

Composition Operators on Function Spaces with Fractional Order of Smoothness

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Abstract

In this paper we will give an overview concerning properties of composition operators $T_f(g) := f \circ g$ in the framework of Besov-Lizorkin-Triebel spaces. Boundedness and continuity will be discussed in a certain detail. In addition we also give a list of open problems.

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1 Introduction

Let E denote a normed space of functions. Composition operators $T_f : g \mapsto f \circ g, g \in E$, are simple examples of nonlinear mappings. It is a little bit surprising that the knowledge about these operators is rather limited. One reason is, of course, that the properties of T_f strongly depend on f and E . Here in this paper we are concerned with E being either a Besov or a Lizorkin-Triebel space (for definitions of these classes we refer to the appendix at the end of this article). These scales of spaces generalize Sobolev spaces $W_p^m(\mathbb{R}^n)$, Bessel potential spaces $H_p^s(\mathbb{R}^n)$, Slobodeckij spaces $W_p^s(\mathbb{R}^n)$ as well as H lder spaces $C^s(\mathbb{R}^n)$ in view of the identities

- $W_p^m(\mathbb{R}^n) = F_{p,2}^m(\mathbb{R}^n), 1 < p < \infty, m \in \mathbb{N};$
- $H_p^s(\mathbb{R}^n) = F_{p,2}^s(\mathbb{R}^n), 1 < p < \infty, s \in \mathbb{R};$

- $W_p^s(\mathbb{R}^n) = F_{p,p}^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n)$, $1 \leq p < \infty$, $s > 0$, $s \notin \mathbb{N}$;
- $C^s(\mathbb{R}^n) = B_{\infty,\infty}^s(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$.

In our opinion there is a very interesting interplay between integrability and regularity properties of f, g and of the composition $f \circ g$. It is our aim to describe this in detail and to give a survey on the state of the art. Many times we will not give proofs but discuss illustrating examples. The theory is far from being complete. However we believe that a discussion of these operators in spaces of fractional order of smoothness $s > 0$ is appropriate and leads to a better understanding of the various phenomena which occur.

Convention: If there is no need for a distinction between Lizorkin-Triebel spaces and Besov spaces we will simply write $E_{p,q}^s(\mathbb{R}^n)$ instead of $F_{p,q}^s(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n)$, respectively. In the same spirit, $E_{p,q}^s(\mathbb{R}^n)$ can denote $W_p^s(\mathbb{R}^n)$ in case $s \in \mathbb{N}$ and $p = 1$ or $+\infty$, although those spaces are not Lizorkin-Triebel nor Besov spaces.

An illustrating example

Let us have a look at the following boundary value problem:

$$\begin{aligned} \Delta u(x) + f(u(x)) &= h(x), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Here Ω is an open and bounded subset of $\mathbb{R}^n$ with smooth boundary. Let L denote the solution operator for the Dirichlet problem for the Laplacian with respect to Ω . Then our boundary value problem can be reformulated as a fixed point problem

$$u = L(h - f(u)). \quad (1)$$

By $\mathring{E}_{p,q}^s(\Omega)$ we denote the collection of all functions in $E_{p,q}^s(\Omega)$ having vanishing boundary values (this makes sense if $s > 1/p$). Since $L : E_{p,q}^{s-2}(\Omega) \rightarrow \mathring{E}_{p,q}^s(\Omega)$ is an isomorphism a discussion of (1) requires

$$T_f(E_{p,q}^s(\Omega)) \subset E_{p,q}^{s-2}(\Omega)$$

including some estimates which relate the norms $\|T_f(u)\|_{E_{p,q}^{s-2}(\Omega)}$ and $\|u\|_{E_{p,q}^s(\Omega)}$.

Remark 1 There are many papers dealing with problems as in our illustrating example. We refer e.g. to [28], [34], [49], where boundedness and continuity of composition operators are treated in connection with the Schrödinger equation.

The main problem

From a little bit more abstract point of view the above example indicates that we have to study the following problem.

Problem 1:

Suppose $E_{p_0, q_0}^{s_0, loc}(\mathbb{R}^n) \subset E_{p_1, q_1}^{s_1, loc}(\mathbb{R}^n)$. Find necessary and sufficient conditions on $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$T_f(E_{p_0, q_0}^{s_0}(\mathbb{R}^n)) \subset E_{p_1, q_1}^{s_1}(\mathbb{R}^n).$$

Remark 2 (i) For a given space of functions E on $\mathbb{R}^n$, the associated space E^{loc} is the collection of all functions g s.t. $u \cdot g \in E$, for all $u \in \mathcal{D}(\mathbb{R}^n)$.

(ii) Our assumption $E_{p_0, q_0}^{s_0, loc}(\mathbb{R}^n) \subset E_{p_1, q_1}^{s_1, loc}(\mathbb{R}^n)$ indicates that there is no hope for an increase of the local regularity of the whole set $T_f(E_{p_0, q_0}^{s_0}(\mathbb{R}^n))$.

(iii) The theory of the operators T_f , as we know it at this moment, does not depend very much on the underlying domain Ω . So we discuss T_f on function spaces defined on $\mathbb{R}^n$.

(iv) Of course, it would make sense to replace T_f by more general mappings like

$$N(g_1, \dots, g_d)(x) := f(x, g_1(x), \dots, g_d(x)), \quad x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^{n+d} \rightarrow \mathbb{R}$. In this generality these mappings N are called Nemytskij operators. Much less is known for these general mappings. In our survey only very few remarks will be made concerning this general situation.

There are hundred's of references dealing with *Problem 1* and its generalizations. However, only in very few cases, e.g. if $f(t) := t^m$, $m \in \mathbb{N}$, one knows the final answer. We will add a few remarks later on. For the moment we will turn to a simplified problem.

Problem 2:

Find necessary and sufficient conditions on $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$T_f(E_{p, q}^s(\mathbb{R}^n)) \subset E_{p, q}^s(\mathbb{R}^n). \tag{2}$$

We will call the property (2) the acting property. In applications one needs more than the acting condition. In general one also needs boundedness and continuity. This justifies to consider the following modified problems.

Problem 2':

Find necessary and sufficient conditions on $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$T_f : E_{p,q}^s(\mathbb{R}^n) \rightarrow E_{p,q}^s(\mathbb{R}^n)$$

is bounded.

Remark 3 (i) Generally speaking, a mapping T of a metric space E to itself is said to be bounded if $T(A)$ is bounded for all bounded set $A \subset E$.

(ii) Coming back to our *illustrating example* it makes sense to ask for optimal inequalities describing the boundedness of T_f . We will take care of this problem as well in our survey.

