

Iteration of Elliptic Functions

By

Toshizô Matsumoto

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1. Let $F(z)$ be an elliptic function with periods ω, ω' and its fundamental parallelogram be D . In the following we shall consider the iteration of $F(z)$. Let $W = m\omega + n\omega'$ be any period, then we consider the point ζ such as

$$F(\zeta) = \zeta + W. \quad (1)$$

The existence of such a point may easily be known provided $|W|$ is great. For let ν be the order of $F(z)$, then in D there are just ν poles. Hence we may suppose $F(z)$ bounded on the contour of D , (or else a little modification of it is sufficient). We consider the integral along the contour

$$\mathfrak{J} = \frac{1}{2\pi i} \oint \frac{F'(z)-1}{F(z)-z-W} dz. \quad (2)$$

$|W|$ being sufficiently great, the denominator shall not be zero at any point of the contour. If we make $|W|$ great, $|\mathfrak{J}|$ becomes however small. Therefore in D there are ν points ζ such as (1).

The set of points ζ for all W is countable and isolated, the poles in D being the points of accumulation.

Now about ζ we have the expansion

$$F(z) = \zeta + W + s(z - \zeta) + \dots, \quad s = F'(\zeta).$$

For n -th iteration we have

$$F_n(z) = \zeta + W + s^n(z - \zeta) + \dots,$$

where

$$F_n(\zeta) = \zeta + W,$$

$$F'_n(\zeta) = F'(F_{n-1}(\zeta))F'(F_{n-2}(\zeta)) \dots F'(F(\zeta))F'(\zeta).$$

If $|s| < 1$, then since in a circle $(\odot)$ sufficiently small, $\left| \frac{F(z) - \zeta - W}{z - \zeta} \right|$ becomes less than a number $\sigma < 1$, we have

$$|F_n(z) - \zeta - W| < \sigma^n |z - \zeta|, \quad n=1, 2, \dots$$

ζ is (the equivalent point in D of) an attractive point and all points of the circle $(\odot)$ by the iteration converge to ζ . All points of a certain domain $(\mathcal{A})$ containing $(\odot)$ in it shall have the same property. This is the immediate domain of attraction.

2. Next let us consider the attractive cycle. If

$$F(\zeta) = \zeta_1 + W, \quad F(\zeta_1) = \zeta_2 + W, \dots, \quad F(\zeta_{m-1}) = \zeta + W, \quad (3)$$

where W mean only certain periods, then $\zeta, \zeta_1, \zeta_2, \dots, \zeta_{m-1}$ are the cycle corresponding to ζ of order m . ζ shall be found from the equation

$$F_m(\zeta) = \zeta + W. \quad (4)$$

The existence of such point may be proved quite in the same way as in the case $m=1$. For that

$$F_m(z) = F_{m-1}(F(z)) = \dots = F(F_{m-1}(z))$$

shall be bounded, z must not pass through the poles of $F(z)$. There are ν poles in D . Let α be one of them and β be such that $F(\beta) = \alpha$. There are such ν points in D . Not only this, we must also avoid such points β in D : $F(\beta) = \alpha + W$. There are ν points β in D . Now varying W , we know that for a pole, there are always countably infinite number of β . Thus for ν poles in D , there are countably infinite number of β in D . Again for each β , we must take care of points γ such that $F(\gamma) = \beta + W$. Continuing this, we have a countable set E of avoidable points in D . These points are isolated but converge to the poles in D . This is clear, since if $|W|$ be bounded, the points $\beta, \gamma, \dots$ are finite in number, therefore, for that they are infinite in number, it must be such that $W \rightarrow \infty$, so that $F \rightarrow \infty$. Therefore we may suppose that any point of E is not on the contour of D , so that $F_n(z)$ is bounded on the contour.

Now we seclude the poles of $F(z)$ by small circles, then there

remain in D a finite number of points of E . We also seclude all the essential singularities of $F_n(z)$ by circles. Let D' be the remaining domain. There remain only a finite number of poles of $F_m(z)$ in D' . Now consider the integral as (2),

$$\mathfrak{J} = \frac{1}{2\pi i} \oint \frac{F'_m(z) - 1}{F_m(z) - z - W} dz, \quad (5)$$

we may conclude (4). As it is known, we have

$$\begin{aligned} F'_m(\zeta) &= F'(\zeta_{m-1})F'_{m-1}(\zeta) = \dots = F'(\zeta_{m-1})F'(\zeta_{m-2})F'(\zeta), \\ F'_m(\zeta) &= F'_m(\zeta_1) = \dots = F'_m(\zeta_{m-1}). \end{aligned}$$

Hence in attractive case, for all z in a circle ($\odot$) about ζ , $F_{pm}(z)$ tend to ζ for $p \rightarrow \infty$; $F_{pm+1}(z)$ to $\zeta_1 + W$; ...; $F_{pm+m-1}(z)$ to $\zeta_{m-1} + W$, which we say $F_n(z)$ converges uniformly to the cycle.

3. Writing $F(\xi) = z + W$, where ζ is an attractive point such as $F(\zeta) = \zeta + W$, we consider the inverse function about $\xi = \zeta$. If ζ be not a branch point, then

$$\xi = F_{-1}(z + W) = \zeta + \bar{s}(z - \zeta) + \dots,$$

where $s\bar{s} = 1$, hence $|\bar{s}| > 1$. If a circle $|z - \zeta| < r$ be in the domain of attraction, then its transformed domain by $\xi = F_{-1}(z + W)$ contains it. Therefore again we may consider the inverse function

$$\xi = F_{-1}(F_{-1}(z + W) + W), \dots$$

If these functions are holomorphic in the circle then they must be bounded in it. Hence they form a normal family. This contradicts the fact that ζ is a repulsive point of these functions. Thus about the attractive point there must be a branch point of the inverse function of $F(\xi) = z + W$, that is $F'(\zeta) = 0$. Since $F'(\xi)$ is elliptic, such points are finite in number in the fundamental parallelogram; hence the number of the attractive points is finite in the parallelogram.

Same consideration will be possible for the cycle.