

Partition properties of subsets of $\mathcal{P}_\kappa\lambda$

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ABSTRACT. Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. The following partition property is shown to be consistent relative to supercompactness: For any $f : \bigcup_{n < \omega} [X]_{\mathcal{C}}^n \rightarrow \gamma$ with $X \subset \mathcal{P}_\kappa\lambda$ unbounded and $1 < \gamma < \kappa$ there is an unbounded $Y \subset X$ with $|f''[Y]_{\mathcal{C}}^n| = 1$ for any $n < \omega$.

Let κ be a regular cardinal $> \omega$, λ a cardinal $\geq \kappa$ and F a filter on $\mathcal{P}_\kappa\lambda$. Partition properties of the form $\mathcal{P}_\kappa\lambda \rightarrow (F^+)_2^2$ (see below for the definition) were introduced by Jech [Je] and successfully used to characterize supercompactness: Menas [Me] proved $\mathcal{P}_\kappa\lambda \rightarrow (C_{\kappa\lambda}^+)_2^2$ for κ $2^{\lambda < \kappa}$ -supercompact via a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ with the partition property. As noted by Kamo [Kam], Menas' argument can be modified to derive $\mathcal{P}_\kappa\lambda \rightarrow (C_{\kappa\lambda}^+)_2^2$ directly from λ -supercompactness of κ . For the converse direction Di Prisco-Zwicker [DZ] and others refined the global result of Magidor [Mag]: λ -supercompactness of κ follows from $\mathcal{P}_\kappa 2^{\lambda < \kappa} \rightarrow (C_{\kappa 2^{\lambda < \kappa}}^+)_2^2$.

Johnson [Jo] studied properties of the form $X \rightarrow (F^+)_2^2$ for $X \in F^+$, which means that for any $f : [X]_{\mathcal{C}}^2 \rightarrow 2$ there is $Y \in F^+$ with $Y \subset X$ and $|f''[Y]_{\mathcal{C}}^2| = 1$. In this note we are concerned with the case where F is canonically defined, in particular $\mathcal{F}_{\kappa\lambda}$, the minimal fine filter on $\mathcal{P}_\kappa\lambda$.

We generally follow the terminology of Kanamori [Kan] with the following exception: For a cardinal $\mu \geq \omega$ we set $[X]^\mu = \{x \subset X : |x| = \mu\}$, $[X]^{<\mu} = \{x \subset X :$

1991 *Mathematics Subject Classification*. 03E05, 03E55.

Part of this work was done during the author's stay at Boston University as one of the Japanese Overseas Research Fellows. He gratefully acknowledges Prof. Kanamori's hospitality. He also wishes to thank Prof. Abe for helpful conversations.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

$|x| < \mu\}$ and $\lim A = \{\alpha < \mu : \sup(A \cap \alpha) = \alpha > 0\}$ for $A \subset \mu$. By $F^+ \rightarrow (F^+)_2^2$ we mean $X \rightarrow (F^+)_2^2$ for any $X \in F^+$. We understand $\bigcup a \subsetneq \bigcap b$ when the union $a \cup b$ of $a \in [\mathcal{P}_\kappa \lambda]_\mathbb{C}^m$ and $b \in [\mathcal{P}_\kappa \lambda]_\mathbb{C}^n$ with $m, n < \omega$ is formed.

Abe [A1] proved $\mathcal{F}_{\kappa\lambda}^+ \not\rightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^2$ under $\lambda^{<\kappa} = 2^\lambda$. Matet [Mat] used Laver's idea (see [JS]) to get the same conclusion from an opposite assumption:

Proposition 1. *Assume $\lambda^\kappa = \lambda$. Then $\mathcal{F}_{\kappa\lambda}^+ \not\rightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^2$.*

Proof. First set $\mathcal{P}_\kappa \lambda = \{x_\xi : \xi < \lambda\}$ and $[\mathcal{P}_\kappa \lambda]^\kappa = \{Y_\alpha : \alpha < \lambda\}$. By induction on $\xi < \lambda$ construct $x_\xi \subset z_\xi \in \mathcal{P}_\kappa \lambda$ mutually distinct and $y_\xi^{\alpha i} \in Y_\alpha$ with $y_\xi^{\alpha i} \subsetneq z_\xi$ mutually distinct for $\alpha \in z_\xi$ and $i < 2$ as follows: At stage $\xi < \lambda$ by induction on $n < \omega$ take $z_{\xi n} \in \mathcal{P}_\kappa \lambda$ and $y_\xi^{\alpha i} \in Y_\alpha$ for $\alpha \in z_{\xi n}$ and $i < 2$ so that $x_\xi \subset z_{\xi 0} \not\subset \bigcup_{\zeta < \xi} z_\zeta$ and $z_{\xi n} \cup \bigcup \{y_\xi^{\alpha i} : \alpha \in z_{\xi n} \wedge i < 2\} \subsetneq z_{\xi n+1}$. Finally set $z_\xi = \bigcup_{n < \omega} z_{\xi n}$. We claim that f defined by $f(\{y_\xi^{\alpha i}, z_\xi\}) = i$ witnesses $\{z_\xi : \xi < \lambda\} \not\rightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^2$.

