{Proj}(G/\text{In}(y_j)G),$$

where $\text{In}(y_j)$ is the initial homogeneous element of y_j . Therefore $\text{In}(y_1), \dots, \text{In}(y_d)$ is a parameter system of G .

(4.4) *Proof of the inequality (ii) of (1).* There is an integer L satisfies the relation $G|_{mL} = (G|_L)^m$, that is $(G^h)^{(L)} = G[G|_L]$. Now we have $e(G|_L, G) = L^d \cdot \lim_{\lambda \rightarrow 1} (1 -$

$\lambda)^d P(G, \lambda)$ by (4.2). We can easily see $G|_{q,L} \subset (G_+)^L$. Hence

$$\begin{aligned} L^d e(G_+, G) &= e((G_+)^L, G) \leq e(G|_{q,L}, G) = e((G|_L)^{q_*}) = q_*^d e(G|_L, G) \\ &= q_*^d \cdot L^d \cdot \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda) \end{aligned}$$

Therefore $e(G_+, G) \leq q_*^d \cdot \lim_{\lambda \rightarrow 1} (1 - \lambda)^d P(G, \lambda)$.

(4.5) *Proof of (3)* By assumption $(G|_L)^{q_*}$ is a reduction of G_+^L . As same as in the arguments of (1) we consider the filtered blowing up ψ of $\text{Spec}(G)$. By the assumption G is described by Demazure's method as $G = R(E, D)$. As in §3, we will represent ψ as :

$$\psi : C = C(E, D) = \text{Spec}_E(\oplus_{k \geq 0} O_E(kD)) \rightarrow \text{Spec}(R).$$

We obtain the relation

$$R_+^L O_C = (R|_L)^{q_*} O_C = O_C(-q_* L E) \text{ on } C.$$

Since x_1^L is not contained in $R|_{L_{q_1+1}}$, we have the relation $R_+^L O_C = O_C(-L_{q_1} E)$. Hence $q_1 = q_*$.

Acknowledgements. この仕事をするのにあたって、全体にわたって渡辺敬一先生より数え切れない貴重な助言を賜りました。深く感謝の意を表したいと思います。

References

- [D] Demazure, M.: Anneaux gradués normaux, in Seminaire Demazure-Giraud-Teissier, Singularites des surfaces. Ecole Polytechnique (1979) in "Introduction a la théorie des singularités II; Méthodes algébriques et géométriques" edited by Le D.T. (1988), 35-68, Travaux En Cours vol. 37, Hermann, Paris
- [Do] Dolgachev, I : Weighted projective spaces. " Group Actions and Vector Fields " Proc. Vancouver 1981. LN in Math. 956. 34-71 Springer Verlag.
- [Fletcher] A. R. Fletcher., Plurigeners of 3-folds and weighted hypersurfaces, Thesis 1988 Aug.,
- [GW] Goto, S., Watanabe, K.-i.: On graded rings, I. J. Math. Soc. Japan., 30, 179-213

- (1978).
- [HIO] Herrmann, M., Ikeda, S., Orbanz, U.: Equimultiplicity and blowing-up. Springer 1988 pp 629.
- [IW] Ishii, S., Watanabe, Kimio : On simple K3-singularities (in Japanese). Note appeared in the proceedings of the conference of Algebraic Geometry at Tokyo Metropolitan Univ. 1988. pp. 20-31.
- [Kl] Kleiman, S.: Toward a numerical theory of ampleness . Ann. of Math. (1966) Vol. 84, pp293-344.
- [L1] Laufer, H. B. : Tangent cones for deformations of two-dimensional quasi-homogeneous singularities. preprint 1986.
- [Lipman] Lipman, J.: Equimultiplicity , reduction and blowing-up, " Commutative Algebra ; Analytic Methods " Dekker, New york (1982) ,111-147.
- [Matsumura] Matsumura, H.: Commutative ring theory, Cambridge studies in advanced mathematics 8 (1980 published in Japan translation by M. Reid and published in 1986).
- [NR] Northcott, D.G., Rees, D.: A note on reduction of ideals with application to the generalized Hilbert function, Proc. Camb. Phil. Soc. 50 (1954), 353-359.
- [R1] Reid, M.: Canonical 3-folds. Journees de geometrie algebrique d'Anges 1979, "Algebraic geometry " edited by A.Beauville, Siythhoff and Noordhoff, 273-310 (1980).
- [Rees] Rees, D.: Lectures on the asymptotic theory of ideals , London Mathematical Society Lecture Note Series 113, Cambridge Univ. Press
- [S1] Saito, K.: Quasihomogene isolierte Singularitäten von Hyperflächen. Invent. Math. 14, 123-142 (1971).
- [S3] —: Regular system of weights and associated singularities. Advanced studies in Math. 8 (1986) Kinokuniya-North-Holland 479-526.
- [T1] Tomari, M : The canonical filtration of higherdimensional purely elliptic singularity of a special type. preprint 1989.
- [TW1] Tomari, M., Watanabe, K.-i.: Filtered rings, filtered blowing-ups and normal two-dimensional singularities with " star-shaped " resolution. to appear in Publ.RIMS.Kyoto Univ.,25-5 (1989).
- [Wagreich] Wagreich, Ph., Elliptic singularities of surfaces. Amer. J. Math., 92. (1970), 419-454.
- [W敬1] Watanabe, K.-i.: Some remarks concerning Demazure's constracution of normal graded rings. Nagoya Math. J. 83, 203-211 (1981).

[W 敬 2] —: Rational singularities with k^* -action. In Commutative algebra. Proc. Trento Conf. edited by S. Greco and G. Valla. Lecture Notes in Pure and applied Math. No.84. (1983) 339-351 Marcel Dekker.

[Y] Yonemura, T.: On hypersurface simple K3-singularities. in preprint 1989.