

# Curvatures on curves passing through Whitney umbrella

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We consider curves passing through Whitney umbrella, which is known as a singularity appearing on stable maps from a surface to the three space. By the study of curvature of such curves, we obtain relations between curves and the differential geometric properties of Whitney umbrella.

## 1 Preliminaries

**Definition 1.1.** Two map-germs  $f, g : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^n, 0)$  are said to be  $\mathcal{A}$ -equivalent if there exist diffeomorphism-germs  $\phi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$  and  $\psi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  such that  $\psi \circ f = g \circ \phi$ .

Using  $\mathcal{A}$ -equivalence, we define Whitney umbrella. Generally, Whitney umbrella is a singular point of maps, but we define it for a map-germ.

**Definition 1.2.** A map-germ  $X : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  is a *Whitney umbrella* (or a *cross cap*) if  $X$  is  $\mathcal{A}$ -equivalent to the map-germ

$$X_0 : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0), (u, v) \mapsto (u, uv, v^2)$$

at the origin.

For a Whitney umbrella, the following normal form is known.

**Theorem 1.3** ([2]). *Let  $W : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  be a Whitney umbrella. Then for any  $k \geq 3$ , there exist an orientation preserving diffeomorphism  $\psi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  and  $T \in SO(3)$  such that*

$$T \circ W \circ \psi(u, v) = \begin{pmatrix} u \\ uv + \frac{b_3}{3!}v^3 + \cdots + O(u, v)^{k+1} \\ \frac{a_{20}}{2}u^2 + a_{11}uv + \frac{a_{02}}{2}v^2 + \cdots + O(u, v)^{k+1} \end{pmatrix},$$

where  $b_j \in \mathbb{R}(j \geq 3), a_{jk} \in \mathbb{R}(j + k \geq 2)$ , and  $a_{02} \neq 0$ . Here,  $O(u, v)^{k+1}$  is the terms whose degrees are greater than or equal to  $k + 1$ .

In particular, the coefficients  $b_j, a_{jk}$  are all differential geometric invariants. On the other hand, the unit normal vector can not be smoothly extended at the singular point of the Whitney umbrella. A map that can smoothly extend the unit normal vector at the singular point is called frontal, and it is defined as follows.

**Definition 1.4.** A map-germ  $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  is a *frontal* if there exists a  $C^\infty$  map-germ  $\nu : (\mathbb{R}^2, 0) \rightarrow S^2$  such that for any  $p \in ((\mathbb{R}^2; u, v), 0)$ ,

$$\nu(p) \cdot f_u(p) = \nu(p) \cdot f_v(p) = 0$$

holds, where  $S^2$  is the unit sphere in  $\mathbb{R}^3$ .

The property that a map-germ is a frontal does not depend on the  $\mathcal{A}$ -equivalence. Since  $X_0(u, v) = (u, uv, v^2)$  satisfies

$$\frac{(X_0)_u \times (X_0)_v}{|(X_0)_u \times (X_0)_v|}(u, v) = \frac{(2v^2, -2v, u)}{\sqrt{4v^4 + 4v^2 + u^2}},$$

Whitney umbrella is not a frontal. On the other hand, the following property is known that for Whitney umbrella there is a well-defined unit normal vector after the bowing-up operation as follows ([1]).

We set  $\tilde{\pi} : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^2; (r, \theta) \rightarrow (r \cos \theta, r \sin \theta)$ , and set  $\mathcal{M}$  which is the quotient set  $\mathbb{R} \times S^1 / \sim$  of the relation  $(r, \theta) \sim (-r, \theta + \pi)$ . Considering

$$\pi : \mathcal{M} \rightarrow \mathbb{R}^2, [(r, \theta)] \mapsto (r \cos \theta, r \sin \theta),$$

one can see

$$\frac{((X_0)_u \times (X_0)_v) \circ \pi}{|((X_0)_u \times (X_0)_v) \circ \pi|}(0, \theta) = \frac{1}{\sqrt{4\sin^2 \theta + \cos^2 \theta}} \begin{pmatrix} 0 \\ -2\sin \theta \\ \cos \theta \end{pmatrix}.$$