Fix an unbounded set $X \subset \{z_\xi : \xi < \lambda\}$. We show $f''[X]_\mathbb{C}^2 = 2$. Take $\alpha < \lambda$ with $Y_\alpha \in [X]^\kappa$, and $\xi < \lambda$ with $\alpha \in z_\xi \in X$. Then $f(\{y_\xi^{\alpha i}, z_\xi\}) = i$ for $i < 2$ by definition, as desired. $\square$

The above proof yields in fact for any $\gamma < \kappa$ $f : [X]_\mathbb{C}^2 \rightarrow \gamma$ with $X \in \mathcal{F}_{\kappa\lambda}^+$ and $f''[Y]_\mathbb{C}^2 = \gamma$ for any $Y \in \mathcal{F}_{\kappa\lambda}^+$ with $Y \subset X$.

It is natural to ask, as did Abe [A1], if $\mathcal{F}_{\kappa\lambda}^+ \not\rightarrow (\mathcal{F}_{\kappa\lambda}^+)_2^2$ holds in general. His answer [A2] to the analogous question would make it more interesting: $\mathcal{C}_{\kappa\lambda}^+ \not\rightarrow (\mathcal{C}_{\kappa\lambda}^+)_2^2$. Appealing more directly to Magidor's idea [Mag], we give a canonical witness to Abe's observation:

Proposition 2. *Let $\mu < \kappa$ be regular. Then $\{x \in \mathcal{P}_\kappa \lambda : \text{cf}(x \cap \kappa) = \mu\} \not\rightarrow (\mathcal{C}_{\kappa\lambda}^+)_2^2$.*

Proof. Set $S = \{x \in \mathcal{P}_\kappa \lambda : \text{cf}(x \cap \kappa) = \mu\}$ and for $x \in S$ fix an unbounded set $c_x \subset$

$x \cap \kappa$ of order type μ . For $\{x, y\} \in [S]_{\mathcal{C}}^2$ let $f(\{x, y\})$ be 0 when $\min(c_x \Delta c_y) \in c_x$, and 1 otherwise. Fix a stationary set $T \subset S$. We show $f''[T]_{\mathcal{C}}^2 = 2$.

First we have $\gamma < \kappa$ such that for any $w \in \mathcal{P}_\kappa \lambda$ there are $w \subset x, y \in T$ with $\gamma \in c_x - c_y$. Suppose to the contrary that we have $g : \kappa \rightarrow \mathcal{P}_\kappa \lambda$ such that for any $\gamma < \kappa$ and $g(\gamma) \subset x, y \in T$, $\gamma \in c_x$ iff $\gamma \in c_y$. Take $x, y \in C(g) \cap T$ with $x \cap \kappa < y \cap \kappa$ by the stationarity of $\{z \cap \kappa : z \in C(g) \cap T\}$. Then $c_x = c_y \cap x \cap \kappa$ has order type μ , contradicting the choice of c_y .

Now let $\gamma < \kappa$ be the minimal as above. Then for $\alpha < \gamma$ we have $w_\alpha \in \mathcal{P}_\kappa \lambda$ such that for any $w_\alpha \subset x, y \in T$, $\alpha \in c_x$ iff $\alpha \in c_y$. Set $w = \bigcup_{\alpha < \gamma} w_\alpha \in \mathcal{P}_\kappa \lambda$. Take $w \subset x \subset y \subset z$ from T with $\gamma \in c_x \cap c_z - c_y$. Then $\min(c_x \Delta c_y) = \min(c_y \Delta c_z) = \gamma$ by $w_\alpha \subset x \subset y \subset z$ for any $\alpha < \gamma$, and hence $f(\{x, y\}) = 0$ and $f(\{y, z\}) = 1$ by definition, as desired. $\square$

The rest of the paper is devoted to a negative answer to Abe's question in the strong sense. We refer to Baumgartner's expository paper [B] for the rudiments of iterated forcings. We are indebted for the definition of the poset Q_f below to Galvin (see [JS]), who proved under MA_{ω_1} that for any $f : [X]_{\mathcal{C}}^m \rightarrow n$ with $X \subset [\omega_1]^{<\omega}$ unbounded and $1 < m, n < \omega$ there is an unbounded $Y \subset X$ with $|f''[Y]_{\mathcal{C}}^m| = 1$.

