

An algorithmic study on the integration of holonomic hyperfunctions — oscillatory integrals and a phase space integral associated with a Feynman diagram

By

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Abstract

Let $u(x, y)$ be a generalized function which satisfies a holonomic system M of linear differential equations with polynomial coefficients. Suppose that $u(x, y)$ is integrable with respect to x and let $v(y)$ be its integral. We give a sufficient condition for $v(y)$ to satisfy the D -module theoretic integration module of M , which can be computed algorithmically. We present some examples related to oscillatory integrals and Cutkosky-type phase space integrals associated with Feynman diagrams

§ 1. Introduction

In this paper, we call a distribution, or more generally, a hyperfunction *holonomic* for short, if it satisfies a holonomic system of linear differential equations with *polynomial* coefficients. The integration of a holonomic function with respect to some of its variables is again holonomic if the integrand is ‘rapidly decreasing’ with respect to the integration variables. Moreover, a holonomic system for the integral is defined naturally as a D -module and is computable, at least in theory, under this condition.

First we give a sufficient condition for the integral to be well-defined and to satisfy the D -module theoretic integration module, or the direct image. This allows us to treat, e.g., the oscillatory integral with a polynomial phase and a holonomic distribution as the amplitude function which is ‘rapidly decreasing’ with respect to the integration variables.

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Then we give some examples of computation of holonomic systems for such oscillatory integrals and their Fourier transforms, as well as what are called Cutkosky-type phase space integrals associated with Feynman diagrams.

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§ 2. Integration of generalized functions

Let $\varpi : \mathbb{R}^{n+d} \ni (x, y) \mapsto y \in \mathbb{R}^d$ be the projection with the standard coordinates $x = (x_1, \dots, x_n)$ of \mathbb{R}^n and $y = (y_1, \dots, y_d)$ of \mathbb{R}^d .

Then the integration along the fibers of ϖ gives a sheaf homomorphism $\varpi_! \mathcal{B}_{\mathbb{R}^{n+d}} \rightarrow \mathcal{B}_{\mathbb{R}^d}$, and hence, in particular, a homomorphism

$$\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}}) \longrightarrow \Gamma(U, \mathcal{B}_{\mathbb{R}^d})$$

for an open set U of \mathbb{R}^d . Here $\mathcal{B}_{\mathbb{R}^n}$ stands for the sheaf of hyperfunctions on \mathbb{R}^n and $\varpi_!$ the sheaf-theoretic direct image with proper supports.

For example, for a real polynomial f in x , and a distribution φ on \mathbb{R}^n , we are interested in the integrals

$$I(f, \varphi)(t) = \int_{\mathbb{R}^n} \delta(t - f(x)) \varphi(x) dx, \quad \hat{I}(f, \varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx.$$

$I(f, \varphi)$ and $\hat{I}(f, \varphi)$ are related by Fourier transformation. If φ is a probability density function, then $I(f, \varphi)$ is that of the random variable $f(x)$, and $\hat{I}(f, \varphi)$ is the characteristic function. We do not assume that φ has a compact support. Hence the integrands do not belong to $\Gamma(\mathbb{R}, \varpi_! \mathcal{B}_{\mathbb{R}^{n+1}})$ in general.

Definition 2.1. We call a pair of classes $(\mathcal{F}_{n,d}, \mathcal{F}_{0,d})$ adapted to the projection $\varpi : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^d$ if the following conditions are satisfied:

1. $\mathcal{F}_{n,d}$ is a left module over the ring $D_{n+d} = \mathbb{C}\langle x, y, \partial_x, \partial_y \rangle$ of differential operators with polynomial coefficients in the variables (x, y) with the notation $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j} = \partial/\partial x_j$.
2. $\mathcal{F}_{0,d}$ is a left module over $D_d = \mathbb{C}\langle y, \partial_y \rangle$.
3. There exists a \mathbb{C} -linear map $\varpi_* : \mathcal{F}_{n,d} \longrightarrow \mathcal{F}_{0,d}$.

4. For any $u \in \mathcal{F}_{n,d}$, $P \in D_d$, and $j = 1, \dots, n$, one has

$$P\varpi_*(u) = \varpi_*(Pu), \quad \varpi_*(\partial_{x_j}u) = 0.$$

The first example of a pair adapted to ϖ is $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ and $\Gamma(U, \mathcal{B}_{\mathbb{R}^d})$ with $\varpi_*(u(x, y)) = \int_{\mathbb{R}^n} u(x, y) dy$ for $u \in \Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$.

As the second example, let $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ be the subspace of $\mathcal{S}'(\mathbb{R}^{n+d})$ consisting of distributions of the form

$$(2.1) \quad u(x, y) = \sum_{j=1}^m u_j(x)v_j(x, y) \quad (m \in \mathbb{N}, u_j \in \mathcal{S}(\mathbb{R}^n), v_j \in \mathcal{S}'(\mathbb{R}^{n+d})),$$

where \mathcal{S} and \mathcal{S}' denote the space of rapidly decreasing functions and that of tempered distributions respectively. Then $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ is a left D_{n+d} -submodule of $\mathcal{S}'(\mathbb{R}^{n+d})$.

As a special case $d = 0$, we denote by $\mathcal{SS}'(\mathbb{R}^n)$ the subspace of $\mathcal{S}'(\mathbb{R}^n)$ consisting of distributions of the form

$$u(x) = \sum_{j=1}^m u_j(x)v_j(x) \quad (m \in \mathbb{N}, u_j \in \mathcal{S}(\mathbb{R}^n), v_j \in \mathcal{S}'(\mathbb{R}^n)).$$

For a distribution $u(x, y)$ in $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$, the integral $\varpi_*(u(x, y)) = \int_{\mathbb{R}^n} u(x, y) dx$ is defined by the pairing

$$\left\langle \int_{\mathbb{R}^n} u(x, y) dx, \varphi(y) \right\rangle = \sum_{j=1}^m \langle v_j(x, y), u_j(x)\varphi(y) \rangle \quad (\forall \varphi \in \mathcal{S}(\mathbb{R}^d)).$$

It does not depend on the choice of expression (2.1). In fact, assume $u(x, y) = 0$ in (2.1) and take $\chi(x)$ which belongs to the space $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable functions with compact support such that $\chi(x) = 1$ if $|x| \leq 1$. Then for an arbitrary constant $r > 0$, we have an equality

$$0 = \left\langle \sum_{j=1}^m u_j(x)v_j(x, y), \chi(x/r)\varphi(y) \right\rangle = \sum_{j=1}^m \langle v_j(x, y), \chi(x/r)u_j(x)\varphi(y) \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $\chi(x/r)u_j(x)\varphi(y)$ converges to $u_j(x)\varphi(y)$ in $\mathcal{S}(\mathbb{R}^{n+d})$ as $r \rightarrow \infty$, we get

$$\sum_{j=1}^m \langle v_j(x, y), u_j(x)\varphi(y) \rangle = 0.$$

Proposition 2.2 (differentiation under the integral sign). *Suppose that $u(x, y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ with an open set U of \mathbb{R}^d , or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then*

$$P(y, \partial_y) \int_{\mathbb{R}^n} u(x, y) dx = \int_{\mathbb{R}^n} P(y, \partial_y)u(x, y) dx$$

holds for any $P = P(y, \partial_y) \in D_d$.