This implies that by composing  $\pi$ , the unit normal vector field is well-defined near a Whitney umbrella on  $\mathcal{M}$ . This property strongly suggests that if we fix the angle of incidence of a Whitney umbrella, a smooth unit normal vector is well-defined. We consider a curve passing through  $(0, 0)$  which satisfies the following general enough condition.

**Definition 1.5.** A  $C^\infty$  curve-germ  $\mathbf{c} : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$  is said to be of *finite multiplicity* if there exist a positive integer  $m$  and a  $C^\infty$ -germ  $\bar{\mathbf{c}} : (\mathbb{R}, 0) \rightarrow \mathbb{R}^2$  such that

$$\mathbf{c}(x) = x^m \bar{\mathbf{c}}(x), \bar{\mathbf{c}}(0) \neq 0.$$

We consider curves of finite multiplicity passing through a Whitney umbrella. Then, we obtain the following lemma.

**Lemma 1.6.** Let  $\mathbf{c} : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$  be a curve of finite multiplicity, and let  $W : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  be a Whitney umbrella. Then the unit normal vector field of  $W$  along  $W \circ \mathbf{c}$  can be smoothly extended across the Whitney umbrella.

Similarly, we obtain the following lemma.

**Lemma 1.7.** *Let  $\mathbf{c} : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$  be a curve of finite multiplicity, and let  $W : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  be a Whitney umbrella. Consider the composition  $W \circ \mathbf{c}$ . Then the unit tangent vector field of  $W \circ \mathbf{c}$  can be smoothly extended across the Whitney umbrella.*

**Example 1.8.** Let us set  $Y_0(x) = (x, x)$  and  $X_0(u, v) = (u, uv, -v^2)$ . We consider the composition  $X_0 \circ Y_0 : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^3, 0)$ . Then we can take a unit normal vector field  $\mathbf{n}_0$  and a unit tangent vector field  $\mathbf{e}_0$  as

$$\mathbf{n}_0(x) = \frac{1}{\sqrt{4x^2 + 5}} \begin{pmatrix} -2x \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{e}_0(x) = \frac{1}{\sqrt{8x^2 + 1}} \begin{pmatrix} 1 \\ 2x \\ -2x \end{pmatrix}$$

along  $X_0 \circ Y_0$ .

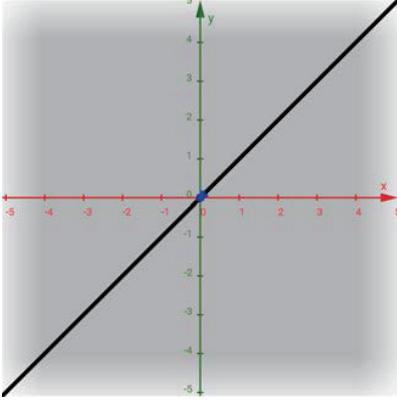


Figure 1.1:  $Y_0(x)$

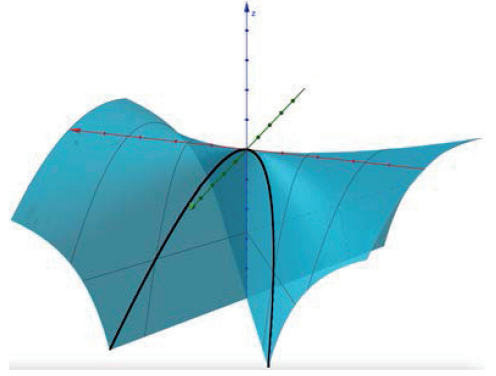


Figure 1.2:  $Y_0 \circ X_0(x)$

## 2 Construction of frame

By the previous section, we have the following lemma.