Assume for the moment that κ is a compact cardinal and $\lambda \leq 2^\kappa$. Fix a coloring $f : \bigcup_{n < \omega} [S]_{\mathcal{C}}^n \rightarrow \gamma$ with $S \subset \mathcal{P}_\kappa \lambda$ unbounded and $1 < \gamma < \kappa$. We define a poset Q_f and establish its basic properties.

Fix a fine ultrafilter U on S and define inductively a κ -complete ultrafilter U_n on $[S]_{\mathcal{C}}^n$ by $U_0 = \{\{\emptyset\}\}$ and $U_{n+1} = \{X : \{x : \{a : \{x\} \cup a \in X\} \in U_n\} \in U\}$. For $n < \omega$ let β_n be the unique $\beta < \gamma$ with $\{a \in [S]_{\mathcal{C}}^n : f(a) = \beta\} \in U_n$. Let

$Q_f = \{p \in [S]^{<\kappa} : \forall m, n < \omega \forall a \in [p]_{\subset}^m (\{b \in [S]_{\subset}^n : f(a \cup b) = \beta_{m+n}\} \in U_n)\}$ and $q \leq p$ iff $q \supset p$ and $y \not\subset x$ for any $x \in p$ and $y \in q - p$.

First for a generic filter $G \subset Q_f$, $\bigcup G$ is unbounded in $\mathcal{P}_\kappa \lambda$ by the density of $\{q \in Q_f : \exists y \in q (x \subset y)\}$ for any $x \in \mathcal{P}_\kappa \lambda$, and homogeneous for f : $f''[\bigcup G]_{\subset}^n = \{\beta_n\}$ for any $n < \omega$.

Next Q_f is κ -centered closed (hence in particular κ -directed closed): A centered subset D of Q_f of size $< \kappa$ has a lower bound $\bigcup D$.

Finally we show that Q_f is κ -linked. Fix an injection $\pi : \mathcal{P}_\kappa \lambda \rightarrow {}^\kappa 2$. For $A \subset {}^\kappa 2$ with $\alpha < \kappa$ set $Q_{f,A} = \{p \in Q_f : \{\pi(x)|_\alpha : x \in p\} = A \wedge \langle \pi(z)|_\alpha : z \in \bigcup_{x \in p} \mathcal{P}x \rangle \text{ is injective}\}$. Then $Q_f = \bigcup \{Q_{f,A} : \exists \alpha < \kappa (A \subset {}^\kappa 2)\}$ by inaccessibility of κ . To see linkedness of $Q_{f,A}$, fix $p, q \in Q_{f,A}$. Then $x \not\subset y$ for any $x \in p - q$ and $y \in q$: Otherwise we would have $x = z$ for some $x \in p - q$, $y \in q$ with $x \subset y$ and $z \in q$ with $\pi(x)|_\alpha = \pi(z)|_\alpha$. Similarly $y \not\subset x$ for any $x \in p$ and $y \in q - p$. Thus $p \cup q \leq p, q$, as desired.

A minor modification of the original proof [B] for $\kappa = \omega_1$ yields the following

Lemma. *Assume $2^{<\kappa} = \kappa$. Let $\langle \dot{Q}_\alpha : \alpha < \beta \rangle$ be a $< \kappa$ -support iteration such that $\Vdash_\alpha$ “ $\dot{Q}_\alpha$ is κ -linked and κ -centered closed” for any $\alpha < \beta$. Then P_β satisfies κ^+ -c.c.*

Proof. Fix $X \in [P_\beta]^{\kappa^+}$. For $\alpha < \beta$ let $\Vdash_\alpha$ “ $\dot{Q}_\alpha = \bigcup_{\gamma < \kappa} \dot{Q}_{\alpha\gamma}$ with $\dot{Q}_{\alpha\gamma}$ linked for any $\gamma < \kappa$.” For $p \in X$ by induction on $\xi < \kappa$ take $p_\xi \leq p$, $\alpha_\xi^p \in \text{supp}(p_\xi)$ and $\gamma_\xi^p < \kappa$ so that $p_\zeta \leq p_\xi$ and $p_{\xi+1}|_{\alpha_\xi^p} \Vdash_{\alpha_\xi^p}$ “ $p_\xi(\alpha_\xi^p) \in \dot{Q}_{\alpha_\xi^p \gamma_\xi^p}$ ” for any $\xi < \zeta < \kappa$, and $\{\xi < \kappa : \alpha_\xi^p = \alpha\}$ is unbounded for any $\alpha \in \bigcup_{\xi < \kappa} \text{supp}(p_\xi)$. Take $Y \in [X]^{\kappa^+}$ and $\delta < \kappa$ so that $\delta \in \Delta_{\xi < \kappa} \cap \{\lim\{\xi < \kappa : \alpha_\xi^p = \alpha\} : \alpha \in \text{supp}(p_\xi)\}$ for any