(iii) We do not know whether *Problems 2* and *2'* have the same solution. Indeed, we do not have a counterexample in the framework of composition operators showing that the acting condition can occur without boundedness. Furthermore, let us mention that it is classically known that the acting condition implies at least a weak form of boundedness, see e.g. [39, pp. 275-276], [55, Lem. 5.2.4, 5.3.1/1] and [14, Section 4.2].

(iv) If we slightly generalize the class of operators by considering $N(g)(x) := f(x, g(x))$, $x \in \mathbb{R}^n$, i.e. Nemytskij operators, then it is well-known that acting conditions and boundedness conditions may be different. For simplicity we only consider $n = 1$ and $\mathbb{R}$ replaced by $[0, 1]$. Then the classical example is given by

$$f_s(x, u) := \begin{cases} 0 & \text{if } u \leq x^{s/2}, \\ \frac{1}{u^{2/s}} - \frac{x}{u^{4/s}} & \text{if } u > x^{s/2}, \end{cases}$$

where $0 < s < 1$. The associated Nemytskij operator maps $B_{\infty,\infty}^s[0, 1]$ into itself, but is not bounded. We refer to [5].

Problem 2'':

Find necessary and sufficient conditions on $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$T_f : E_{p,q}^s(\mathbb{R}^n) \rightarrow E_{p,q}^s(\mathbb{R}^n)$$

is continuous.

Remark 4 Problem 2 (Problem 2') and Problem 2'' have different answers in general. Below we will show that for some Lizorkin-Triebel spaces there exist noncontinuous bounded composition operators, see Corollary 1.

In our opinion also the following problem is of interest.

Problem 3:

Characterize all function spaces E , where the following assertions are equivalent

- $T_f(E) \subset E$,
- $T_f : E \rightarrow E$ is bounded,
- $T_f : E \rightarrow E$ is continuous,

whatever be $f : \mathbb{R} \rightarrow \mathbb{R}$.

At the end of our paper, see Section 7, we will give a list of spaces for which the acting property is equivalent to the boundedness and to the continuity. We will also produce a negative list, it means, we will collect also those spaces, for which such equivalences do not hold.

Both authors have written surveys with respect to this topic in earlier times, see [11], [55, Chapt. 5] and [15]. To increase the readability of this survey we allow some overlap with those articles. However, we mainly concentrate on the progress since 1995. In addition, a number of open problems is formulated within this text.

This survey is organized as follows. After the Introduction we collect a number of necessary conditions for the acting property to hold. This will make the splitting in the discussion of sufficient conditions more transparent. We start with this discussion in Section 3 by concentrating on Sobolev spaces. In Section 4 we will discuss the case of fractional order of smoothness. The results of this section motivate to study composition operators on intersections which we will do in Section 5. Some generalizations to the vector-valued situation are considered in Section 6 (but here we concentrate on Sobolev spaces). As mentioned above, there will be a section devoted to *Problem 3*, namely Section 7. In the very short final section we make some concluding remarks.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. If E and F are two Banach spaces, then the symbol $E \hookrightarrow F$ indicates that the embedding is continuous. The symbol c denotes a positive constant which depends only on the fixed parameters n, s, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols “ $\lesssim$ ” and “ $\gtrsim$ ” instead

of “ $\leq$ ” and “ $\geq$ ”, respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leq cB$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.

We denote by $C_b^m(\mathbb{R}^n)$ the Banach space of functions on $\mathbb{R}^n$ which are continuous and bounded, together with their derivatives up to order m , and by $C_{ub}(\mathbb{R}^n)$ the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^n$. The classical Sobolev spaces are denoted by $W_p^m(\mathbb{R}^n)$, and their homogeneous counterparts by $\dot{W}_p^m(\mathbb{R}^n)$, for $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. Inhomogeneous Besov and Lizorkin-Triebel spaces are denoted by $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$, respectively. We use the notation $\dot{F}_{p,q}^s(\mathbb{R}^n)$ and $\dot{B}_{p,q}^s(\mathbb{R}^n)$ for the homogeneous Lizorkin-Triebel and Besov spaces. For their definition, we refer to Section 9. General information about these function spaces, as well as the Wiener classes BV_p , can be found e.g. in [55, 63, 64] ($F_{p,q}^s(\mathbb{R}^n)$), [33, 37, 63] ($\dot{F}_{p,q}^s(\mathbb{R}^n)$), and [20, 21, 67] (BV_p). Let us mention that we exclude the spaces $F_{\infty,q}^s(\mathbb{R}^n)$, even in case when we write $E_{p,q}^s(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$. If an equivalence class $[f]$, for the a.e. equality, contains a continuous representative, then we call the class continuous and speak of values of f at any point (by taking the values of the continuous representative).

If f is a function defined on $\mathbb{R}^n$, and if $h \in \mathbb{R}^n$, we put $\Delta_h f(x) := f(x+h) - f(x)$. Throughout the paper $\psi \in \mathcal{D}(\mathbb{R}^n)$ denotes a specific cut-off function, i.e., $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 3/2$.

2 Necessary conditions

We begin with a collection of necessary conditions for the acting property. They are all found essentially before 1990. However, they give a good idea concerning the expectable solution of *Problem 2*.

Proposition 1 *Let $s > 0$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. The acting property $T_f(\mathcal{D}(\mathbb{R}^n)) \subset E_{p,q}^s(\mathbb{R}^n)$ implies $f \in E_{p,q}^{s,loc}(\mathbb{R})$.*

The proof follows easily by testing T_f on functions $u \in \mathcal{D}(\mathbb{R}^n)$ such that $u(x) = x_1$ on some ball of $\mathbb{R}^n$, see e.g. [4, Thm. 3.5] or [55, Thm. 5.3.1/2].

Proposition 2 *Let $s > 0$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function.*

- (i) *The acting property $T_f(E_{p,q}^s \cap L_\infty(\mathbb{R}^n)) \subset B_{p,\infty}^s(\mathbb{R}^n)$ implies $f \in W_\infty^{1,loc}(\mathbb{R})$.*
- (ii) *In case $E_{p,q}^s(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, the acting property $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset B_{p,\infty}^s(\mathbb{R}^n)$ implies $f \in \dot{W}_\infty^1(\mathbb{R})$.*

Remark 5 (i) The necessity of (local) Lipschitz continuity for the acting condition in the framework of function spaces with fractional order of smoothness has been observed for the first time by Igari [36] ($p = q = 2$, $0 < s < 1$, $s \neq 1/2$). Then the result has been extended to Besov and Lizorkin-Triebel spaces by the first named author and some co-authors. We refer to [12] for more detailed references and for the proof of Prop. 2. Let us also mention the recent publication [4, Thm. 3.1 and 3.2], where a proof in a more general context is given.

