

**A sufficient condition for a bipartite graph to have
a k -factor.**

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In this paper, we consider only finite undirected simple graphs. A graph denoted by $(X, Y; E)$ is a bipartite graph with partite sets X and Y and edge set $E \subset X \times Y$. If A is a subset of vertices, $N(A)$ denotes the set of vertices adjacent to one of the vertices of A . For two disjoint subsets of vertices A and B , $e(A, B)$ denotes the number of the edges joining A and B . A vertex x is often identified with $\{x\}$. So $e(x, B)$ means $e(\{x\}, B)$ and $N(x)$ means $N(\{x\})$. The other notations may be found in [1].

A k -regular spanning subgraph is called a k -factor. In a bipartite graph $(X, Y; E)$, a complete k -matching from X to Y is defined as a spanning subgraph such that the degree of each vertex of X is k , and the degree of each vertex of Y is at most k . We abbreviate the complete 1-matching from X to Y as a complete matching from X to Y .

Theorem A (Hall[2]). *A bipartite graph $(X, Y; E)$ has a complete matching from X to Y if and only if $|N(S)| \geq |S|$ holds for all $S \subset X$.*

The next theorem, first proved by Ore and Ryser, gives a necessary and sufficient condition for a bipartite graph $(X, Y; E)$ to have a complete k -matching from X to Y . Now, for $S \subset X$ and $T \subset Y$, we define

$$\delta(S, T) := e(S, Y - T) + k|T| - k|S|.$$

Theorem B (Ore, Ryser[4]). *A bipartite graph $(X, Y; E)$ has a complete k -matching from X to Y if and only if $\delta(S, T) \geq 0$ holds for all $S \subset X$ and all $T \subset Y$.*

In this paper, we give a sufficient condition for the existence of a complete k -matching in a bipartite graph, which is an extension of Hall's theorem (Theorem A). Katerinis proved the following theorem.

Theorem C (Katerinis[3]). *If a bipartite graph $(X, Y; E)$ satisfies (C.1) and (C.2), then $(X, Y; E)$ has a 2-factor.*

$$(C.1) \quad |X| = |Y| \geq 2.$$

$$(C.2) \quad \text{For all } M \subset X,$$

$$\begin{aligned} |N(M)| &\geq \frac{3}{2}|M| && \text{if } |M| < \left\lfloor \frac{2}{3}|Y| \right\rfloor, \\ |N(M)| &= |Y| && \text{if } |M| \geq \left\lfloor \frac{2}{3}|Y| \right\rfloor. \end{aligned}$$

As a generalization of Theorem C, we give our main result in this paper.

Theorem 1. *Suppose $k \geq 2$. If a bipartite graph $(X, Y; E)$ satisfies (1.1), (1.2) and (1.3), then $(X, Y; E)$ has a complete k -matching from X to Y .*

$$(1.1) \quad |X| \leq |Y|, \quad |Y| \geq k.$$

$$(1.2) \quad \text{For every } M \subset X \text{ satisfying } |M| < \left\lfloor (k-1 + \frac{1}{k})^{-1} |Y| \right\rfloor,$$

$$|N(M)| \geq (k-1 + \frac{1}{k})|M| \text{ holds.}$$

$$(1.3) \quad \text{For every } M \subset X \text{ satisfying } |M| \geq \left\lfloor (k-1 + \frac{1}{k})^{-1} |Y| \right\rfloor, |N(M)| = |Y| \text{ holds.}$$

In case of $|X| = |Y|$, a complete k -matching from X to Y is equivalent to a

k -factor. Therefore, Theorem 1 also gives a sufficient condition on the existence of a k -factor. Hence, in case of $k=2$, Theorem 1 implies Theorem C. Moreover, if we apply Theorem 1 to the case of $k=1$, then we have the non-trivial implication of Hall's theorem.

The next theorem is slightly stronger than Theorem 1. Hence, we prove Theorem 2 instead of Theorem 1.

