

Asymptotic Analysis of Confluent Hypergeometric Partial Differential Equations in Many Variables

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1 Introduction

The confluent differential equation in one variable, known as the Kummer differential equation

$$x \frac{d^2}{dx^2} w + (\gamma - x) \frac{d}{dx} w - \beta w = 0,$$

is studied by several authors in various ways. Among those, the so-called Borel-Laplace-Ecalle method is a powerful one, which is explained for example in [1]. This method is applicable to an analysis of the Humbert confluent hypergeometric differential equations Φ_2 in 2 variables

$$\begin{aligned} x \frac{\partial^2}{\partial x^2} w + y \frac{\partial}{\partial y} \frac{\partial}{\partial x} w + (\gamma - x) \frac{\partial}{\partial x} w - \beta w &= 0, \\ y \frac{\partial^2}{\partial y^2} w + x \frac{\partial}{\partial x} \frac{\partial}{\partial y} w + (\gamma - y) \frac{\partial}{\partial y} w - \beta' w &= 0, \end{aligned}$$

and we can obtain formal solutions, asymptotic solutions and so-called Stokes multipliers (see [2], [3]).

It is also applicable to an asymptotic analysis of the Humbert confluent hypergeometric partial differential equations in $m(> 2)$ variables. Here, the author gives an overview of it.

2 Humbert confluent hypergeometric partial differential equations Φ_D

The system of Humbert confluent hypergeometric partial differential equations Φ_D is as follows:

$$x_k \frac{\partial^2 u}{\partial x_k^2} + \sum_{l \neq k} x_l \frac{\partial^2 u}{\partial x_k \partial x_l} + (\gamma - x_k) \frac{\partial u}{\partial x_k} - \beta_k u = 0,$$

where $\beta_k (k = 1, \dots, m)$ and γ are not non-negative integers.

We consider this system in $M = (P^1(\mathbb{C}))^m$. The system has irregular singularities on $H = \bigcup_{k=1}^m H_k$, where $H_k = P^1(\mathbb{C}) \times \dots \times \{\infty\} \times \dots \times P^1(\mathbb{C})$.

For simplicity, let p be a point in $H \setminus \bigcup_{k \neq l} (H_k \cap H_l)$, we consider the formal solutions and asymptotic solutions to Φ_D near the point.

Proposition 1. We have $(m+1)$ linearly independent formal solutions. Among them, $(m-1)$ formal solutions are convergent and 2 formal solutions are divergent.

Near a point $(\infty, x_2, \dots, x_m)$ with bounded $x_2, \dots, x_m$, we have divergent solutions of the following forms

$$e^{x_1} x_1^{\beta_1 - \gamma} \hat{V}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}),$$

and

$$x_1^{-\beta_1} \hat{U}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}).$$

Here, we put

$$\begin{aligned} &\hat{V}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}) \\ &= \sum_{n=0}^{\infty} P_n(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m) x_1^{-n}, \end{aligned}$$

with the polynomials

$$P_n(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m) \\ = \sum_{\ell=0}^n \frac{(\gamma - \beta_1 + \ell)_{n-\ell} (1 - \beta_1)_{n-\ell}}{(n - \ell)! \ell!} \sum_{j_2 + \dots + j_m = \ell} \frac{(\beta_2)_{j_2} \dots (\beta_m)_{j_m} \ell!}{j_2! \dots j_m!} x_2^{j_2} \dots x_m^{j_m}$$

and

$$\hat{U}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}) \\ = \sum_{n=0}^{\infty} \frac{(\beta_1)_n (\beta_1 - \gamma + 1)_n}{n!} \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1 - n; x_2, \dots, x_m) (-x_1)^{-n},$$

where $\Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1 - n; x_2, \dots, x_m)$ is the Humbert confluent hypergeometric function in $(m-1)$ variables with the parameter $(\beta_2, \dots, \beta_m; \gamma - \beta_1 - n)$,

$$\Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1 - n; x_2, \dots, x_m) \\ = \sum_{j_2=0}^{\infty} \dots \sum_{j_m=0}^{\infty} \frac{(b_2)_{j_2} \dots (b_m)_{j_m} x_2^{j_2} \dots x_m^{j_m}}{(\gamma - \beta_1 - n)_{j_2 + \dots + j_m} j_2! \dots j_m!}.$$

In the above, we use the Pochhammer symbol $(b)_s = (b+1) \dots (b+s-1)$.