(ii) Observe that $E_{p,q}^s(\mathbb{R}^n) \hookrightarrow B_{p,\infty}^s(\mathbb{R}^n)$. Hence, the Nikol'skij-Besov space $B_{p,\infty}^s(\mathbb{R}^n)$ is the largest space within the family $E_{p,q}^s(\mathbb{R}^n)$.

Proposition 3 *Suppose $1 + 1/p < s < n/p$. Then the acting property $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset B_{p,\infty}^s(\mathbb{R}^n)$ implies $f(t) = ct$ for some constant c .*

The phenomenon described in Proposition 3 is well known since Dahlberg [30] had published his short note in 1979. He had proved that the implication $T_f(W_p^m(\mathbb{R}^n)) \subset W_p^m(\mathbb{R}^n)$ requires $f(t) = ct$ for some constant c . Extensions to Besov and Triebel-Lizorkin spaces have been given by Bourdaud [8, 11]. Extensions to values $p, q < 1$ can be found in Runst [54] and in [59]. Runst has also been the first who had investigated implications of $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset B_{p,\infty}^s(\mathbb{R}^n)$. A proof of Prop. 3 can be also found in [55, 5.3.1] and [4, Thm. 3.3].

As many times in the theory of Besov-Lizorkin-Triebel spaces in limiting situations the third index q has some influence.

Proposition 4 *Suppose $1 + 1/p = s < n/p$.*

- (i) *Let $q > 1$. Then the acting property $T_f(B_{p,q}^s(\mathbb{R}^n)) \subset B_{p,q}^s(\mathbb{R}^n)$ implies $f(t) = ct$ for some constant c .*
- (ii) *Let $p > 1$. Then the acting property $T_f(F_{p,q}^s(\mathbb{R}^n)) \subset F_{p,q}^s(\mathbb{R}^n)$ implies $f(t) = ct$ for some constant c .*

This has been proved by the first named author in [10] and [11], see also [55, Lem. 5.3.1/2] and [4, Thm. 3.3]. The existence of nontrivial composition operators on $B_{p,1}^{1+(1/p)}(\mathbb{R}^n)$, $n > p + 1$, and on $F_{1,q}^2(\mathbb{R}^n)$, $n > 2$, is an open problem.

Remark 6 The degeneracy is connected to the existence of unbounded functions in $E_{p,q}^s(\mathbb{R}^n)$. Instead, if we consider $E_{p,q}^s \cap L_\infty(\mathbb{R}^n)$, the Dahlberg phenomenon disappears. Indeed $E_{p,q}^s \cap L_\infty(\mathbb{R}^n)$ is known to be a Banach algebra for the pointwise product. Hence, all entire functions, vanishing at 0, act on it. For this problem, we refer also to Section 5.

Remark 7 Notice that the embedding $W_\infty^{1,\ell oc}(\mathbb{R}) \hookrightarrow E_{p,q}^{s,\ell oc}(\mathbb{R})$ holds if $0 < s < 1$, while the reverse embedding holds for $s > 1 + (1/p)$. In view of the four preceding propositions, the values $s = 1$, $s = 1 + (1/p)$ and $s = n/p$ appear as the critical ones for *Problem 2*.

3 Sobolev spaces

For an easier reading of the paper we first discuss *Problems 2, 2', 2''* for Sobolev spaces. According to Remark 3, we present boundedness in the same section than the acting condition.

3.1 The acting condition on Sobolev spaces

In view of Section 2, in particular Remarks 6 and 7, it makes sense to separate the discussion into the cases

- $m = 1$;
- $m \geq 2$ and $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$;
- $m \geq 2$ and $W_p^m(\mathbb{R}^n) \not\hookrightarrow L_\infty(\mathbb{R}^n)$.

Theorem 1 *Suppose $1 \leq p \leq \infty$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function s.t. $f(0) = 0$. Then it holds*

$$T_f(W_p^1(\mathbb{R}^n)) \subset W_p^1(\mathbb{R}^n) \iff \begin{cases} f' \in L_\infty^{\text{loc}}(\mathbb{R}) & p > n \\ \text{or } p = 1 = n & \\ f' \in L_\infty(\mathbb{R}) & \text{otherwise.} \end{cases}$$

In either case we have

$$\|f \circ g\|_{W_p^1(\mathbb{R}^n)} \leq \|f'\|_{L_\infty(\mathbb{R})} \|g\|_{W_p^1(\mathbb{R}^n)}. \quad (3)$$

Remark 8 (i) For a proof we refer to Marcus and Mizel [42], see also [6, Chapt. 9].

(ii) If f and g are C^1 functions, and if $f(0) = 0$, then

$$\begin{aligned} \int_{\mathbb{R}^n} |f(g(x))|^p dx &= \int_{\mathbb{R}^n} |f(g(x)) - f(0)|^p dx \leq \|f'\|_{L_\infty(\mathbb{R})}^p \int_{\mathbb{R}^n} |g(x)|^p dx, \quad (4) \\ \int_{\mathbb{R}^n} \left| f'(g(x)) \frac{\partial g}{\partial x_i}(x) \right|^p dx &\leq \|f'\|_{L_\infty(\mathbb{R})}^p \int_{\mathbb{R}^n} \left| \frac{\partial g}{\partial x_i}(x) \right|^p dx, \quad i = 1, \dots, n. \end{aligned}$$

Hence, (3) follows under these extra conditions. By (4) also the role of the condition $f(0) = 0$ becomes clear. The extension of these estimates to the general case is more complicated.

Theorem 2 *Let $m = 2, 3, \dots$, and let $1 \leq p < \infty$. We suppose that $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function s.t. $f(0) = 0$.*

- (i) Then the composition operator T_f maps $W_p^m(\mathbb{R}^n)$ into itself if, and only if, $f \in W_p^{m,loc}(\mathbb{R})$.
(ii) For $f \in W_p^{m,loc}(\mathbb{R})$ we have

$$\|f \circ g\|_{W_p^m(\mathbb{R}^n)} \leq C(f, g) \left(\|g\|_{W_p^m(\mathbb{R}^n)} + \|g\|_{W_p^m(\mathbb{R}^n)}^m \right), \quad (5)$$

where

$$\begin{aligned} C(f, g) &:= c \left(\|f\|_{W_\infty^{m-1}(I_g)} + \|f^{(m)}\|_{L_p(I_g)} \right) \\ I_g &:= \{t \in \mathbb{R} : |t| \leq \|g\|_{L_\infty(\mathbb{R}^n)}\}, \end{aligned}$$

and c is independent of f and g .