Theorem 2. *Suppose $k \geq 2$. If a bipartite graph $(X, Y; E)$ satisfies (2.1), (2.2) and (2.3), then $(X, Y; E)$ has a complete k matching from X to Y .*

$$(2.1) \quad |X| \leq |Y|, \quad |Y| \geq k.$$

$$(2.2) \quad \text{For every } M \subset Y \text{ satisfying } |M| < \left\lceil \left(k-1 + \frac{1}{k}\right)^{-1} |Y| \right\rceil \quad \text{and}$$

$$|M| \equiv 1 \pmod{k}, \quad |N(M)| \geq \left(k-1 + \frac{1}{k}\right) |M|.$$

$$(2.3) \quad \text{For every } M \subset Y \text{ satisfying } |M| \geq \left\lceil \left(k-1 + \frac{1}{k}\right)^{-1} |Y| \right\rceil, \quad |N(M)| = |Y| \text{ holds.}$$

Before proving Theorem 2, we state the following lemma.

Lemma 3. *Let $k \geq 2$ be an integer, and $G = (X, Y; E)$ be a bipartite graph satisfying $|X| \leq |Y|$ and $|Y| \geq k$. Suppose there exist $S \subset X$ and $T \subset Y$ such that $\delta(S, T) < 0$. If we choose such S and T so that $S \cup (Y - T)$ is minimal, then (3.1), (3.2), (3.3) and (3.4) hold.*

$$(3.1) \quad \text{For any vertex } x \text{ of } S, \quad e(x, Y - T) \leq k - 1 \text{ holds. Therefore} \\ e(S, Y - T) \leq (k - 1) |S| \text{ holds.}$$

$$(3.2) \quad \text{For any vertex } y \text{ of } Y - T, \quad e(S, y) \leq k - 1 \text{ holds.}$$

$$(3.3) \quad |N(S)| < \left(k-1 + \frac{1}{k}\right) |S|.$$

$$(3.4) \quad \text{There exists a subset } M \text{ of } S \text{ such that } |M| \equiv 1 \pmod{k} \text{ and}$$

$$|N(M)| < \left\lceil \left(k-1 + \frac{1}{k}\right) |M| \right\rceil \text{ holds.}$$

Proof. Suppose there exists a vertex x of S such that $e(x, Y-T) \geq k$. Let $S' := S - \{x\}$. Then

$$\begin{aligned}\delta(S', T) &= k|T| + e(S', Y-T) - k|S'| \\ &\leq k|T| + e(S, Y-T) - k - k|S| + k \\ &= \delta(S, T) < 0.\end{aligned}$$

This contradicts the minimality of $S \cup (Y-T)$. Thus we obtain (3.1).

Similarly, suppose there exists $y \in Y-T$ such that $e(S, y) \geq k$, and let $T' := T \cup \{y\}$. Then $\delta(S, T') \leq \delta(S, T) < 0$, contradicting the minimality of $S \cup (Y-T)$, and (3.2) follows.

Since G is a bipartite graph, $|N(S)| \leq |T| + e(S, Y-T)$. By the fact that $\delta(S, T) < 0$ and (3.1),

$$\begin{aligned}|N(S)| &\leq |T| + e(S, Y-T) \\ &< |S| + (1 - \frac{1}{k})e(S, Y-T) \\ &\leq |S| + (1 - \frac{1}{k})(k-1)|S| \\ &= (k-1 + \frac{1}{k})|S|.\end{aligned}$$

Thus (3.3) is obtained.

If $|S| \equiv 1 \pmod{k}$, then immediately (3.4) holds. By the fact that $\delta(S, T) < 0$, $S \neq \emptyset$. Hence let $|S| \equiv 1+r \pmod{k}$ where $1 \leq r \leq k-1$ and R be a subset of S such that $e(R, Y-T)$ is maximum over $|R| = r$ and $R \subset S$. Let $d := \min\{e(x, Y-T); x \in R\}$ and $M := S - R$. Then we have

$$\begin{aligned}e(x, Y-T) &\geq d \quad \text{for all } x \in R \\ e(x', Y-T) &\leq d \quad \text{for all } x' \in M\end{aligned}$$