Proof. First we suppose $u(x, y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ and is defined by

$$u(x, y) = \sum_{j=1}^m u_j(x) v_j(x, y)$$

with $u_j \in \mathcal{S}(\mathbb{R}^n)$ and $v_j \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} \left\langle \partial_{y_i} \int_{\mathbb{R}^n} u(x, y) dx, \varphi(y) \right\rangle &= - \left\langle \int_{\mathbb{R}^n} u(x, y) dx, \partial_{y_i} \varphi(y) \right\rangle \\ &= - \sum_{j=1}^m \langle v_j(x, y), u_j(x) \partial_{y_i} \varphi(y) \rangle = - \sum_{j=1}^m \langle v_j(x, y), \partial_{y_i} (u_j(x) \varphi(y)) \rangle \\ &= \sum_{j=1}^m \langle \partial_{y_i} v_j(x, y), u_j(x) \varphi(y) \rangle = \left\langle \int_{\mathbb{R}^n} \partial_{y_i} u(x, y) dx, \varphi(y) \right\rangle \end{aligned}$$

and

$$\begin{aligned} \left\langle y_i \int_{\mathbb{R}^n} u(x, y) dx, \varphi(y) \right\rangle &= \sum_{j=1}^m \langle v_j(x, y), y_i u_j(x) \varphi(y) \rangle \\ &= \sum_{j=1}^m \langle y_i v_j(x, y), u_j(x) \varphi(y) \rangle = \left\langle \int_{\mathbb{R}^n} y_i u(x, y) dx, \varphi(y) \right\rangle. \end{aligned}$$

Next, let us assume $u(x, y)$ to belong to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$. Moreover, by induction on n , we may assume $n = 1$. Since the statement is local in U , we may suppose that U is convex and the support of $u(x, y)$ is contained in $[-R/2, R/2] \times U$ with $R > 0$. Then there exists a hyperfunction $v(x, y)$ on $\mathbb{R} \times U$ such that $\partial_x v(x, y) = u(x, y)$ whose singular spectrum S.S. $v(x, y)$ does not contain the points $(\pm R, y; \pm \sqrt{-1} dx)$ with $y \in U$ in the purely imaginary cotangent bundle $\sqrt{-1} T^*(\mathbb{R} \times \mathbb{R}^d)$. See Proposition 3.2.1 and subsequent arguments in Kashiwara-Kawai-Kimura [3] on integration of hyperfunctions. This implies that x is a real analytic parameter of $v(x, y)$ on a neighborhood of $\{\pm R\} \times U$. Hence $v(\pm R, y)$ are well-defined as hyperfunctions on U and one has

$$\int_{-\infty}^{\infty} u(x, y) dx = v(R, y) - v(-R, y)$$

by the definition. One also has

$$\int_{-\infty}^{\infty} P(y, \partial_y) u(x, y) dx = P(y, \partial_y) v(R, y) - P(y, \partial_y) v(-R, y)$$

for any $P \in D_d$ since $\partial_x P(y, \partial_y) v(x, y) = P(y, \partial_y) u(x, y)$. Thus we get

$$\begin{aligned} P(y, \partial_y) \int_{-\infty}^{\infty} u(x, y) dx &= P(y, \partial_y) v(R, y) - P(y, \partial_y) v(-R, y) \\ &= \int_{-\infty}^{\infty} P(y, \partial_y) u(x, y) dx. \end{aligned}$$

□

Proposition 2.3. Suppose that $u(x, y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ with an open set U of \mathbb{R}^d , or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then one has

$$\int_{\mathbb{R}^n} \partial_{x_j} u(x, y) dx = 0 \quad (j = 1, \dots, n).$$

Proof. First suppose that $u(x, y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. We may assume, without loss of generality, that $u(x, y) = v(x)w(x, y)$ with $v \in \mathcal{S}(\mathbb{R}^n)$ and $w \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} & \left\langle \int_{\mathbb{R}^n} \partial_{x_j} (v(x)w(x, y)) dx, \varphi(y) \right\rangle \\ &= \left\langle \int_{\mathbb{R}^n} (\partial_{x_j} v(x))w(x, y) dx, \varphi(y) \right\rangle + \left\langle \int_{\mathbb{R}^n} v(x)(\partial_{x_j} w(x, y)) dx, \varphi(y) \right\rangle \\ &= \langle w(x, y), (\partial_{x_j} v(x))\varphi(y) \rangle + \langle \partial_{x_j} w(x, y), v(x)\varphi(y) \rangle \\ &= \langle w(x, y), (\partial_{x_j} v(x))\varphi(y) \rangle - \langle w(x, y), \partial_{x_j} (v(x)\varphi(y)) \rangle = 0. \end{aligned}$$

Next, let us assume that $u(x, y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$. We may also assume that U is convex, $n = 1$, and the support of $u(x, y)$ is contained in $[-R/2, R/2] \times U$ with $R > 0$. Then by the definition of the integration, we have

$$\int_{-\infty}^{\infty} \partial_x u(x, y) dx = u(R, y) - u(-R, y) = 0.$$

□

Hence the pairs $(\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}}), \Gamma(U, \mathcal{B}_{\mathbb{R}^d}))$ and $(\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ are adapted to the projection ϖ of \mathbb{R}^{n+d} to \mathbb{R}^d .

§ 3. Integration of D -modules

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$ be (complex or real) variables. We set $X = \mathbb{C}^{n+d}$ and $Y = \mathbb{C}^d$ and let $\varpi_{\mathbb{C}} : X \ni (x, y) \mapsto y \in Y$ be the projection. We denote by $D_X = D_{n+d}$ the ring of differential operators in the variables (x, y) , and by $D_Y = D_d$ that in the variables y . The module

$$D_{Y \leftarrow X} := D_X / (\partial_{x_1} D_X + \dots + \partial_{x_n} D_X)$$

has a structure of (D_Y, D_X) -bimodule. The *integral* of a left D_X -module M along the fibers of $\varpi_{\mathbb{C}}$, or the *direct image* by $\varpi_{\mathbb{C}}$ is defined to be the left D_Y -module

$$(\varpi_{\mathbb{C}})_* M := D_{Y \leftarrow X} \otimes_{D_X} M = M / (\partial_{x_1} M + \dots + \partial_{x_n} M).$$

For an element u of M , let $[u]$ be its residue class in $(\varpi_{\mathbb{C}})_*M$. If M is generated by u_1, \dots, u_r over D_X , then $(\varpi_{\mathbb{C}})_*M$ is generated by the set $\{x^\alpha[u_j] \mid 1 \leq j \leq r, \alpha \in \mathbb{N}^n\}$ over D_Y .