Remark 9 (i) For a proof we refer to [9], see also [55, 5.2.4].

(ii) The estimate (5) will be typical for the supercritical case, i.e., the case $E_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Recall the chain rule

$$D^\gamma (f \circ g) = \sum_{\ell=1}^{|\gamma|} \sum_{\substack{\alpha^1 + \dots + \alpha^\ell = \gamma \\ |\alpha^i| \neq 0}} c_{\gamma, \ell, \alpha^1, \dots, \alpha^\ell} (f^{(\ell)} \circ g) D^{\alpha^1} g \dots D^{\alpha^\ell} g, \quad (6)$$

where $\gamma := (\gamma_1, \dots, \gamma_n)$, $\alpha^i := (\alpha_1^i, \dots, \alpha_n^i)$, $i = 1, \dots, \ell$, are multi-indices and $c_{\gamma, \ell, \alpha^1, \dots, \alpha^\ell}$ are certain combinatorial constants. Let $\gamma = (m, 0, \dots, 0)$. Then the sum on the right-hand side contains the terms

$$(f^{(m)} \circ g) \left(\frac{\partial g}{\partial x_1} \right)^m \quad \text{and} \quad (f' \circ g) \frac{\partial^m g}{\partial x_1^m}.$$

From this point of view an estimate as given in (5) looks natural.

We need a further class of functions. If E is a normed space of functions on $\mathbb{R}^n$, then the space E_{unif} is the collection of all $g \in E^{loc}$ s.t.

$$\|g\|_{E_{unif}} := \sup_{a \in \mathbb{R}^n} \|g \psi(\cdot - a)\|_E < \infty,$$

where ψ is the cut-off function from our list of conventions.

Theorem 3 Let $m = 2, 3, \dots$, and let $1 \leq p < \infty$. We suppose that $W_p^m(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function s.t. $f(0) = 0$.

- (i) If $m = n/p \geq 2$, then $T_f(W_p^m(\mathbb{R}^n)) \subset W_p^m(\mathbb{R}^n)$ holds if, and only if, $f' \in W_{p,unif}^{m-1}(\mathbb{R})$.
(ii) Let $m = n/p \geq 2$ and $f' \in W_{p,unif}^{m-1}(\mathbb{R})$. Then

$$\|f \circ g\|_{W_p^m(\mathbb{R}^n)} \leq C_f \left(\|g\|_{W_p^m(\mathbb{R}^n)} + \|g\|_{W_p^m(\mathbb{R}^n)}^m \right) \quad (7)$$

where

$$C_f := c \left(\|f\|_{\dot{W}_\infty^1(\mathbb{R})} + \sup_{a \in \mathbb{R}} \left(\int_a^{a+1} |f^{(m)}(t)|^p dt \right)^{1/p} \right) \quad (8)$$

and c is independent of f and g .

(iii) Let either $1 < p < \infty$ and $2 \leq m < n/p$, or $p = 1$ and $3 \leq m < n$. Then $T_f(W_p^m(\mathbb{R}^n)) \subset W_p^m(\mathbb{R}^n)$ holds if, and only if, $G(t) = ct$, $t \in \mathbb{R}$, for some $c \in \mathbb{R}$.

(iv) If $n \geq 3$, then $T_f(W_1^2(\mathbb{R}^n)) \subset W_1^2(\mathbb{R}^n)$ holds if, and only if, $f'' \in L_1(\mathbb{R})$.

(v) Let $f'' \in L_1(\mathbb{R})$. Then

$$\|f \circ g\|_{W_1^2(\mathbb{R}^n)} \leq c (\|f'\|_{L_\infty(\mathbb{R})} + \|f''\|_{L_1(\mathbb{R})}) \|g\|_{W_1^2(\mathbb{R}^n)} \quad (9)$$

and c is independent of f and g .

Remark 10 (i) For a proof we refer to [9], see also [55, 5.2.4] and [14]. The crucial idea in the proofs of Thm. 2 and Thm. 3 (except part (iii)) consists in an integration by parts involving the norm of the Sobolev space. Part (iii) in Thm. 3 is a particular case of Proposition 3, see some further comments in Subsection 4.3.

(ii) The class $W_{p,unif}^m(\mathbb{R})$ plays a crucial role in the composition problem, not only in the critical case $m = n/p$. This becomes clear if we consider instead of the usual Sobolev space $W_p^m(\mathbb{R}^n)$ the so-called Adams-Frazier space $W_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, see Theorem 25.

We turn back to the problem touched in Remark 9(ii). Here is an improvement of the estimate (5) in the supercritical case.

Theorem 4 Let $m = 2, 3, \dots$, and let $1 \leq p < \infty$. We suppose that $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be in $W_p^m(\mathbb{R})$ s.t. $f(0) = 0$. Then there exists a constant c s.t.

$$\|f \circ g\|_{W_p^m(\mathbb{R}^n)} \leq c \|f\|_{W_p^m(\mathbb{R})} \left(\|g\|_{W_p^m(\mathbb{R}^n)} + \|g\|_{W_p^m(\mathbb{R}^n)}^{m-1/p} \right). \quad (10)$$

Here c is independent of f and $g \in W_p^m(\mathbb{R}^n)$.

Remark 11 A proof of (10) has been given in [14]. The exponent $m - 1/p$ is the optimal one, see Proposition 6 in Section 4. However, the result does not extend to all f in $W_p^{m,loc}(\mathbb{R})$. A counterexample is given by $f(t) = \sin t$, $t \in \mathbb{R}$, see [14].

3.2 Continuity of composition operators on Sobolev spaces

Theorem 5 (i) Let $1 \leq p < \infty$. Every composition operator T_f , which maps $W_p^1(\mathbb{R}^n)$ into itself, is continuous.

(ii) Let $1 < p < \infty$, $m \in \mathbb{N}$ and $m > n/p$. Then every composition operator T_f , which maps $W_p^m(\mathbb{R}^n)$ into itself, is continuous.

Proof We concentrate on (ii). Under the given restrictions we have $W_p^m(\mathbb{R}^n) = F_{p,2}^m(\mathbb{R}^n)$ in the sense of equivalent norms. From Thm. 2 we derive the equivalence of the acting condition $T_f(W_p^m(\mathbb{R}^n)) \subset W_p^m(\mathbb{R}^n)$ and $f \in W_p^{m,loc}(\mathbb{R})$, $f(0) = 0$. Furthermore, the boundedness of T_f follows from the estimate (5). In this situation we can apply Proposition 7, see paragraph 4.2.3, and obtain that T_f must be continuous as well.