$p \in Y$. Next take $Z \in [Y]^{\kappa^+}$ so that $\{\{\alpha_\xi^p : \xi < \delta\} : p \in Z\}$ forms a Δ -system with root $d \in [\beta]^{<\kappa}$. Finally take $W \in [Z]^{\kappa^+}$ and $H \in [\delta \times d \times \kappa]^{<\kappa}$ so that $H = \{(\xi, \alpha_\xi^p, \gamma_\xi^p) : \xi < \delta \wedge \alpha_\xi^p \in d\}$ for any $p \in W$.

To see that W is linked, fix $p, q \in W$. Inductively we construct a lower bound r of $\{p_\xi, q_\xi : \xi < \delta\} \subset P_\beta$ with support $s = \bigcup_{\xi < \delta} \text{supp}(p_\xi) \cup \bigcup_{\xi < \delta} \text{supp}(q_\xi)$. At stage $\alpha \in s$ it suffices to show $r|_\alpha \Vdash_\alpha \text{"}\{p_\xi(\alpha), q_\xi(\alpha) : \xi < \delta\} \subset \dot{Q}_\alpha \text{"}$ is centered."

When $\alpha \in d = \bigcup_{\xi < \delta} \text{supp}(p_\xi) \cap \bigcup_{\xi < \delta} \text{supp}(q_\xi)$, for unboundedly many $\xi < \delta$ $\alpha = \alpha_\xi^p$ by the choice of δ , and hence $r|_\alpha \leq p_{\xi+1}|_\alpha, q_{\xi+1}|_\alpha$ forces $p_\xi(\alpha), q_\xi(\alpha) \in \dot{Q}_{\alpha\gamma}$, where $(\xi, \alpha, \gamma) \in H$, as desired. Otherwise the claim follows, since $r|_\alpha \Vdash_\alpha \text{"}\{p_\xi(\alpha), q_\xi(\alpha) : \xi < \delta\} \subset \dot{Q}_\alpha \text{"}$ is descending." $\square$

We are now ready to prove our main result. By $F^+ \rightarrow (F^+)_\gamma^{<\omega}$ we mean that for any $f : \bigcup_{n < \omega} [X]_\mathbb{C}^n \rightarrow \gamma$ with $X \in F^+$ there is $Y \in F^+$ with $Y \subset X$ and $|f"[Y]_\mathbb{C}^n| = 1$ for any $n < \omega$. Note that $\mathcal{F}_{\kappa\kappa}^+ \rightarrow (\mathcal{F}_{\kappa\kappa}^+)_\gamma^{<\omega}$ iff κ is Ramsey for any $1 < \gamma < \kappa$.

Theorem. *Let κ be a supercompact cardinal and λ a cardinal $> \kappa$. Then there is a κ^+ -c.c. poset forcing supercompactness of κ and $\mathcal{F}_{\kappa\lambda}^+ \rightarrow (\mathcal{F}_{\kappa\lambda}^+)_\gamma^{<\omega}$ for any $1 < \gamma < \kappa$.*

Proof. First we force with the Laver poset $[L]$ for κ and then add λ Cohen subsets of κ to ensure supercompactness of κ and $\lambda \leq 2^\kappa$ in the further extensions. Next we perform a $< \kappa$ -support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < 2^{\lambda^{<\kappa}} \rangle$ with $\Vdash_\alpha \text{"}\dot{Q}_\alpha = \dot{Q}_f \text{"}$ for some canonical P_α -name f for a coloring. The standard inductive argument, together with κ -closure and κ^+ -c.c. of P_α shows that P_α is of size $\leq 2^{\lambda^{<\kappa}}$, and so is the set of canonical P_α -names for colorings for any $\alpha < 2^{\lambda^{<\kappa}}$, whose union can be identified with that of canonical $P_{2^{\lambda^{<\kappa}}}$ -names for colorings. Thus the iteration can

be arranged so that a homogeneous set for a coloring in the final model by $P_{2^{\lambda < \kappa}}$ appears in an intermediate model, which remains unbounded by absoluteness of $\mathcal{P}_\kappa\lambda$, as desired. $\square$

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