Let $(\mathcal{F}_{n,d}, \mathcal{F}_n)$ be a pair adapted to ϖ . Let h be a D_X -homomorphism from M to $\mathcal{F}_{n,d}$. Let us define a \mathbb{C} -linear map h' from M to \mathcal{F}_d by

$$h'(u) = \varpi_*(h(u)) \quad (\forall u \in M),$$

which is D_Y -linear and we have

$$\partial_{x_1}M + \dots + \partial_{x_n}M \subset \ker h'.$$

Hence h' induces a D_Y -homomorphism

$$\varpi_*(h) : (\varpi_{\mathbb{C}})_*M \longrightarrow \mathcal{F}_d.$$

In conclusion, we have defined a \mathbb{C} -linear map

$$\varpi_* : \text{Hom}_{D_X}(M, \mathcal{F}_{n,d}) \longrightarrow \text{Hom}_{D_Y}((\varpi_{\mathbb{C}})_*M, \mathcal{F}_d).$$

If M is a holonomic D_X -module, then $(\varpi_{\mathbb{C}})_*M$ is a holonomic D_Y -module. An algorithm to compute $(\varpi_{\mathbb{C}})_*M$, which works at least if M is holonomic, was given in [9], [10]; see also [8]. For practical computation, we use a library file `nk_restriction.rr` by Hiromasa Nakayama for the computer algebra system Risa/Asir [7].

§ 4. Oscillatory integrals

Let $f(x)$ be a real polynomial in the real variables $x = (x_1, \dots, x_n)$. Suppose that $\varphi(x)$ belongs to $\mathcal{SS}'(\mathbb{R}^n)$. Let t and τ be real variables. Since both $\delta(t - f(x))\varphi(x)$ and $e^{itf(x)}\varphi(x)$ belong to $\mathcal{SS}'(\mathbb{R}_x^n \times \mathbb{R}_t)$, the integrals

$$F(t) = I(f, \varphi)(t) = \int_{\mathbb{R}^n} \delta(t - f(x))\varphi(x) dx, \quad G(t) = \hat{I}(f, \varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x) dx$$

are well-defined as elements of $\mathcal{S}'(\mathbb{R})$. The integral $\hat{I}(f, \varphi)(t)$ is called the oscillatory integral with the phase function $f(x)$ and the amplitude function $\varphi(x)$, which is usually assumed to belong to $C_0^\infty(\mathbb{R}^n)$ in the literature (see e.g., [5]).

Proposition 4.1. *Define $F(t)$ and $G(\tau)$ as above with $\varphi \in \mathcal{SS}'(\mathbb{R}^n)$ and $f \in \mathbb{R}[x]$. Then $F(t)$ and $G(\tau)$ are related by*

$$G(\tau) = \hat{F}(\tau) := \int_{-\infty}^{\infty} e^{it\tau} F(t) dt, \quad F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\tau} G(\tau) d\tau,$$

where the integrals make sense as Fourier transformation in $\mathcal{S}'(\mathbb{R})$. Moreover $G(\tau)$ belongs to $C^\infty(\mathbb{R})$.

Proof. We may assume that $\varphi(x) = \psi(x)u(x)$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then by the definition of the integral of an element of $\mathcal{SS}'(\mathbb{R}_x^n \times \mathbb{R}_t)$, we get, for any $\chi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \langle \hat{F}, \chi \rangle &= \left\langle \int_{\mathbb{R}^n} \psi(x) \delta(t - f(x)) u(x) dx, \hat{\chi} \right\rangle = \langle \delta(t - f(x)) u(x), \psi(x) \hat{\chi}(t) \rangle \\ &= \langle \delta(t) u(x), \psi(x) \hat{\chi}(t + f(x)) \rangle = \langle u(x), \psi(x) \hat{\chi}(f(x)) \rangle_x \\ &= \left\langle u(x), \psi(x) \int_{-\infty}^{\infty} e^{itf(x)} \chi(t) dt \right\rangle_x \\ &= \int_{-\infty}^{\infty} \left\langle u(x), \psi(x) e^{itf(x)} \chi(t) \right\rangle_x dt = \int_{-\infty}^{\infty} \left\langle u(x), \psi(x) e^{itf(x)} \right\rangle_x \chi(t) dt \end{aligned}$$

in view of the lemma below. This implies

$$\hat{F}(\tau) = \left\langle u(x), \psi(x) e^{i\tau f(x)} \right\rangle_x = \int_{\mathbb{R}^n} \psi(x) e^{i\tau f(x)} u(x) dx = G(\tau)$$

and that $G(\tau)$ belongs to $C^\infty(\mathbb{R})$. □

Lemma 4.2. *Assume that $f(x)$ is a real polynomial in x , χ belongs to $\mathcal{S}(\mathbb{R})$, and that ψ belongs to $\mathcal{S}(\mathbb{R}^n)$. Then*

$$\psi(x) \int_{-\infty}^{\infty} e^{itf(x)} \chi(t) dt = \lim_{R \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{R}{N} \sum_{k=-N}^{N-1} \psi(x) \exp\left(i \frac{Rk}{N} f(x)\right) \chi\left(\frac{Rk}{N}\right)$$

holds in the topology of $\mathcal{S}(\mathbb{R}^n)$.

Proof. Since it is easy to see that

$$\psi(x) \int_{-\infty}^{\infty} e^{itf(x)} \chi(t) dt = \psi(x) \lim_{R \rightarrow \infty} \int_{-R}^R e^{itf(x)} \chi(t) dt$$

holds in the topology of $\mathcal{S}(\mathbb{R}^n)$, let us show

$$\psi(x) \int_{-R}^R e^{itf(x)} \chi(t) dt = \lim_{N \rightarrow \infty} \frac{R}{N} \sum_{k=-N}^{N-1} \psi(x) \exp\left(i \frac{Rk}{N} f(x)\right) \chi\left(\frac{Rk}{N}\right)$$

in $\mathcal{S}(\mathbb{R}^n)$. For integers k, ν with $-N \leq k \leq N-1$ and $\nu \geq 0$, there exist t_k, t'_k in the interval $\left[\frac{R}{N}k, \frac{R}{N}(k+1)\right]$, which depend on x , such that

$$\int_{-R}^R e^{itf(x)} t^\nu \chi(t) dt = \frac{R}{N} \sum_{k=-N}^{N-1} \left\{ \cos(t_k f(x)) t_k^\nu \chi(t_k) + i \sin(t'_k f(x)) t_k'^\nu \chi(t'_k) \right\}.$$