Remark 12 Part (i) is a famous result of Marcus and Mizel [43]. Part (ii) seems to be a novelty.

4 Spaces of fractional order of smoothness

According to Section 2, it makes sense to separate the discussion into the cases

- $0 < s < 1$;
- $1 < s < 1 + 1/p$;
- $1 + 1/p < s < n/p$;
- $\max(1 + 1/p, n/p) < s$.

4.1 The case of low smoothness

In case $0 < s < 1$, the only known necessary condition, see Remark 7, turns out to be also sufficient.

Theorem 6 *Let $1 \leq p, q \leq \infty$ and $0 < s < 1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function s.t. $f(0) = 0$.*

(i) The following assertions are equivalent:

- (a) $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset E_{p,q}^s(\mathbb{R}^n)$;*
- (b) $T_f : E_{p,q}^s(\mathbb{R}^n) \rightarrow E_{p,q}^s(\mathbb{R}^n)$ is bounded;*
- (c) Either $f' \in L_\infty(\mathbb{R})$ if $E_{p,q}^s(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$ or $f' \in L_\infty^{loc}(\mathbb{R})$ if $E_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$.*

(ii) Let f' be as in (c). Then

$$\|f \circ g\|_{E_{p,q}^s(\mathbb{R}^n)} \leq \|f'\|_{L_\infty(I_g)} \|g\|_{E_{p,q}^s(\mathbb{R}^n)}$$

holds for all $g \in E_{p,q}^s(\mathbb{R}^n)$ and I_g is defined as in Thm. 2.

Proof The space $E_{p,q}^s(\mathbb{R}^n)$ is defined by first order differences, see Definition 2 in Section 9. For those differences we have the obvious inequality $|\Delta_h(f \circ g)| \leq \|f'\|_\infty |\Delta_h g|$. For the estimate of the L_p -term we refer to (4). Both inequalities together prove (ii) and at the same time the implications (c) $\Rightarrow$ (b), (a). The nontrivial implication (a) $\Rightarrow$ (c) follows by Proposition 2. ■

By using real interpolation for Lipschitz-continuous operators, see [52], one can add continuity to the list in part (i) of the theorem.

Theorem 7 *Let $1 \leq p, q < \infty$ and $0 < s < 1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function s.t. $f(0) = 0$ and either $f' \in L_\infty(\mathbb{R})$ if $B_{p,q}^s(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$ or $f' \in L_\infty^{\text{loc}}(\mathbb{R})$ if $B_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Then $T_f : B_{p,q}^s(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n)$ is continuous.*

The proof in [55, Thm. 5.5.2/3] uses the real interpolation formula

$$\left(L_p(\mathbb{R}^n), W_p^1(\mathbb{R}^n) \right)_{s,q} = B_{p,q}^s(\mathbb{R}^n), \quad 0 < s < 1, \quad 1 \leq q \leq \infty,$$

in connection with a result of Maligranda [41] concerning the continuity of a nonlinear operator T with respect to real interpolation, see [55, Sect. 2.5].

4.2 The case of high smoothness

In view of the results presented in Section 2, there is a natural conjecture.

Conjecture 1 *Let $s > 1 + (1/p)$. The composition operator T_f , associated to a Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, maps $E_{p,q}^s \cap L_\infty(\mathbb{R}^n)$ to $E_{p,q}^s(\mathbb{R}^n)$ if, and only if, $f \in E_{p,q}^{s,\text{loc}}(\mathbb{R})$ and $f(0) = 0$.*

The validity of the above Conjecture is known in case of Sobolev spaces (recall $W_p^m(\mathbb{R}^n) = F_{p,2}^m(\mathbb{R}^n)$, $1 < p < \infty$, $m \in \mathbb{N}$), see Thm. 1 and [14, Thm. 2]. For $n = 1$, it is also valid for any Lizorkin-Triebel space and for some Besov spaces, see Subsection 4.2.1. The extension to dimensions $n > 1$ is an open question.

4.2.1 The acting condition in the one-dimensional situation

To begin with we deal with a simplified situation where we give a sketch of the proof. However, the used arguments are also typical for the more general results which will be mentioned below.

We need some more classes of functions. For a function $g : \mathbb{R} \rightarrow \mathbb{R}$, we denote by $\|g\|_{BV_p}$ the supremum of numbers

$$\left(\sum_{k=1}^N |g(b_k) - g(a_k)|^p \right)^{1/p},$$

taken over all finite sets $\{[a_k, b_k]; k = 1, \dots, N\}$ of pairwise disjoint open intervals. A function g is said to be of *bounded p -variation* if $\|g\|_{BV_p} < +\infty$. The collection of all such functions is called the Wiener class and denoted by BV_p . We refer to [20, 21, 67] for a discussion of these classes. Their importance in the composition problem is related to the Peetre's embedding

$$B_{p,1}^{1/p}(\mathbb{R}) \hookrightarrow BV_p(\mathbb{R}), \quad (11)$$

see e.g. [53] and [20, Thm. 5].

Proposition 5 *Let $1 \leq p < \infty$, $p \leq q \leq \infty$ and $1 + 1/p < s < 2$. Let $f' \in B_{p,q}^{s-1}(\mathbb{R})$ s.t. $f(0) = 0$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{B_{p,q}^s(\mathbb{R})} \leq c \|f'\|_{B_{p,q}^{s-1}(\mathbb{R})} \left(\|g\|_{B_{p,q}^s(\mathbb{R})} + \|g'\|_{BV_{sp-1}}^{s-(1/p)} \right) \quad (12)$$

holds for all real analytic functions g in $B_{p,q}^s(\mathbb{R})$.

Remark 13 By (11) and by assumption $s > 1 + (1/p)$, a weaker version of (12) is given by

$$\|f \circ g\|_{B_{p,q}^s(\mathbb{R})} \leq c \|f'\|_{B_{p,q}^{s-1}(\mathbb{R})} \left(\|g\|_{B_{p,q}^s(\mathbb{R})} + \|g'\|_{B_{p,q}^{s-1}(\mathbb{R})}^{s-(1/p)} \right). \quad (13)$$

Proof Because of

$$B_{p,q}^{s-1}(\mathbb{R}) \hookrightarrow B_{\infty,\infty}^{s-1-1/p}(\mathbb{R}) \hookrightarrow C_{ub}^{s'}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R}), \quad 0 < s' < s - 1 - \frac{1}{p},$$

the function f' is a continuous. Hence we may apply the chain rule and obtain

$$(f \circ g)' = (f' \circ g) g'$$

in the pointwise sense. Furthermore, we will use that

$$\|u\|_{B_{p,q}^s(\mathbb{R})} \asymp \|u\|_{L_p(\mathbb{R})} + \|u'\|_{B_{p,q}^{s-1}(\mathbb{R})} \quad (14)$$

for all distribution u if $s > 0$, see [63, 2.3.8].