and by (3.1), $d \leq k-1$. On the other hand, $|N(M)| \leq |T| + e(M, Y-T)$. Therefore,

$$\begin{aligned}|N(M)| &\leq |T| + e(M, Y-T) \\ &< |S| - \frac{1}{k}e(S, Y-T) + e(M, Y-T) \\ &= (1 - \frac{1}{k})e(M, Y-T) - \frac{1}{k}e(R, Y-T) + |S|\end{aligned}$$

$$\begin{aligned}
&\leq (1 - \frac{1}{k})d|M| - \frac{1}{k}d|R| + |S| \\
&= \frac{d}{k}((k-1)|M| - |R|) + |S| \\
&\leq \frac{(k-1)^2}{k}|M| - \frac{k-1}{k}|R| + |M| + |R| \\
&= (k-1 + \frac{1}{k})|M| + \frac{1}{k}|R| \\
&\leq \left\lceil (k-1 + \frac{1}{k})|M| \right\rceil
\end{aligned}$$

and (3.4) follows. ▀

Proof of Theorem 2. We assume that $(X, Y; E)$ has no complete k -matching from X to Y . By Theorem B, there exist $S \subset X$ and $T \subset Y$ satisfying $\delta(S, T) < 0$. We may assume that the situations of (3.1) – (3.4) occur. By the assumption (2.2) and (2.3), we have $N(M) = Y$. Since $M \subset S$, also we have $N(S) = Y$.

Let $z := \min\{e(S, y); y \in Y - T\}$. Note that $Y - T \neq \emptyset$, by the assumption that $\delta(S, T) < 0$. Hence z is well-defined, and $1 \leq z \leq k-1$, by (3.2) and the fact that $|N(S)| = Y$. The neighborhood of S , that is Y , is at most $|T| + \frac{1}{z}e(S, Y - T)$.

Hence by (3.1) and the fact that $\delta(S, T) < 0$,

$$\begin{aligned}
|Y| = |N(S)| &\leq |T| + \frac{1}{z}e(S, Y - T) \\
&< |S| + (\frac{1}{z} - \frac{1}{k})e(S, Y - T) \\
&\leq |S| + (\frac{1}{z} - \frac{1}{k})(k-1)|S| \\
&= \frac{k^2 - k + z}{kz}|S|. \tag{1}
\end{aligned}$$

Let y_0 be a vertex of $Y - T$ such that $e(S, y_0) = z$, and let $S_0 := S - N(y_0)$. Since $N(S_0) \subset Y - \{y_0\}$, S_0 cannot satisfy the conclusion of (2.3). Hence

$$|S_0| = |S| - z \leq \frac{k}{k^2 - k + 1}|Y| - 1. \tag{2}$$

By (1) and (2),

$$\begin{aligned}
k(k-1)(z-1)|S| &< z(z-1)(k^2 - k + 1) \\
|S| &< (k-1 + \frac{1}{k}) \cdot \frac{z}{k-1}
\end{aligned}$$

$$\leq k-1 + \frac{1}{k}.$$

Thus $|S| < k$ and therefore $|M| = 1$. But (3.4) and the fact that $N(M) = Y$ imply that $|Y| = |N(M)| < \left\lfloor k-1 + \frac{1}{k} \right\rfloor = k$. This contradicts the assumption that $|Y| \geq k$. ■

Theorem 2 is in some sense best possible. The graph $(X, Y; E)$ defined as the following shows that the condition $|N(M)| \geq (k-1 + \frac{1}{k})|M|$ of (2.2) cannot be replaced by $|N(M)| \geq \left\lfloor (k-1 + \frac{1}{k})|M| \right\rfloor - 1$ (see Fig. 1).

$$X := A \cup A'$$

$$\text{where } A = \{a_1, \dots, a_{km+1}\}$$

$$A' = \{a_{km+2}, \dots, a_n\} \quad n \geq (k^2 + k + 1)m + 2k - 1.$$

$$Y := B \cup C \cup D$$

$$\text{where } B = \{b_{ij} \mid 1 \leq i \leq km+1, 1 \leq j \leq k-1\}$$

$$C = \{c_1, \dots, c_m\}$$

$$D = \{d_1, \dots, d_l\} \quad (k-1)(km+1) + m + l \geq n.$$

$$E = \{a_i b_{ij} \mid 1 \leq i \leq km+1, 1 \leq j \leq k-1\} \cup (A \times C) \cup (A' \times Y).$$

Moreover, in this graph all but one $M (= A)$ satisfy (2.2).