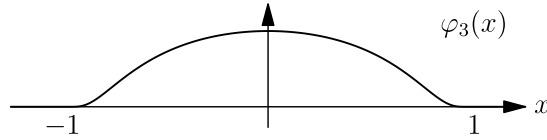
Hence

$$\begin{aligned}
& \left| \int_{-R}^R e^{itf(x)} t^\nu \chi(t) dt - \frac{R}{N} \sum_{k=-N}^{N-1} \exp\left(i \frac{Rk}{N} f(x)\right) \left(\frac{Rk}{N}\right)^\nu \chi\left(\frac{Rk}{N}\right) \right| \\
& \leq \frac{R}{N} \sum_{k=-N}^{N-1} \left| \left\{ \cos(t_k f(x)) t_k^\nu \chi(t_k) + i \sin(t'_k f(x)) t_k'^\nu \chi(t'_k) \right\} \right. \\
& \quad \left. - \exp\left(i \frac{Rk}{N} f(x)\right) \left(\frac{Rk}{N}\right)^\nu \chi\left(\frac{Rk}{N}\right) \right| \\
& \leq \frac{R}{N} \sum_{k=-N}^{N-1} C(|f(x)| + 1) \max \left\{ \left| t_k - \frac{Rk}{N} \right|, \left| t'_k - \frac{Rk}{N} \right| \right\} \\
& \leq \frac{2R^2}{N} C(|f(x)| + 1)
\end{aligned}$$

with some constant C independent of x . This implies the assertion. \square

Example 4.3. Set $n = 1$ and $x = x_1$. Let us choose

$$\varphi_1(x) = \exp\left(-\frac{x^2}{2}\right), \quad \varphi_2(x) = Y(1 - x^2), \quad \varphi_3(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$



as amplitude functions and set

$$F_k(t) = \int_{-\infty}^{\infty} \delta(t - x^2) \varphi_k(x) dx, \quad G_k(\tau) = \int_{-\infty}^{\infty} e^{i\tau x^2} \varphi_k(x) dx \quad (k = 1, 2, 3).$$

Here $Y(x)$ denotes the Heaviside function. Then $F_k(t)$ satisfy differential equations

$$(2t\partial_t + t + 1)F_1(t) = 0, \quad (t - 1)(2t\partial_t + 1)F_2(t) = 0, \quad (2t(t - 1)^2\partial_t + t^2 + 1)F_3(t) = 0$$

respectively. The point 0 is a regular singular point of the three differential equations with characteristic exponent $-1/2$. The point 1 is a regular singular point of the second equation with characteristic exponent 0, but is an irregular singular point of the last equation. Consequently we get

$$F_1(t) = t_+^{-1/2} e^{-t/2}, \quad F_2(t) = t_+^{-1/2} Y(1 - t)$$

in view of

$$\int_{-\infty}^{\infty} F_k(t) dt = \int_{-\infty}^{\infty} \varphi_k(x) dx,$$

and

$$F_3(t) = C_3 t_+^{-1/2} Y(1-t) \exp\left(-\frac{1}{1-t}\right)$$

with some constant C_3 .

On the other hand, G_k ($k = 1, 2, 3$) belong to $\mathcal{S}'(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and satisfy differential equations

$$\begin{aligned} ((2\tau + i)\partial_\tau + 1)G_1(\tau) &= 0, \\ (2\tau\partial_\tau^2 + (-2i\tau + 3)\partial_\tau - i)G_2(\tau) &= 0, \\ (2\tau\partial_\tau^3 + (-4i\tau + 5)\partial_\tau^2 + (-2\tau - 8i)\partial_\tau - 1)G_3(\tau) &= 0 \end{aligned}$$

respectively. In particular, we have $G_1(\tau) = \sqrt{2\pi}(1-2i\tau)^{-1/2}$. The equations for $G_2(\tau)$ and for $G_3(\tau)$ have 0 as a regular singular point and the point at infinity as an irregular singular point. Note that $G_2(\tau)$ and $G_3(\tau)$ are entire, i.e., holomorphic on \mathbb{C} .

Example 4.4. Set

$$F(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{2}\right) \delta(t - f(x)) dx$$

with a quadratic form $f(x) = \sum_{i,j=1}^n a_{ij}x_i x_j$. If the absolute values of all the eigenvalues of the matrix (a_{ij}) are the same, then $F(t)$ satisfies a linear differential equation of the second order. We may assume

$$f(x) = a(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2)$$

with a constant $a > 0$. Then the integrand $u = u(x, t)$ satisfies

$$(t - f(x))u = (\partial_{x_i} + 2ax_i\partial_t + x_i)u = (\partial_{x_j} - 2ax_j\partial_t + x_j)u = 0 \quad (1 \leq i \leq p < j \leq n).$$

The following operators P and Q annihilate u :

$$\begin{aligned} P &= \sum_{i=1}^p x_i(\partial_{x_i} + 2ax_i\partial_t + x_i) + \sum_{i=p+1}^n x_i(\partial_{x_i} - 2ax_i\partial_t + x_i) \\ &= \sum_{i=1}^n \partial_{x_i} x_i + 2f\partial_t + |x|^2 - n = \sum_{i=1}^n \partial_{x_i} x_i + 2\partial_t t + |x|^2 - n - 2\partial_t(t - f), \\ Q &= \sum_{i=1}^p x_i(\partial_{x_i} + 2ax_i\partial_t + x_i) - \sum_{i=p+1}^n x_i(\partial_{x_i} - 2ax_i\partial_t + x_i) \\ &= \sum_{i=1}^p \partial_{x_i} x_i - \sum_{i=p+1}^n \partial_{x_i} x_i + 2a|x|^2\partial_t + \frac{1}{a}f + n - 2p. \end{aligned}$$

Hence

$$2a\partial_t P - Q = 2a \sum_{i=1}^p \partial_{x_i} x_i \partial_t - \sum_{i=1}^p \partial_{x_i} x_i + \sum_{i=p+1}^n \partial_{x_i} x_i + (-4a\partial_t^2 + \frac{1}{a})(t - f) \\ + 4a\partial_t^2 t - 2na\partial_t - \frac{1}{a}t + (2p - n)$$

implies

$$\{4a^2 t \partial_t^2 + 2a^2(4 - n)\partial_t - t + (2p - n)a\}F(t) = 0.$$

The solutions of this differential equation are expressed as

$$P \left\{ \begin{array}{ccc} \overbrace{\frac{1}{2a} \quad 1 - \frac{p}{2}}^{\infty} & 0 & \\ -\frac{1}{2a} & -\frac{1}{4}(2n - 2p - 4) & \frac{n-2}{2} \end{array} \quad t \right\}.$$

On the other hand,

$$G(\tau) = \int_{\mathbb{R}^n} \exp\left(i\tau f(x) - \frac{|x|^2}{2}\right)$$

satisfies a differential equation

$$\{(4a^2\tau^2 + 1)\partial_\tau + a(2na\tau + (n - 2p)i)\}G(\tau) = 0.$$

It follows that

$$G(\tau) = \exp\left(i\left(p - \frac{n}{2}\right) \tan^{-1}(2a\tau)\right) (4a^2\tau^2 + 1)^{-n/4}.$$

More generally, if $f(x)$ is a general quadratic form with eigenvalues a_1, \dots, a_n , then one has

$$G(\tau) = \prod_{k=1}^n (1 - 2ia_k\tau)^{-1/2} = \prod_{k=1}^n (4a_k^2\tau^2 + 1)^{-1/4} \exp\left(-\frac{i}{2} \tan^{-1}(2a_k\tau)\right)$$

since $G(\tau) = (1 - 2i\tau)^{-1/2}$ if $n = 1$ and $a = a_1 = 1$.