Step 1. Since the estimate of the L_p -term is as in (4), we only have to deal with the homogeneous part of the Besov norm. For brevity we put

$$\omega_p(f, h) := \left(\int_{\mathbb{R}} |\Delta_h f(x)|^p dx \right)^{1/p}, \quad h \in \mathbb{R}.$$

Using (14) we have to estimate

$$\left(\int_{-1}^1 \left(\frac{\omega_p((f \circ g)', h)}{|h|^{s-1}} \right)^q \frac{dh}{|h|} \right)^{1/q}.$$

Since

$$\omega_p((f' \circ g)g', h) \leq \|f'\|_\infty \omega_p(g', h) + U(h),$$

where

$$U(h) := \left(\int_{\mathbb{R}} |\Delta_h(f' \circ g)(x)|^p |g'(x)|^p dx \right)^{1/p},$$

we are reduced to prove that

$$\left(\int_{-1}^1 \left(\frac{U(h)}{|h|^{s-1}} \right)^q \frac{dh}{|h|} \right)^{1/q}$$

can be estimated by the right-hand side of (12).

Step 2. Without loss of generality we may assume $h > 0$. The set of zeros of g' is discrete, and its complement in $\mathbb{R}$ is the union of a family $(I_l)_l$ of nonempty open disjoint intervals. For any $h > 0$ we denote by I'_l the (possibly empty) set of $x \in I_l$ whose distance to the right endpoint of I_l is greater than h , and we set

$$I''_l := I_l \setminus I'_l, \quad a_l := \sup_{I_l} |g'|.$$

By g_l we mean the restriction of g to I_l , hence a strictly monotone smooth function. If $I'_l \neq \emptyset$, then we have

$$|g(g_l^{-1}(y) + h) - y| \leq a_l h \quad \text{for } y \in g_l(I'_l), \quad (15)$$

where g_l^{-1} denotes the inverse function of g_l .

Substep 2.1. Let

$$\Omega_p(f, t) := \left(\int_{\mathbb{R}} \sup_{|h| \leq t} |\Delta_h f(x)|^p dx \right)^{1/p}, \quad t > 0. \quad (16)$$

By (15) and by a change of variable we find

$$\int_{I'_l} |\Delta_h(f' \circ g)(x)|^p |g'(x)|^p dx \leq a_l^{p-1} \Omega_p^p(f', a_l h). \quad (17)$$

By the Minkowski inequality w.r.t. $L_{q/p}$, and by Proposition 8 in Section 9, we obtain

$$\begin{aligned}
& \left(\int_0^\infty \left(\frac{1}{h^{(s-1)p}} \sum_l a_l^{p-1} \Omega_p^p(f', a_l h) \right)^{q/p} \frac{dh}{h} \right)^{1/q} \\
& \leq \left(\sum_l \left(\int_0^\infty \left(\frac{1}{h^{p(s-1)}} a_l^{p-1} \Omega_p^p(f', a_l h) \right)^{q/p} \frac{dh}{h} \right)^{p/q} \right)^{1/p} \\
& = \left(\sum_l a_l^{p-1+p(s-1)} \right)^{1/p} \left(\int_0^\infty \left(\frac{\Omega_p(f', t)}{t^{s-1}} \right)^q \frac{dt}{t} \right)^{1/q} \\
& \leq c \|f'\|_{B_{p,q}^{s-1}(\mathbb{R})} \left(\sum_l \left(\sup_{I_l} |g'| \right)^{sp-1} \right)^{1/p}.
\end{aligned}$$

Now we follow [20, proof of Thm. 7]. By definition, the function g' vanishes at the endpoints of I_l . Let β_l be one of these endpoints. Furthermore, there is at least one point $\xi_l \in I_l$ such that

$$|g'(\xi_l)| = \sup_{I_l} |g'|.$$

Hence

$$\sum_l \sup_{I_l} |g'|^{sp-1} = \sum_l |g'(\xi_l) - g'(\beta_l)|^{sp-1} \leq \|g'\|_{BV_{sp-1}}^{sp-1}.$$

By (17), we conclude that

$$\begin{aligned}
& \left(\int_0^\infty \left(\frac{1}{h^{(s-1)p}} \sum_l \int_{I_l'} |\Delta_h(f' \circ g)(x)|^p |g'(x)|^p dx \right)^{q/p} \frac{dh}{h} \right)^{1/q} \\
& \leq c \|f'\|_{B_{p,q}^{s-1}(\mathbb{R})} \|g'\|_{BV_{sp-1}}^{s-(1/p)}.
\end{aligned} \tag{18}$$

Substep 2.2. Since g' vanishes at the right endpoint of I_l'' , it holds

$$|g'(x)| \leq \sup_{|v| \leq h} |g'(x) - g'(x+v)|$$

for all $x \in I_l''$. Thus we obtain

$$\sum_l \int_{I_l''} |\Delta_h(f' \circ g)(x)|^p |g'(x)|^p dx \leq (2\|f'\|_\infty)^p \Omega_p(g', h)^p. \tag{19}$$

By Proposition 8, we conclude that

$$\begin{aligned}
& \left(\int_0^\infty \left(\frac{1}{h^{(s-1)p}} \sum_l \int_{I_l''} |\Delta_h(f' \circ g)(x)|^p |g'(x)|^p dx \right)^{q/p} \frac{dh}{h} \right)^{1/q} \\
& \leq c \|f'\|_\infty \|g\|_{B_{p,q}^s(\mathbb{R})}.
\end{aligned} \tag{20}$$

Putting (4), (17), (18) and (20) together we obtain (12). The proof is complete. $\blacksquare$

To derive the acting property with respect to $B_{p,q}^s(\mathbb{R})$ one uses a specific density argument (to get rid of the restriction to real analytic g), the Fatou property, see, e.g., [22] and [32]. To obtain the acting property for $B_{p,q}^s(\mathbb{R})$ for higher values of s , $m + 1/p < s < m + 1$, $m \geq 2$, one can use an induction argument in combination with

$$\|u\|_{B_{p,q}^s(\mathbb{R})} \asymp \|u\|_{L_p(\mathbb{R})} + \|u^{(m)}\|_{B_{p,q}^{s-m}(\mathbb{R})} \quad (21)$$

if $s > 0$, see [63, 2.3.8]. However, to get an optimal inequality for those values of s and to deal with $m \leq s \leq m + 1/p$, $m \geq 2$, more effort is needed. To describe this we have to introduce a further space of functions. Since from now on we do not give proofs and the arguments in the Lizorkin-Triebel case are similar (however, a bit more sophisticated) we turn to the general notation. The outcome is the following, see [23], [24], [25] and [48] for all details.