Besides, the conditions (1.2) and (1.3) of Theorem 1 cannot be unified to the condition:

$$|N(M)| \geq \min\{|Y|, (k-1 + \frac{1}{k})|M|\} \quad \text{for all } M \subset X. \quad (1.4)$$

The graph in Fig. 2 satisfies (1.1) and (1.4) but has no complete k -matching from X to Y .

But the graphs which satisfy (1.1) and (1.4) and have no complete k -matching from X to Y have a similar induced subgraph. Finally, we prove the next theorem.

Theorem 4. Suppose $k \geq 2$. And also suppose that $G = (X, Y; E)$ is a bipartite graph such that $|X| \leq |Y|$ and $|Y| \geq k$. If G satisfies (1.4) and G has no

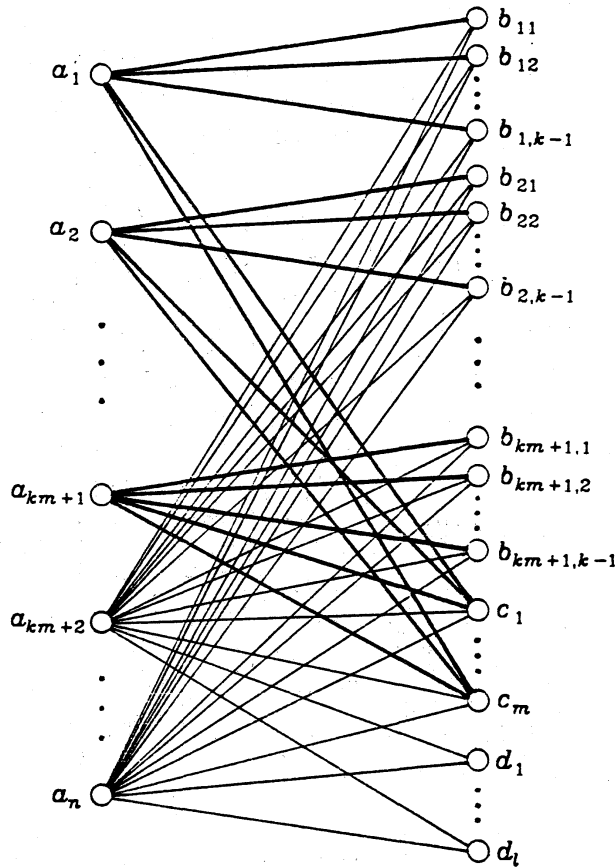


Fig. 1.

complete k -matching from X to Y , then there exist $S \subset X$ and $T \subset Y$ such that (4.1), (4.2) and (4.3) hold.

$$(4.1) \quad |S| = k|T| + 1.$$

$$(4.2) \quad e(S, y) = 1 \quad \text{for all } y \in Y - T.$$

$$(4.3) \quad e(x, Y - T) = k - 1 \quad \text{for all } x \in S.$$

Proof. Since G has no complete k -matching from X to Y , we may assume that we have (3.1) – (3.4). Therefore we have $N(S) = Y$. Let $z := \min\{e(S, y); y \in Y - T\}$. Since $Y - T \neq \emptyset$, z is well-defined, and $1 \leq z \leq k - 1$. Now, we have