Example 4.5. Set $f(x, y) = x^3 - y^2$ and

$$F(t) = (2\pi)^{-1} \int_{\mathbb{R}^2} \exp\left(-\frac{x^2 + y^2}{2}\right) \delta(t - f(x, y)) dx dy.$$

Then $F(t)$ satisfies

$$\{108t^2\partial_t^5 + (-108t^2 + 648t)\partial_t^4 + (27t^2 - 486t + 627)\partial_t^3 + (85t - 303)\partial_t^2 + (-4t + 21)\partial_t + t - 3\}F(t) = 0.$$

It has a regular singularity at 0 with the indicial equation $s(s-1)(s-2)(6s+1)(6s-7)$. Note that the b -function of f is $(s+1)(6s+5)(6s+7)$.

Example 4.6. Set $f(x, y, z) = x^2 - y^2z$ and

$$F(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp\left(-\frac{x^2 + y^2 + z^2}{2}\right) \delta(t - f(x, y, z)) dx dy dz.$$

Then $F(t)$ satisfies

$$\{16t^2\partial_t^5 + (16t^2 + 96t)\partial_t^4 + (4t^2 + 72t + 96)\partial_t^3 + (16t + 48)\partial_t^2 + (4t + 9)\partial_t + t + 3\}F(t) = 0.$$

It has a regular singularity at 0 with the indicial equation $s^2(s-1)^2(s-2)$. Note that the b -function of f is $(s+1)^2(2s+3)$.

Example 4.7. Set $f(x, y, z) = x^3 - y^2z^2$ and

$$F(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp\left(-\frac{x^2 + y^2 + z^2}{2}\right) \delta(t - f(x, y, z)) dx dy dz.$$

Then $F(t)$ satisfies a linear ordinary differential equation of order 10 which has a regular singularity at 0 with the indicial equation

$$s(s-1)(s-2)(s-3)(s-4)(s-5)(6s+1)^2(6s-7)^2 = 0.$$

Note that the b -function of f is $(s+1)(3s+4)(3s+5)(6s+5)^2(6s+7)^2$.

§ 5. Cutkosky-type phase space integrals associated with Feynman diagrams

The Cutkosky-type phase space integral associated with a Feynman diagram G , which we will call the phase space integral for short, describes the discontinuity of the Feynman integral $F_G(p)$ along its singularity locus. We consider simple Feynman diagrams in two-dimensional space-time for the sake of simplicity in actual computation, inspired by the recent work by Honda and Kawai (see e.g., [1], [2]) on the Landau-Nakanishi surface associated with G .

In general, for a two-dimensional vector $\mathbf{p} = (p_0, p_1)$, we denote $\mathbf{p}^2 = p_0^2 - p_1^2$ for the Lorentz norm and $d\mathbf{p} = dp_0 dp_1$ for the volume element. Let m be a positive constant.

Then the delta function $\delta(\mathbf{p}^2 - m^2)$ is well-defined and its support coincides with the curve $\mathbf{p}^2 - m^2 = 0$ in the 2-dimensional space-time \mathbb{R}^2 . We set

$$\delta_+(\mathbf{p}^2 - m^2) = Y(p_0)\delta(\mathbf{p}^2 - m^2),$$

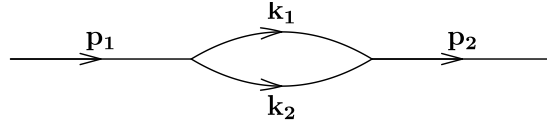
which is well-defined since the line $p_0 = 0$ is disjoint from the curve $\mathbf{p}^2 = m^2$. Its support is contained in $\{\mathbf{p} \mid \mathbf{p}^2 = m^2, p_0 \geq m\}$. Moreover, if $P \in D_2$ annihilates $\delta(\mathbf{p}^2 - m^2)$, then it also annihilates $\delta_+(\mathbf{p}^2 - m^2)$. More precisely it is easy to see that

$$\text{Ann}_{D_2}\delta_+(\mathbf{p}^2 - m^2) = \text{Ann}_{D_2}\delta(\mathbf{p}^2 - m^2)$$

holds, where D_2 is the ring of differential operators with polynomial coefficients with respect to the variables p_0, p_1 . On the other hand, we denote by $\delta(\mathbf{p})$ the delta function $\delta(p_0)\delta(p_1)$ supported at the origin of \mathbb{R}^2 .

We give a precise definition of the Cutkosky-type phase space integral associated with a Feynman diagram in each example instead of presenting a general formulation.

Example 5.1. Let us study the Feynman diagram G with two vertices, two external lines, and two internal lines as below:



Let us assign 2-vectors $\mathbf{p}_1 = (p_{10}, p_{11})$ and $\mathbf{p}_2 = (p_{20}, p_{21})$ to the external lines, $\mathbf{k}_1 = (k_{10}, k_{11})$ and $\mathbf{k}_2 = (k_{20}, k_{21})$ to the internal lines. Then the phase space integral associated with this diagram is defined to be

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2)\delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2) d\mathbf{k}_1 d\mathbf{k}_2.$$

It is easy to see that the product in the integrand makes sense as a hyperfunction by considering the singular spectrum (the analytic wave front set) of each factor. Integration with respect to \mathbf{k}_2 yields the expression

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)I_G(\mathbf{p}_1)$$

with

$$I_G(\mathbf{p}_1) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) d\mathbf{k}_1.$$

Since $I_G(\mathbf{p}_1)$ is invariant under Lorentz transformations of \mathbf{p}_1 , we may put $\mathbf{p}_1 = (x, 0)$. Then the support of the integrand of $I_G((x, 0))$ is contained in the set

$$\begin{aligned} & \{(x, \mathbf{k}_1) \mid \mathbf{k}_1^2 - m_1^2 = (x - k_{10})^2 - k_{11}^2 - m_1^2 = 0, k_{10} > 0, x - k_{10} > 0\} \\ & \subset \{(x, \mathbf{k}_1) \mid k_{10} \geq m_1, x - k_{10} \geq m_2, |k_{11}| < k_{10}\} \\ & \subset \{(x, \mathbf{k}_1) \mid x \geq m_1 + m_2, m_1 \leq k_{10} \leq x - m_2, |k_{11}| < k_{10}\}. \end{aligned}$$

Hence the support of the integrand is proper with respect to the projection $\varpi : \mathbb{R}^3 \ni (x, \mathbf{k}_1) \mapsto x \in \mathbb{R}$. It follows that $I_G((x, 0))$ is well-defined as a hyperfunction on \mathbb{R} and its support is contained in $\{x \in \mathbb{R} \mid x \geq m_1 + m_2\}$.

Let us consider the second local cohomology group $H_I^2(\mathbb{C}[x, \mathbf{k}_1])$ with the ideal I generated by two polynomials $f_1 := \mathbf{k}_1^2 - m_1^2$ and $f_2 := (x - k_{10})^2 - k_{11}^2 - m_2^2$. Then we can identify the integrand with the cohomology class $\delta(f_1, f_2) = [1/(f_1 f_2)]$ in this local cohomology group. Since the variety $f_1 = f_2 = 0$ is non-singular, the annihilator in the ring D_3 of the integrand coincides with that of $\delta(f_1, f_2)$, which consists of first order operators together with f_1, f_2 and can be computed easily.