Proposition 2. The divergent formal series

$$\hat{V}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1})$$

and

$$\hat{U}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1})$$

are of Gevrey order 1 as $x_1 \rightarrow \infty$ uniformly on a bounded domain D in the $(x_2, \dots, x_m)$ -space.

Definition. For a formal expression $e^{\rho x_1} x_1^{-\lambda} \hat{p}(x)$ with a complex number ρ , a non-negative integer λ and a formal series $\hat{p}(x) = \sum_{n=0}^{\infty} c_n(x_2, \dots, x_m) x_1^{-n}$, we define the Borel transform as follows,

$$\hat{B}_1(e^{\rho x_1} \sum_{n=0}^{\infty} c_n x_1^{-\lambda-n})(\xi_1) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+\lambda)} (\rho + \xi_1)^{n+\lambda-1}.$$

Proposition 3. The Borel transforms of divergent solutions are holomorphic functions in a domain in $\mathbb{C}^m$, which are analytically prolongable.

In fact,

$$\hat{B}_1(e^{x_1} x_1^{\beta_1 - \gamma} \hat{V}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}))(\xi_1) \\ = \frac{1}{\Gamma(\gamma - \beta_1)} (-\xi_1)^{\beta_1 - 1} (1 + \xi_1)^{\gamma - \beta_1 - 1} \\ \times \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \dots, (1 + \xi_1)x_m),$$

$$\hat{B}_1(x_1^{-\beta_1} \hat{U}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}))(\xi_1) \\ = \frac{1}{\Gamma(\beta_1)} \xi_1^{\beta_1 - 1} (1 + \xi_1)^{\gamma - \beta_1 - 1} \\ \times \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \dots, (1 + \xi_1)x_m).$$

Between

$$B_1(\hat{v})(\xi_1) = \hat{B}_1(e^{x_1} x_1^{\beta_1 - \gamma} \hat{V}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}))(\xi_1)$$

and

$$B_1(\hat{u})(\xi_1) = \hat{B}_1(x_1^{-\beta_1} \hat{U}(\beta_1, \beta_2, \dots, \beta_m, \gamma, x_2, \dots, x_m, x_1^{-1}))(\xi_1),$$

we have a relation

$$\Gamma(\beta_1)B_1(\hat{u})(\xi_1) = \Gamma(\gamma - \beta_1)(-1)^{-\beta_1+1}B_1(\hat{v})(\xi_1),$$

Definition. Consider a function $f(\xi_1, x_2, \dots, x_m)$ which is holomorphic and exponentially small in a tubular neighborhood in the first variable and a bounded domain in the other variables. We define the generalized Laplace transforms of $f(\xi_1, x_2, \dots, x_m)$, as follows

$$\int_{C(q, \theta)} \exp(-x_1 \xi_1) f(\xi_1, x_2, \dots, x_m) d\xi_1,$$

where $C(q, \theta)$ is a following path of integral. For a point q in the tubular neighborhood, $C(q, \theta)$ is a path on which $\arg(\xi_1 - q)$ is taken to be initially θ and finally $\theta + 2\pi$.

Proposition 4. The Laplace transforms of Borel transforms of divergent solutions are holomorphic functions in a suitable angular domain with the summit p in $P^1(\mathbb{C}) \times \mathbb{C}^{m-1}$, where they are actual solutions to the system Φ_D with asymptotic expansions of Gevrey order 1. Here, the asymptotic expansions coincide with the divergent solutions, respectively.