**Lemma 2.1.** *The vectors*

$$\frac{(W \circ \mathbf{c})'}{|(W \circ \mathbf{c})'|}(x), \frac{(W_u \times W_v) \circ \mathbf{c}}{|(W_u \times W_v) \circ \mathbf{c}|}(x)$$

*are well-defined across the singular point ( $x = 0$ ) by taking an appropriate reduction. Setting  $\mathbf{b}(x) = \mathbf{n}(x) \times \mathbf{e}(x)$ , one obtain a Darboux frame  $\{\mathbf{e}(x), \mathbf{b}(x), \mathbf{n}(x)\}$  along  $W \circ \mathbf{c}$ .*

We assume the condition

$$\text{rank}(\mathbf{c}(0), \mathbf{c}'(0), \mathbf{c}''(0), \dots) = 2$$

for a curve  $\mathbf{c}(x)$ . Then we can set

$$\mathbf{c}(x) = \begin{pmatrix} d_1(x)x^{m_1} \\ d_2(x)x^{m_2} \end{pmatrix}, \quad d_1(0) \neq 0, \quad d_2(0) \neq 0,$$

where  $m_1, m_2 \in \mathbb{Z}_{>0}$ ,  $d_1, d_2 : (\mathbb{R}, 0) \rightarrow \mathbb{R}$ . The following is an example of  $\{\mathbf{e}(x), \mathbf{b}(x), \mathbf{n}(x)\}$ .

**Example 2.2.** We set

$$m_1 = 2m_2, \quad d_2 = 1, \quad d_1(x) = \sum_{i=0} c_i x^i,$$

then  $(W \circ \mathbf{c})'(x)$  and  $(W_u \times W_v) \circ \mathbf{c}(x)$  satisfy

$$(W \circ \mathbf{c})'(x) = \begin{pmatrix} \{2m_2 c_0 + O(x)\} \\ \{3m_2 c_0 + \frac{1}{2}m_2 b_3 + O(x)\}x^{m_2} \\ \{m_2 a_{02} + O(x)\} \end{pmatrix} x^{2m_2-1} = \mathcal{A}(x)x^{2m_2-1},$$

$$(W_u \times W_v) \circ \mathbf{c}(x) = \begin{pmatrix} \{a_{02} + O(x)\}x^{m_2} \\ \{-a_{02} + O(x)\} \\ \{c_0 + \frac{b_3}{2} + O(x)\}x^{m_2} \end{pmatrix} x^{m_2} = \mathcal{B}(x)x^{m_2},$$

where  $|\mathcal{A}(0)| \neq 0, |\mathcal{B}(0)| \neq 0$ . Therefore, we have

$$\begin{aligned} \mathbf{e}(x) &= \frac{\mathcal{A}}{|\mathcal{A}|} = \frac{1}{|\mathcal{A}|} \begin{pmatrix} e_1(x) \\ e_2(x)x^{m_2} \\ e_3(x) \end{pmatrix}, \quad \begin{cases} e_1(0) = 2m_2 c_0, \\ e_2(0) = m_2(3c_0 + \frac{1}{2}b_3), \\ e_3(0) = m_2 a_{02}, \end{cases} \\ \mathbf{b}(x) &= \frac{1}{|\mathcal{A}||\mathcal{B}|} \begin{pmatrix} b_1(x) \\ b_2(x)x^{m_2} \\ b_3(x) \end{pmatrix}, \quad \begin{cases} b_1(0) = -m_2 a_{02}^2, \\ b_2(0) = m_2(2c_0^2 + b_3 c_0 - a_{02}^2), \\ b_3(0) = 2m_2 a_{02} c_0, \end{cases} \\ \mathbf{n}(x) &= \frac{\mathcal{B}}{|\mathcal{B}|} = \frac{1}{|\mathcal{B}|} \begin{pmatrix} n_1(x)x^{m_2} \\ n_2(x) \\ n_3(x)x^{m_2} \end{pmatrix}, \quad \begin{cases} n_1(0) = a_{02}, \\ n_2(0) = -a_{02}, \\ n_3(0) = c_0 + \frac{1}{2}b_3. \end{cases} \end{aligned}$$