By virtue of Propositions 2.2 and 2.3, the integration algorithm described in [8] gives us a differential equation $PI_G((x, 0)) = 0$ with

$$P = (x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x + 2(x^2 - m_1^2 - m_2^2)x.$$

If $m_1 \neq m_2$, then we get

$$I_G((x, 0)) = C(x - m_1 + m_2)^{-1/2}(x + m_1 - m_2)^{-1/2}(x + m_1 + m_2)^{-1/2}(x - m_1 - m_2)_+^{-1/2}$$

with some constant C by quadrature noting that its support is contained in the interval $[m_1 + m_2, \infty)$ and that there is no hyperfunction solution of the differential equation above whose support is the point $m_1 + m_2$.

On the other hand, if $m_1 = m_2$, then $I_G((x, 0))$ satisfies

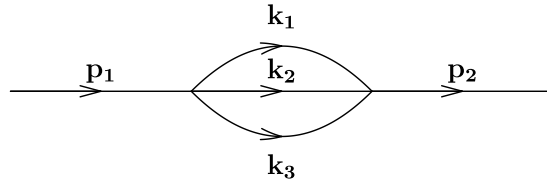
$$\{x(x^2 - 4m_1^2)\partial_x + 2(x^2 - 2m_1^2)\}I_G((x, 0)) = 0.$$

It follows that

$$I_G((x, 0)) = Cx^{-1}(x + 2m_1)^{-1/2}(x - 2m_1)_+^{-1/2}$$

with some constant C .

Example 5.2. The phase space integral associated with the Feynman diagram G with two vertices, two external lines, and three internal lines as below



is given by

$$\begin{aligned} \tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = & \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ & \times \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+(\mathbf{k}_3^2 - m_3^2) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \end{aligned}$$

with variables $\mathbf{p}_1 = (p_{10}, p_{11})$, $\mathbf{p}_2 = (p_{20}, p_{21})$, $\mathbf{k}_1 = (k_{10}, k_{11})$, $\mathbf{k}_2 = (k_{20}, k_{21})$, $\mathbf{k}_3 = (k_{30}, k_{31})$ and positive constants m_1, m_2, m_3 . We rewrite this integral as

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)I_G(\mathbf{p}_1)$$

with

$$I_G(\mathbf{p}_1) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2) d\mathbf{k}_1 d\mathbf{k}_2.$$

As in the previous example we can set $\mathbf{p}_1 = (x, 0)$. The support of the integrand is contained in

$$\{(x, \mathbf{k}_1, \mathbf{k}_2) \mid k_{10} \geq m_1, k_{20} \geq m_2, x - k_{10} - k_{20} \geq m_3, |k_{11}| < k_{10}, |k_{21}| < k_{20}\}.$$

Hence $I_G((x, 0))$ is well-defined as a hyperfunction on \mathbb{R} and its support is contained in the interval $[m_1 + m_2 + m_3, \infty)$.

Since the computation for general m_1, m_2, m_3 is intractable, let us set $m_1 = m_2 = m_3 = 1$. Then by the integration algorithm we obtain

$$\{(x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x\}I_G((x, 0)) = 0.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2 . It follows that $I_G((x, 0))$ is a locally integrable function of the form

$$I_G((x, 0)) = a(x)Y(x-3)$$

with a real analytic function $a(x)$ on the interval $(1, \infty)$ in view of the lemma below and analytic continuation. Here we have $a(3) \neq 0$ unless $I_G((x, 0))$ vanishes everywhere. The point at infinity is also a regular singular point with the indicial equation $(s-2)^2$.

On the other hand, if we set $m_1 = 1, m_2 = 2, m_3 = 3$, then we obtain

$$\begin{aligned} & \{56x^2(x-2)(x+2)(x-4)(x+4)(x-6)(x+6)\partial_x^3 \\ & + (-15x^9 + 1680x^7 - 46256x^5 + 341888x^3 - 387072x)\partial_x^2 \\ & + (-75x^8 + 5544x^6 - 98000x^4 + 404480x^2 - 96768)\partial_x \\ & - 60x^7 + 3192x^5 - 33712x^3 + 44608x\}I_G((x, 0)) = 0. \end{aligned}$$

The points $0, \pm 2, \pm 4, \pm 6$ are regular singular points. The indicial equation at 6 is $s^2(s-1)$. It follows that $I_G((x, 0))$ is a locally integrable function of the form

$$I_G((x, 0)) = a(x)Y(x-6)$$

with a real analytic function $a(x)$ on the interval $(4, \infty)$ such that $a(6) \neq 0$ unless $I_G((x, 0))$ vanishes everywhere, in view of the lemma below. The point at infinity is an irregular singular point.

Lemma 5.3. *Let P be a differential operator of the form*

$$P = x\partial_x^m + a_1(x)\partial_x^{m-1} + \cdots + a_m(x)$$

with a positive integer m and analytic functions $a_1(x), \dots, a_m(x)$ defined on a neighborhood of $x = 0$. Assume $a_1(0) = m - 1$. Let $u(x)$ be a hyperfunction defined on a neighborhood of 0 whose support is contained in $\{x \in \mathbb{R} \mid x \geq 0\}$ such that $Pu(x) = 0$. Then $u(x)$ is written in the form

$$u(x) = a(x)Y(x)$$

with a real analytic function $a(x)$ on a neighborhood of 0 such that $Pa(x) = 0$. Moreover, we have $a(0) \neq 0$ unless $u(x)$ vanishes everywhere.

Proof. Note that 0 is a regular singular point of P and its indicial polynomial at 0 is given by

$$\begin{aligned} b(\lambda) &= \lambda(\lambda - 1) \cdots (\lambda - m + 1) + a_1(0)\lambda(\lambda - 1) \cdots (\lambda - m + 2) \\ &= \lambda(\lambda - 1) \cdots (\lambda - m + 2)(\lambda - m + 1 + a_1(0)) \\ &= \lambda^2(\lambda - 1) \cdots (\lambda - m + 2). \end{aligned}$$

Since $b(\lambda) = 0$ has no integer roots greater than $m - 2$, it follows that the homomorphism $P : \mathbb{C}\{x\} \rightarrow \mathbb{C}\{x\}$ is surjective and the dimension of its kernel is $m - 1$.