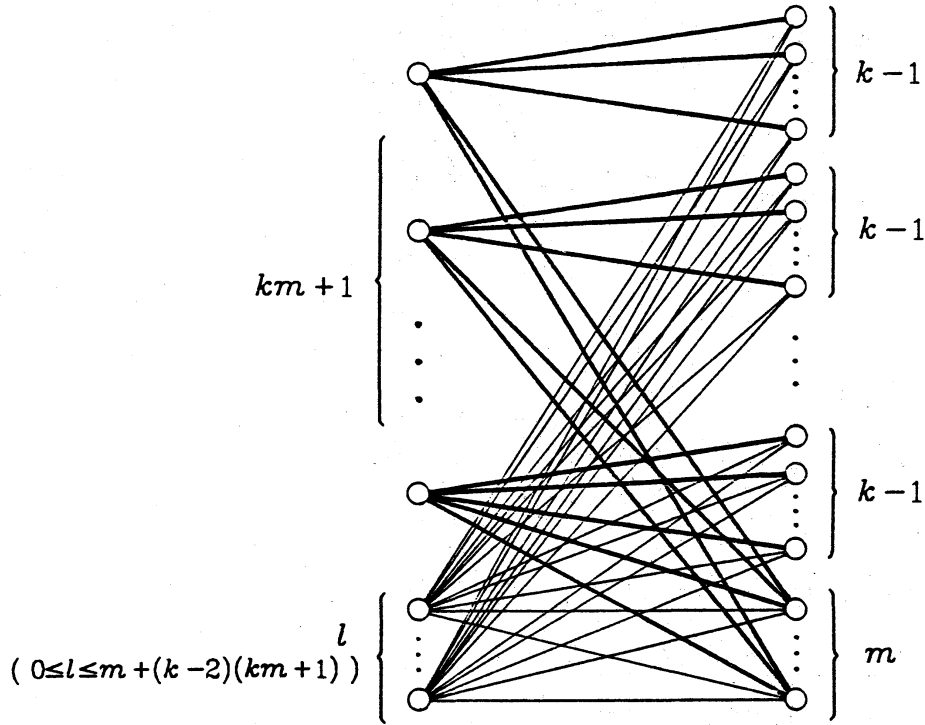


Fig. 2.

$$\begin{aligned}
 |Y| = |N(S)| &\leq |T| + \frac{1}{z} e(S, Y-T) \\
 &< \frac{k^2 - k + z}{kz} |S|.
 \end{aligned}$$

Let $y_0 \in Y-T$ such that $e(S, y_0) = z$, and let $S_0 := S - N(y_0)$. Since $N(S_0) \subset Y - \{y_0\}$, we have

$$|Y| - 1 \geq |N(S_0)| \geq (k-1 + \frac{1}{k}) |S_0| = (k-1 + \frac{1}{k})(|S| - z).$$

By the above two inequalities, we have

$$\begin{aligned}
 \frac{k^2 - k + z}{kz} |S| &> |Y| \geq (k-1 + \frac{1}{k})(|S| - z) + 1, \\
 (k^2 - k + z) |S| &> z(k^2 - k + 1)(|S| - z) + kz, \\
 z^2(k^2 - k + 1) - kz &> (z-1)k(k-1)|S|,
 \end{aligned}$$

Since $|Y| \geq k$, $|S| \geq |M| \geq k+1$. Hence

$$z^2(k^2-k+1) - kz > (z-1)k(k-1)(k+1),$$

$$z^2(k^2-k+1) - zk^3 + k(k^2-1) > 0.$$

We claim that the only situation that $z = 1$ makes this inequality true. Suppose

$z \geq 2$, and let $f_k(z) := z^2(k^2-k+1) - zk^3 + k(k^2-1)$. Then, since $k \geq z+1 \geq 3$,

$$\begin{aligned} f_k(2) &= 4(k^2-k+1) - 2k^3 + k(k^2-1) \\ &= -k^3 + 4k^2 - 5k + 4 \\ &= -k(k-2)^2 - k + 4 < 0, \end{aligned}$$

and

$$\begin{aligned} f_k(k-1) &= (k-1)^2(k^2-k+1) - (k-1)k^3 + k(k^2-1) \\ &= (k-1)\{1 - (k-1)(k-2)\} < 0. \end{aligned}$$

Hence, $2 \leq z \leq k-1$ implies $f_k(z) < 0$, which is a contradiction. Thus the claim follows.