### 3 Curvature

We define functions  $\kappa_1, \kappa_2, \kappa_3$  by

$$\frac{d}{dx} \begin{pmatrix} \mathbf{e}(x) \\ \mathbf{b}(x) \\ \mathbf{n}(x) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(x) & \kappa_2(x) \\ -\kappa_1(x) & 0 & \kappa_3(x) \\ -\kappa_2(x) & -\kappa_3(x) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}(x) \\ \mathbf{b}(x) \\ \mathbf{n}(x) \end{pmatrix}.$$

We see the relations between  $\kappa_1, \kappa_2, \kappa_3$  and the geodesic curvature  $\kappa_g$ , the normal curvature  $\kappa_\nu$ , and the geodesic torsion  $\kappa_t$  which are defined on a set of regular points.

**Lemma 3.1.** *We set*

$$\begin{aligned}\gamma'(x) &= \mathcal{A}(x)x^\alpha, \mathcal{A}(0) \neq 0, \\ (W_u \times W_v) \circ \mathbf{c}(x) &= \mathcal{B}(x)x^\beta, \mathcal{B}(0) \neq 0.\end{aligned}$$

*Then it holds that*

$$\begin{aligned}\kappa_1(x) &= \operatorname{sgn}(x^{\alpha+\beta})|\mathcal{A}(x)|x^\alpha\kappa_g(x), \\ \kappa_2(x) &= \operatorname{sgn}(x^\beta)|\mathcal{A}(x)|x^\alpha\kappa_\nu(x), \\ \kappa_3(x) &= |\mathcal{A}(x)|x^\alpha\kappa_t(x).\end{aligned}$$

By lemma 3.1, we can see a boundedness of  $\kappa_g, \kappa_\nu, \kappa_t$  if we give degrees and top terms of  $\kappa_1, \kappa_2, \kappa_3$ . We consider a curve  $\mathbf{c}(x)$  such that

$$\mathbf{c}'(0) \in \operatorname{Ker} dW.$$

Then  $\mathbf{c}(x)$  satisfies

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{m_1} \\ x^{m_2} \end{pmatrix}, c(0) \neq 0, m_1 > m_2.$$

Therefore, we set

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{mp+q} \\ x^m \end{pmatrix}, 1 \leq p, 1 \leq q < m, \quad (3.1)$$

or

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{mp} \\ x^m \end{pmatrix}, 2 \leq p \quad (3.2)$$

where  $c(x) = c_0 + c_1x + c_2x^2 + \cdots$ ,  $c_0 \neq 0$ .

**Theorem 3.2.** *We assume that  $\mathbf{c}(x)$  satisfies (3.1). Then it holds that*

$$\kappa_1(x) = \begin{cases} \tilde{\kappa}_1(x)x^{m-q-1} & (p=1), \\ \tilde{\kappa}_1(x)x^{m(p-2)+q-1} & (2 \leq p < 4), \\ \tilde{\kappa}_1(x)x^{2m-1} & (4 \leq p), \end{cases}$$

$$\kappa_2(x) = \tilde{\kappa}_2(x)x^{m-1}$$

$$\kappa_3(x) = \begin{cases} \tilde{\kappa}_3(x)x^{q-1} & (p=1), \\ \tilde{\kappa}_3(x)x^{m-1} & (2 \leq p). \end{cases}$$