By the assumption, there exists a holomorphic function $F(z)$ on the set $\{z \in \mathbb{C} \mid |z| < \varepsilon\} \setminus [0, \infty]$ with $\varepsilon > 0$ such that

$$u(x) = F(x + \sqrt{-1}0) - F(x - \sqrt{-1}0).$$

Since $P : \mathbb{C}\{x\} \rightarrow \mathbb{C}\{x\}$ is surjective, we may assume $PF(z) = 0$. Let us rewrite P as

$$P = z\partial_z^m + (m - 1)\partial_z^{m-1} + P_1\partial_z^{m-2} + \cdots + P_{m-2}\partial_z + P_{m-1} + P_m + \cdots,$$

where P_k is a differential operator of order at most $\min\{k, m - 1\}$ such that

$$P_k z^\lambda = p_k(\lambda) z^{\lambda + \max\{0, k - m + 1\}}$$

with a polynomial $p_k(\lambda)$ of λ . Following the Frobenius method, we can construct a series

$$v(z, \lambda) = \sum_{n=0}^{\infty} c_n(\lambda) z^{\lambda+n}$$

with rational functions $c_n(\lambda)$ of λ such that

$$(5.1) \quad Pv(z, \lambda) = b(\lambda)z^\lambda$$

and $c_0(\lambda) = 1$ by the recursion formula

$$\begin{aligned}
c_n(\lambda) &= - \sum_{k=1}^{\max\{m-2, n\}} \frac{(\lambda + n - k) \cdots (\lambda + n - m + 2)}{b(\lambda + n)} p_k(\lambda + n - m + 1) c_{n-k}(\lambda) \\
&\quad - \sum_{k=m-1}^n \frac{p_k(\lambda + n - k)}{b(\lambda + n)} c_{n-k}(\lambda) \\
&= - \sum_{k=1}^{\max\{m-2, n\}} \frac{p_k(\lambda + n - m + 1)}{(\lambda + n)^2 (\lambda + n - 1) \cdots (\lambda + n - k + 1)} c_{n-k}(\lambda) \\
&\quad - \sum_{k=m-1}^n \frac{p_k(\lambda + n - k)}{b(\lambda + n)} c_{n-k}(\lambda)
\end{aligned}$$

for $n = 1, 2, 3, \dots$. This implies that $c_n(\lambda)$ are regular at $\lambda = 0$. Differentiating (5.1) with respect to λ and substituting 0 for λ , we get

$$P \frac{\partial v}{\partial \lambda}(z, 0) = \frac{\partial b}{\partial \lambda}(0) + b(0) \log z = 0$$

with

$$\frac{\partial v}{\partial \lambda}(z, 0) = \sum_{n=0}^{\infty} \frac{\partial c_n}{\partial \lambda}(0) z^n + \sum_{n=0}^{\infty} c_n(0) z^n \log z.$$

Hence $F(z)$ is written in the form

$$F(z) = G(z) + a \sum_{n=0}^{\infty} c_n(0) z^n \log z$$

with a holomorphic function $G(z)$ on a neighborhood of 0 and $a \in \mathbb{C}$. Hence we get

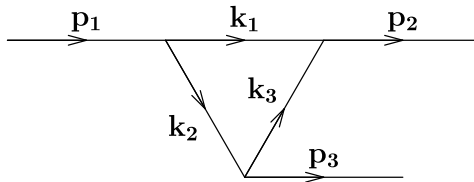
$$u(x) = 2\pi\sqrt{-1}a \sum_{n=0}^{\infty} c_n(0) x^n Y(x) = 2\pi\sqrt{-1}av(x, 0)Y(x)$$

with $c_0(0) = 1$ and $Pv(x, 0) = b(0) = 0$. □

Example 5.4. Let us consider the phase space integral

$$\begin{aligned}
\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) &= \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2) \delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_3) \delta(-\mathbf{p}_3 + \mathbf{k}_2 - \mathbf{k}_3) \\
&\quad \times \delta_+(\mathbf{k}_1^2 - m_1^2) \delta_+(\mathbf{k}_2^2 - m_2^2) \delta_+(\mathbf{k}_3^2 - m_3^2) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3
\end{aligned}$$

associated with the diagram G below.



Performing the integration with respect to \mathbf{k}_2 and \mathbf{k}_3 , we obtain

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) I_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$I_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta_+(\mathbf{k}_1^2 - m_1^2) \delta_+((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) \delta_+((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2) d\mathbf{k}_1.$$

The support of the integrand is contained in

$$\{(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1) \mid k_{10} \geq m_1, p_{10} - k_{10} \geq m_2, p_{20} - k_{10} \geq m_3, |k_{11}| < k_{10}\}.$$

Hence $I_G(\mathbf{p}_1, \mathbf{p}_2)$ is well-defined as a hyperfunction on \mathbb{R}^4 and one has

$$\text{supp } I_G(\mathbf{p}_1, \mathbf{p}_2) \subset \{(\mathbf{p}_1, \mathbf{p}_2) \mid p_{10} \geq m_1 + m_2, p_{20} \geq m_1 + m_3\}.$$

Let us set $m_1 = m_2 = m_3 = 1$, $\mathbf{p}_1 = (x, 0)$ and $\mathbf{p}_2 = ((y+z)/2, (y-z)/2)$ following Honda-Kawai-Stapp [2] and set

$$I_G(x, y, z) = I_G((x, 0), ((y+z)/2, (y-z)/2))$$

by abuse of notation. Then the integration algorithm returns a holonomic system $M = D_3/I$ for $I_G(x, y, z)$ with a left ideal I of D_3 , which is too complicated to show here. The characteristic variety of M is given by

$$\begin{aligned} \text{Char}(M) = & T_{\{f=0\}}^* \mathbb{C}^3 \cup T_{\{x=0\}}^* \mathbb{C}^3 \cup T_{\{x=f_0=0\}}^* \mathbb{C}^3 \cup T_{\{x=yz-4=0\}}^* \mathbb{C}^3 \\ & \cup T_{\{x=y=0\}}^* \mathbb{C}^3 \cup T_{\{x=z=0\}}^* \mathbb{C}^3 \cup T_{\{x=y=z=0\}}^* \mathbb{C}^3 \end{aligned}$$

with

$$f(x, y, z) = yzx^2 - yz(y+z)x + y^2z^2 + (y-z)^2, \quad f_0(y, z) = f(0, y, z),$$

where we denote by $T_Z^* \mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of \mathbb{C}^3 . The decomposition of $\text{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir [7] for prime and primary decomposition of polynomial ideals developed by M. Noro (see e.g., [4] for algorithms); he also computed a primary decomposition of the symbol ideal of I , which enabled us to compute the multiplicity of each component of $\text{Char}(M)$. Thus the characteristic cycle, i.e., the characteristic variety with multiplicity of each component, of M is

$$\begin{aligned} & T_{\{f=0\}}^* \mathbb{C}^3 + 2 T_{\{x=0\}}^* \mathbb{C}^3 + T_{\{x=f_0=0\}}^* \mathbb{C}^3 + T_{\{x=yz-4=0\}}^* \mathbb{C}^3 \\ & + T_{\{x=y=0\}}^* \mathbb{C}^3 + T_{\{x=z=0\}}^* \mathbb{C}^3 + 2 T_{\{x=y=z=0\}}^* \mathbb{C}^3. \end{aligned}$$