Theorem 8 *Let $1 < p < \infty$, $1 \leq q \leq \infty$ ($p \leq q$ if $E = B$), and $s > 1 + (1/p)$. The composition operator T_f , associated to a Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, acts on $E_{p,q}^s(\mathbb{R})$ if, and only if, $f(0) = 0$ and $f \in E_{p,q}^{s,loc}(\mathbb{R})$.*

Remark 14 We believe that the restriction $p \leq q$ in the B -case is connected with the method of proof but not with the problem itself.

Acting property and boundedness are equivalent in this situation. To describe this, we use the smooth cut-off function ψ and define $\psi_t(x) := \psi(x/t)$, $t > 0$. We denote by $\dot{\mathcal{E}}_{p,q}^s(\mathbb{R})$ the space of all functions in $L_\infty(\mathbb{R})$ which belong to the *homogeneous* Besov-Lizorkin-Triebel space $\dot{E}_{p,q}^s(\mathbb{R})$, and endow it with the natural norm

$$\|f\|_{\dot{\mathcal{E}}_{p,q}^s(\mathbb{R})} := \|f\|_{\dot{E}_{p,q}^s(\mathbb{R})} + \|f\|_{L_\infty(\mathbb{R})}.$$

Theorem 9 *Let $1 < p < \infty$, $1 \leq q \leq \infty$ ($p \leq q$ if $E = B$), and $s > 1 + (1/p)$.*

(i) *Let $f \in E_{p,q}^{s,loc}(\mathbb{R})$ and $f(0) = 0$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{E_{p,q}^s(\mathbb{R})} \leq c \|(f \psi \|g\|_{L_\infty(\mathbb{R})})'\|_{\dot{\mathcal{E}}_{p,q}^{s-1}(\mathbb{R})} \left(\|g\|_{E_{p,q}^s(\mathbb{R})} + \|g\|_{E_{p,q}^s(\mathbb{R})}^{s-(1/p)} \right) \quad (22)$$

holds for all such functions f and all $g \in E_{p,q}^s(\mathbb{R})$.

(ii) *There exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{E_{p,q}^s(\mathbb{R})} \leq c \|f\|_{E_{p,q}^s(\mathbb{R})} \left(\|g\|_{E_{p,q}^s(\mathbb{R})} + \|g\|_{E_{p,q}^s(\mathbb{R})}^{s-(1/p)} \right) \quad (23)$$

holds for all functions $f \in E_{p,q}^s(\mathbb{R})$, $f(0) = 0$, and all functions $g \in E_{p,q}^s(\mathbb{R})$.

4.2.2 Optimal inequalities (I)

We would like to discuss the quality of the estimate (23). However, for doing that there is no need to concentrate on $n = 1$. The following simple arguments work for any n .

Proposition 6 *Let $1 \leq p, q \leq \infty$ and $s > 1 + 1/p$. Let $h : [0, \infty[\rightarrow [0, \infty[$ be a nondecreasing function satisfying $h(x) = o(x^{s-(1/p)})$ for $x \rightarrow +\infty$. Let N be any semi-norm on $\mathcal{D}(\mathbb{R}^n)$. If f is a continuous function such that, for some constant $c > 0$, the inequality*

$$\|f \circ g\|_{B_{p,\infty}^s(\mathbb{R}^n)} \leq c \left(N(g) + h(N(g)) \right) \quad (24)$$

holds for all $g \in \mathcal{D}(\mathbb{R}^n)$, then f is a polynomial of degree $\leq s$.

Proof We take the natural number $m \geq 1$ such that $m - 1 \leq s < m$. Let φ be a function in $\mathcal{D}(\mathbb{R}^n)$ s.t. $\varphi(x) = x_1$ for all $x \in [-1, 1]^n$. Then it holds $(f \circ a\varphi)(x) = f(ax_1)$ for all $x \in [-1, 1]^n$ and all $a > 0$. Hence

$$\Delta_{te_1}^m(f \circ a\varphi)(x) = \Delta_{at}^m f(ax_1), \quad \forall x \in [-1/2, 1/2]^n, \forall t \in [0, 1/2m], \forall a > 0.$$

Using this identity and (24), we deduce

$$\left(\int_{-a/2}^{a/2} |\Delta_t^m f(x)|^p dx \right)^{1/p} \leq c t^s (a^{(1/p)+1-s} N(\varphi) + h(a N(\varphi)) a^{(1/p)-s})$$

for all $t \in]0, 1]$ and $a > 2m$. By taking a to $+\infty$, and by applying the assumption on h , we deduce

$$\Delta_t^m f(x) = 0 \quad \text{a.e.}, \quad \forall t \in]0, 1].$$

Then by a standard argument, we deduce that f is a polynomial of degree at most $m - 1$.

Remark 15 Proposition 6 yields that the exponent $s - 1/p$ in (13), (22), and (23) is optimal. It also implies the optimality of the estimate (10).

4.2.3 Continuity of composition operators in the one-dimensional situation

There is a general continuity theorem in [18, Cor. 2] which can be applied for all n , if Conjecture 1 is valid.

Proposition 7 *Let $s > 1 + 1/p$. If Conjecture 1 is valid for $E_{p,q}^s(\mathbb{R}^n)$, then a composition operator T_f is continuous if, and only if, $f(0) = 0$ and f belongs to the closure of the smooth functions in $E_{p,q}^{s,\text{loc}}(\mathbb{R})$.*

In consequence of Thm. 8 and this proposition we immediately get the following.

Corollary 1 *Let $1 < p < \infty$ and $s > 1 + (1/p)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable.*

(i) *Let $1 \leq q < \infty$ ($p \leq q$ if $E = B$). Then the following assertions are equivalent :*

- T_f satisfies the acting condition $T_f(E_{p,q}^s(\mathbb{R})) \subset E_{p,q}^s(\mathbb{R})$.
- T_f is a bounded mapping of $E_{p,q}^s(\mathbb{R})$ to itself.
- $T_f : E_{p,q}^s(\mathbb{R}) \rightarrow E_{p,q}^s(\mathbb{R})$ is continuous.
- $f \in E_{p,q}^{s,loc}(\mathbb{R})$ and $f(0) = 0$.