**Theorem 3.3.** *We assume that  $\mathbf{c}(x)$  satisfies (3.2). Then it holds that*

$$\kappa_1(x) = \begin{cases} \tilde{\kappa}_1(x)x^0 & (p=2), \\ \tilde{\kappa}_1(x)x^{m-1} & (p=3), \\ \tilde{\kappa}_1(x)x^{2m-1} & (4 \leq p), \end{cases}$$

$$\kappa_2(x) = \tilde{\kappa}_2(x)x^{m-1},$$

$$\kappa_3(x) = \tilde{\kappa}_3(x)x^{m-1}.$$

In particular, if

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{2m} \\ x^m \end{pmatrix},$$

the top terms of  $\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$  satisfy

$$\tilde{\kappa}_1(x) = \begin{cases} \frac{1}{|\mathcal{A}|^2|\mathcal{B}|}a_{02}(6a_{11}c_0^2 + a_{03}c_0 - 3a_{02}c_1) + O(x)(m_2 = 1), \\ -\frac{1}{|\mathcal{A}|^2|\mathcal{B}|}m_2(2m_2 + 1)a_{02}^2c_1 + O(x)(2 \leq m_2), \end{cases} \quad (3.3)$$

$$\tilde{\kappa}_2(x) = -\frac{1}{|\mathcal{A}||\mathcal{B}|}m_2^2a_{02}\left(3c_0 + \frac{b_3}{2}\right) + O(x), \quad (3.4)$$

$$\tilde{\kappa}_3(x) = -\frac{1}{|\mathcal{A}||\mathcal{B}|^2}m_2^2a_{02}(2c_0^2 + b_3c_0 - a_{02}^2) + O(x). \quad (3.5)$$

If we assume these top terms vanish, then we have the following relations in (3.3), (3.4), and (3.5),

$$6a_{11}c_0^2 + a_{03}c_0 - 3a_{02}c_1 = 0, \quad (3.6)$$

$$3c_0 + \frac{b_3}{2} = 0, \quad (3.7)$$

$$2c_0^2 + b_3c_0 - a_{02}^2 = 0, \quad (3.8)$$

where  $c_0, c_1$  are differential geometric invariants of curve  $\mathbf{c}$ ,  $a_{02}, a_{11}, a_{03}, b_3$  are differential geometric invariants of Whitney umbrella  $W$ . When a Whitney umbrella and a curve satisfy these relations, the degrees of  $\kappa_1, \kappa_2, \kappa_3$  are at least greater than 1 for the non-zero cases. Thus we can consider that these relations have a geometric meaning.

**Example 3.4.** Let us set

$$\mathbf{c}^1(x) = \begin{pmatrix} x^2 \\ x \end{pmatrix}, \quad (3.9)$$

$$W_1(u, v) = \begin{pmatrix} u \\ uv + \frac{b_3}{6}v^3 \\ a_{11}uv + \frac{a_{02}}{2}v^2 + \frac{a_{03}}{6}v^3 \end{pmatrix}. \quad (3.10)$$

We consider the composition  $W_1 \circ \mathbf{c}^1$ . We give examples of the cases

$$(1) \quad \tilde{\kappa}_1(0) \neq 0, \quad \tilde{\kappa}_2(0) \neq 0, \quad \tilde{\kappa}_3(0) \neq 0,$$

$$(2) \quad \tilde{\kappa}_1(0) = 0, \quad \tilde{\kappa}_2(0) \neq 0, \quad \tilde{\kappa}_3(0) \neq 0,$$

$$(3) \quad \tilde{\kappa}_1(0) \neq 0, \quad \tilde{\kappa}_2(0) = 0, \quad \tilde{\kappa}_3(0) \neq 0,$$

$$(4) \quad \tilde{\kappa}_1(0) \neq 0, \quad \tilde{\kappa}_2(0) \neq 0, \quad \tilde{\kappa}_3(0) = 0,$$

by

$$(1) \quad (b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -1),$$

$$(2) \quad (b_3, a_{11}, a_{02}, a_{03}) = (1, 1/6, -2, -1),$$

$$(3) \quad (b_3, a_{11}, a_{02}, a_{03}) = (-6, -1, -4, -1),$$

$$(4) \quad (b_3, a_{11}, a_{02}, a_{03}) = (2, -1, -2, -1).$$

The figures of  $W_1$  with  $W_1 \circ \mathbf{c}^1$  of the cases (1) - (4) are given in Figure 3.1 - 3.4.