In fact, for

$$-2\pi < \theta < 0,$$

the Laplace integral

$$\frac{1}{\Gamma(\gamma - \beta_1)} \int_{C(-1, \theta)} \exp(-x_1 \xi_1) (-\xi_1)^{\beta_1-1} (1 + \xi_1)^{\gamma-\beta_1-1} \\ \times \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \dots, (1 + \xi_1)x_m) d\xi_1$$

is defined and represents a holomorphic function in the first variable x_1 in the angular domain (mod. 2π)

$$\frac{\pi}{2} < \arg(-\xi_1 x_1) < \frac{3\pi}{2},$$

namely,

$$-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,$$

because $\exp(-x_1 \xi_1)$ tends to 0 as ξ_1 tends to the infinity. By considering the analytic prolongation, we obtain an actual solution v in the angular domain

$$-\frac{5\pi}{2} < \arg x_1 < \frac{\pi}{2}.$$

For

$$-\pi < \theta < \pi,$$

the Laplace integral

$$\frac{1}{\Gamma(\beta_1)} \int_{C(0, \theta)} \exp(-x_1 \xi_1) \xi_1^{\beta_1-1} (1 + \xi_1)^{\gamma-\beta_1-1} \\ \times \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \dots, (1 + \xi_1)x_m) d\xi_1$$

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$$\frac{\pi}{2} < \arg(-\xi_1 x_1) < \frac{3\pi}{2},$$

namely,

$$-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,$$

because $\exp(-x_1 \xi_1)$ tends to 0 as ξ_1 tends to the infinity. By considering the analytic prolongation, we obtain an actual solution u in the angular domain

$$-\frac{3\pi}{2} < \arg x_1 < \frac{3\pi}{2}.$$

Proposition 5 We have fundamental systems of solutions

$$(e_1 u, e_2 v, w_2, \dots, w_m)$$

in the angular domain

$$-\frac{3\pi}{2} < \arg x_1 < \frac{\pi}{2}.$$

and

$$(e_1 u, e_2 v(x_1 e^{-2i\pi}) e^{2i\pi(-\gamma+\beta_1)}, w_2, \dots, w_m)$$

in the angular domain

$$-\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}.$$

where

$$w_2 = x_1^{-\beta_1} x_2^{\beta_1-\gamma+1} h_2, \dots, w_m = x_1^{-\beta_1} x_m^{\beta_1-\gamma+1} h_m$$

with holomorphic functions $h_2, \dots, h_m$ at the point p .

Then, we have the relations

$$\begin{aligned} & (e_1 u, e_2 v(x_1 e^{-2i\pi}) e^{2i\pi(-\gamma+\beta_1)}) \\ &= (e_1 u, e_2 v) \begin{pmatrix} 1 & c_{12} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

in the angular domain

$$-\frac{\pi}{2} < \arg x_1 < \frac{\pi}{2}.$$

and

$$\begin{aligned} & (e_1 u(x_1 e^{-2i\pi}) e^{2i\pi(-\beta_1)}, e_2 v(x_1 e^{-2i\pi}) e^{2i\pi(-\gamma+\beta_1)}) \\ &= (e_1 u, e_2 v(x_1 e^{-2i\pi}) e^{2i\pi(-\gamma+\beta_1)}) \begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix} \end{aligned}$$

in the angular domain

$$\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}.$$

In the above, we use the following constants

$$\begin{aligned} e_1 &= (e^{2i\pi\beta_1} - 1)^{-1}, \\ e_2 &= (e^{2i\pi(\gamma-\beta_1)} - 1)^{-1}, \\ c_{12} &= \frac{-2i\pi}{\Gamma(1-\beta_1)\Gamma(\gamma-\beta_1)}, \\ c_{21} &= \frac{-2i\pi e^{i\pi(\gamma-2\beta_1)}}{\Gamma(\beta_1)\Gamma(1-\gamma+\beta_1)}. \end{aligned}$$

References

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