In particular, the support of M as D -module is the hypersurface of \mathbb{C}^3 defined by $xf(x, y, z) = 0$. The singular locus of the complex hypersurface $f = 0$ is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$. There is a stratification of the hypersurface $f = 0$ of \mathbb{C}^3 with respect to the (local) b -function $b_{f,p}(s)$ of f at a point p of each stratum as follows:

strata	$b_{f,p}(s)$
$\{(0, 0, 0)\}$	$(s + 1)^3(2s + 3)$
$\{(2, 0, 0), (-2, 0, 0), (2, 2, 2), (-2, -2, -2)\}$	$(s + 1)^2(2s + 3)$
$\{x = y = z\} \cup \{y = z = 0\} \setminus \{(0, 0, 0), (\pm 2, 0, 0), \pm(2, 2, 2)\}$	$(s + 1)^2$
$\{f = 0\} \setminus (\{x = y = z\} \cup \{y = z = 0\})$	$s + 1$

Note that the b -function of f at the points $(\pm 2, 0, 0)$ and $\pm(2, 2, 2)$ coincides with that of $g := x^2 - y^2z$ at the origin, which defines what is called the Whitney umbrella. More precisely, the b -function of g at each stratum is as follows:

stratum	$b_{g,p}(s)$
$\{(0, 0, 0)\}$	$(s + 1)^2(2s + 3)$
$\{x = y = 0\} \setminus \{(0, 0, 0)\}$	$(s + 1)^2$
$\{g = 0\} \setminus \{x = y = 0\}$	$s + 1$

However, the present author does not know whether the germ of analytic function $(f, (2, 2, 2))$, for example, is (real) analytically equivalent to $(g, (0, 0, 0))$. We used a library file `mn_ndbf.rr` of Risa/Asir [7] for the computation of the stratifications above (see e.g., [6] for algorithms).

The fiber of the characteristic variety of M at each zero-dimensional stratum of $f = 0$ is given by

$$\begin{aligned} \pi^{-1}((0, 0, 0)) \cap \text{Char}(M) &= \{(0, 0, 0; \xi, \eta, \zeta) \mid \xi, \eta, \zeta \in \mathbb{C}\}, \\ \pi^{-1}(p) \cap \text{Char}(M) &= \{p; \xi, \eta, \zeta \mid \xi, \eta, \zeta \in \mathbb{C}, \eta = \zeta\} \end{aligned}$$

with $p = (\pm 2, 0, 0)$ or $p = \pm(2, 2, 2)$, where $\pi : T^*\mathbb{C}^3 \rightarrow \mathbb{C}^3$ is the projection of the cotangent bundle to the base space. The fiber of $\text{Char}(M)$ at a point in a one-dimensional stratum, e.g., $x = y = z$, consists of two complex lines while that at a regular point of $f = 0$ consists of one complex line.

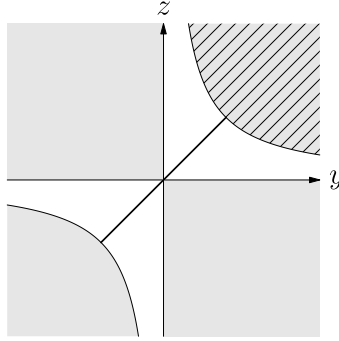
Next let us consider the real hypersurface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$$

of \mathbb{R}^3 . Since the discriminant of f with respect to x is $yz(y - z)^2(yz - 4)$, the projection of S to the yz -plane is given by

$$S_{yz} = \{(y, z) \in \mathbb{R}^2 \mid yz(yz - 4) \geq 0\} \cup \{(y, z) \mid y = z\}$$

as shown by the grey regions and the line segment in the figure below:



Except at the origin, the fiber of the projection of S to S_{yz} consists of one or two points. In particular, the fiber at a point in the line $y = z$ consists of only one point. Hence S has a line segment connecting the two points $\pm(2, 2, 2)$ as a one-dimensional component.

Since the support of $I_G(x, y, z)$ is contained in the support of M , i.e., $xf(x, y, z) = 0$, as well as in the set $\{(x, y, z) \in \mathbb{R}^3 \mid x \geq 2, y + z \geq 4\}$, we have

$$\text{supp } I_G(x, y, z) \subset \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0, x \geq 2, y + z \geq 4\}.$$

In addition, since $I_G(x, y, z)$ coincides with a finite sum of derivatives of $\delta(f)$ multiplied by real analytic functions on the regular part S_{reg} of S , the intersection of the support of $I_G(x, y, z)$ and S_{reg} is a union of connected components of S_{reg} . Thus we conclude that the support of $I_G(x, y, z)$ satisfies

$$\text{supp } I_G(x, y, z) \subset \{(x, y, z) \mid f(x, y, z) = 0, x \geq 2, y > 0, z > 0, yz \geq 4\},$$

the projection of which to the yz -plane is shown as the hatched region in the figure above. More precisely, we can confirm by direct computation that $xf(x, y, z)$ belongs to the left ideal I , which implies $f(x, y, z)I_G(x, y, z) = 0$. It follows that $I_G(x, y, z)$ is the product of a real analytic function and $\delta(f)$ on S_{reg} .

We remark that the hypersurface S of \mathbb{R}^3 coincides with the Landau-Nakanishi surface associated with the triangle diagram T_1 studied by Honda-Kawai-Stapp in Appendix A of [2].

References

- [1] Honda, N., Kawai, T., An invitation to Sato's postulates in micro-analytic S -matrix theory, RIMS Kôkyûroku Bessatsu **B61** (2017), 23–56.
- [2] Honda, N., Kawai, T., Stapp, H. P., On the geometric aspect of Sato's postulates on the S -matrix, RIMS Kôkyûroku Bessatsu **B52** (2014), 11–53.

- [3] Kashiwara, M., Kawai, T., Kimura, T., ‘Foundations of Algebraic Analysis’, Kinokuniya, Tokyo, 1980 (in Japanese).
- [4] Kawazoe, T., Noro, M., Algorithms for computing a primary ideal decomposition without producing intermediate redundant components, *J. Symbolic Computation* **46** (2011), 1158–1172.
- [5] Malgrange, B., Intégrales asymptotiques et monodromie. *Ann. Sci. École Normal Sup.*, 4^e série, **7** (1974), 405–430.
- [6] Nishiyama, K., Noro, M., Stratification associated with local b -functions, *J. Symbolic Computation* **45** (2010), 462–480.
- [7] Noro, M., Takayama, N., Nakayama, H., Nishiyama, K., Ohara, K, Risa/Asir: a computer algebra system, <http://www.math.kobe-u.ac.jp/Asir/asir.html>.
- [8] Oaku, T., Algorithms for D -modules, integration, and generalized functions with applications to statistics, in Proceedings of “The 50th Anniversary of Gröbner Bases”, *Advanced Studies in Pure Mathematics* **77**, Mathematical Society of Japan (2018), 253–352.
- [9] Oaku, T., Takayama, N., An algorithm for de Rham cohomology groups of the complement of an affine variety. *J. Pure Appl. Algebra* **139** (1999), 201–233.
- [10] Oaku, T., Takayama, N., Algorithms for D -modules — restriction, tensor product, localization, and local cohomology groups. *J. Pure Appl. Algebra* **156** (2001), 267–308.