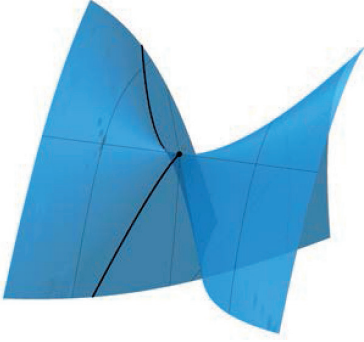


Figure 3.1:  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$     Figure 3.2:  $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

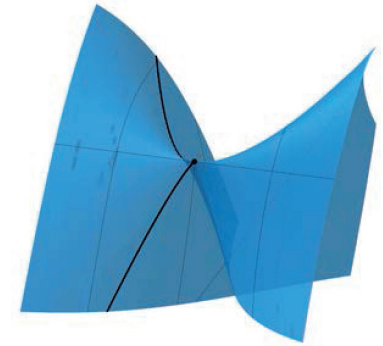


Figure 3.3:  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0$     Figure 3.4:  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0$

**Example 3.5.** Let us set

$$\mathbf{c}^2(x) = \begin{pmatrix} x^4 + x^5 \\ x^2 \end{pmatrix}, \quad (3.11)$$

We consider the composition  $W_1 \circ \mathbf{c}^2$ . We give examples of the cases

- (1)  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0,$
- (2)  $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0,$
- (3)  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0,$
- (4)  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0,$

by

- (1)  $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -1),$
- (2)  $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -0),$
- (3)  $(b_3, a_{11}, a_{02}, a_{03}) = (-6, -1, -4, -1),$
- (4)  $(b_3, a_{11}, a_{02}, a_{03}) = (2, -1, -2, -1).$

The figures of  $W_1$  with  $W_1 \circ \mathbf{c}^2$  of the cases (1) - (4) are given in Figure 3.5 - 3.8.

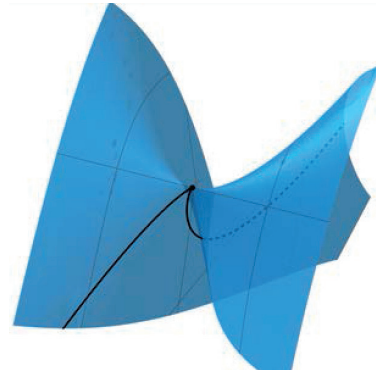
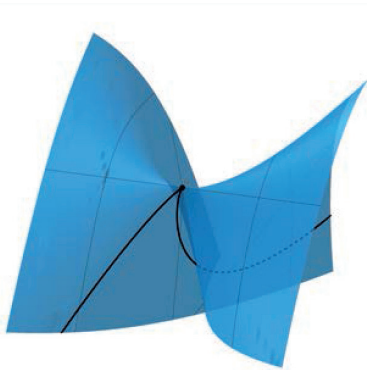


Figure 3.5:  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$     Figure 3.6:  $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

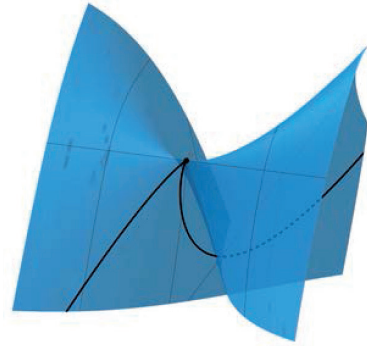


Figure 3.7:  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0$     Figure 3.8:  $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0$

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## References

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