Define

$$U := \{u \in Y-T; e(S, u) = 1\},$$

$$W := \{w \in Y-T; e(S, w) \geq 2\} = Y-T-U.$$

Since $z=1$, $U \neq \emptyset$. We choose $u \in U$ arbitrarily, and let x_u be the only neighborhood of u in S . Now, define α , β and γ as the following non-negative integers (especially, note that $\gamma \geq 1$).

$$\alpha := \sum_{w \in W} (e(S, w) - 1),$$

$$\beta := \sum_{x \in S} (k - 1 - e(x, Y-T)),$$

$$\gamma := e(x_u, U).$$

By these definitions, we have

$$|Y| = |N(S)| = |T| + e(S, Y-T) - \alpha, \quad (3)$$

$$e(S, Y-T) = (k-1)|S| - \beta, \quad (4)$$

$$|Y| - \gamma \geq |N(S - \{x_u\})| \geq (k-1 + \frac{1}{k})(|S| - 1). \quad (5)$$

By the definitions of α and β ,

$$\alpha \geq e(x_u, W), \quad (6)$$

$$\beta \geq k - 1 - e(x_u, Y-T). \quad (7)$$

Thus we have

$$\alpha + \beta + \gamma \geq k-1. \quad (8)$$

By (3) and the fact that $\delta(S, T) < 0$,

$$|Y| < |S| + (1 - \frac{1}{k})e(S, Y-T) - \alpha.$$

And by (4),

$$|Y| < (k-1 + \frac{1}{k})|S| - \alpha - (1 - \frac{1}{k})\beta. \quad (9)$$

Thus with (5), we have

$$\alpha + \gamma + (k-1)(\alpha + \beta + \gamma) < k^2 - k + 1.$$

If $\alpha + \beta + \gamma \geq k$, we have $\alpha + \gamma < 1$. This contradicts the fact that $\gamma \geq 1$.

Therefore in (8), hence also in (6) and (7), the equality holds.

From the equality of (7), we have $e(x, Y-T) = k-1$ for all $x \in S - \{x_u\}$. From the equality of (6), for all $w \in W$, $e(S, w) = 2$ and $x_u \in N(w) \cap S$, and hence $\alpha = |W|$.

First we claim that $W = \emptyset$. In case of $k = 2$, $W = \emptyset$ is an immediate consequence of (3.2). Thus it suffices to show the claim in case of $k \geq 3$. Assume $W \neq \emptyset$ and let $w_0 \in W$. Since $|S| \geq k+1 \geq 4$ and $e(S, w_0) = 2$, there exists $x_0 \in S - N(w_0)$. We note that $x_0 \neq x_u$. Because of the fact that $e(x_0, Y-T) = k-1 > \alpha = |W|$, there exists $v \in U \cap N(x_0)$. Especially $v \neq u$, and the only neighborhood of v in S , say x_v , is x_0 . Hence the similar arguments lead us to the fact that $x_v \in N(w) \cap S$ for all $w \in W$. But $x_v = x_0 \notin S \cap N(w_0)$. This is a contradiction. Thus we have the claim, and therefore (4.2) holds.

Next we prove (4.3). It suffices to show that $e(x_u, Y-T) = k-1$. Since $|S| \geq k+1 \geq 3$, we can take $x' \in S - \{x_u\}$. From the above arguments, we can regard x' as the only neighborhood in S of some vertex $v \in U = Y-T$. Then by the similar arguments, $e(x, Y-T) = k-1$ holds for all $x \in S - \{x'\}$. Especially $e(x_u, Y-T) = k-1$ holds.

The results given above show that $\alpha = \beta = 0$ and $\gamma = k-1$. Hence by (5) and (9),

$$(k-1+\frac{1}{k})|S| - \frac{1}{k} \leq |Y| < (k-1+\frac{1}{k})|S|,$$

and so

$$|Y| = (k-1)|S| + \frac{|S|-1}{k}.$$

This implies (4.1), for we have $|Y-T| = (k-1)|S|$ from (4.2) and (4.3). And we complete the proof. ■

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