(ii) *The following assertions are equivalent :*

- T_f satisfies the acting condition $T_f(E_{p,\infty}^s(\mathbb{R})) \subset E_{p,\infty}^s(\mathbb{R})$.
- T_f is a bounded mapping of $E_{p,\infty}^s(\mathbb{R})$ to itself.
- $f \in E_{p,\infty}^{s,loc}(\mathbb{R})$ and $f(0) = 0$.

(iii) *There exist functions f in $E_{p,\infty}^s(\mathbb{R})$, $f(0) = 0$, such that $T_f : E_{p,\infty}^s(\mathbb{R}) \rightarrow E_{p,\infty}^s(\mathbb{R})$ is not continuous.*

4.2.4 Acting conditions in the general n -dimensional situation

In dimension $n > 1$, Conjecture 1 has not been proved up to now. We give here two sufficient acting conditions. In the first one, we try to approach the minimal assumptions on f . In the second one, we obtain an optimal estimate, but with stronger regularity assumptions on f .

Theorem 10 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\max(n/p, 1) < s < \mu$. We suppose $f \in C^\mu(\mathbb{R})$ and $f(0) = 0$. Then T_f maps $E_{p,q}^s(\mathbb{R}^n)$ into $E_{p,q}^s(\mathbb{R}^n)$.*

Remark 16 (i) Of course, we have $C^\mu(\mathbb{R}) \subset E_{p,q}^{s,loc}(\mathbb{R})$. For a proof of Thm. 10 we refer to [55, Sect. 5.3.6, Thm. 2].

(ii) Let $E = B$. Under some extra conditions on s , it is proved in [13] that one can replace $C^\mu(\mathbb{R})$ by $B_{p,q}^{s+\varepsilon,loc}(\mathbb{R})$ with $\varepsilon > 0$ arbitrary.

Theorem 11 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $m \in \mathbb{N}$ and $\max(m, n/p) < s < m + 1$. We suppose $f \in C^{m+1}(\mathbb{R})$ and $f(0) = 0$. Then there exists a constant c s.t.*

$$\|f \circ g\|_{E_{p,q}^s(\mathbb{R}^n)} \leq c \|f\|_{C_b^{m+1}(\mathbb{R})} (\|g\|_{E_{p,q}^s(\mathbb{R}^n)} + \|g\|_{E_{p,q}^s(\mathbb{R}^n)}^s) \quad (25)$$

holds for all such f and all $g \in E_{p,q}^s(\mathbb{R}^n)$.

Proof The theorem has been proved essentially in [55, 5.3.7, Thm. 1 and 2]. Only one further remark is needed. Because of $s > \max(1, n/p)$ we have

$$s - \frac{n}{p} = s \left(1 - \frac{n}{ps}\right) > 1 - \frac{n}{ps}$$

and therefore, the continuous embeddings

$$B_{p,q}^s(\mathbb{R}^n) \hookrightarrow B_{ps,v}^1(\mathbb{R}^n) \quad \text{and} \quad F_{p,q}^s(\mathbb{R}^n) \hookrightarrow F_{ps,v}^1(\mathbb{R}^n) \quad (26)$$

for arbitrary $v \in]0, \infty]$. ■

Remark 17 (i) Probably the exponent s in (25) can be improved to $s - 1/p$ under additional assumptions on f , see Proposition 6 and Thm. 4.

(ii) The estimation of the norm of $\|f \circ g\|$ in Theorem 11 is better than in Theorem 10. In the detailed statement of Theorem 10, the norm $\|f \circ g\|$ is controlled by $\|g\|^\mu$ instead of $\|g\|^s$ for $\|g\| \geq 1$.

(iii) Both theorems have a long list of forerunners. Let us mention at least a few: Moser 1960 (Sobolev spaces) [47], Mizohata 1965 (Bessel potential spaces with $p = 2$) [46], Peetre 1970 (Besov spaces, nonlinear interpolation) [52], Adams 1976 (Bessel potential spaces) [1], Meyer 1981 (Bessel potential spaces) [45], Runst 1986 (Besov-Lizorkin-Triebel spaces) [54], Adams & Frazier 1992 (Bessel potential spaces) [2, 3].

4.3 The intermediate case (I)

We consider the case $1 + 1/p < s < n/p$. By Proposition 3, the composition operator necessarily lowers the regularity. The study of T_f for $f(t) := \sin t$, $t \in \mathbb{R}$, is particularly enlightening in this respect. Let us define

$$g_\alpha(x) = \psi(x) |x|^{-\alpha}, \quad x \in \mathbb{R}^n, \quad \alpha > 0. \quad (27)$$

Then it is well known (and not very complicated to prove) that in case $s > 0$

$$g_\alpha \in F_{p,q}^s(\mathbb{R}^n) \quad \Longleftrightarrow \quad 0 < \alpha < \frac{n}{p} - s \quad (28)$$

and

$$g_\alpha \in B_{p,q}^s(\mathbb{R}^n) \quad \Longleftrightarrow \quad \text{either } 0 < \alpha < \frac{n}{p} - s \quad \text{or} \quad \alpha = \frac{n}{p} - s \quad \text{and} \quad q = \infty \quad (29)$$

hold, cf. [55, Lem. 2.3.1/1]. Then

$$\frac{\partial^m}{\partial x_1^m} \sin(g_\alpha(x)) \asymp \sin^{(m)}(|x|^{-\alpha}) (|x|^{-\alpha-2} x_1^m + \text{lower order terms}),$$

in a certain neighborhood of the origin. Compare this with the m -th order derivative of the original function. It turns out that

$$\frac{\partial^m}{\partial x_1^m} g_\alpha(x) \asymp |x|^{-\alpha-2m} x_1^m + \text{lower order terms} \quad (|x| \rightarrow 0).$$

Since we have $(\alpha + 1)m > \alpha + m$ if $m \geq 2$ this shows that the local singularity of the composition becomes stronger than the singularity of the original function g_α .

Going back to the general situation, a natural question consists in asking for the best possible image space, hence we turn to *Problem 1* for the rest of this subsection. The following strange number will play a certain role

$$\varrho^* = \varrho^*(s, p, n) := \frac{\frac{n}{p} + \frac{1}{p} \left(\frac{n}{p} - s \right)}{\frac{n}{p} - s + 1}. \quad (30)$$

Theorem 12 *Suppose $1 + 1/p < s < n/p$. Let f be a non-polynomial Borel measurable function. Then for every $r > \varrho^*(s, p, n)$ there exists a compactly supported function $g_r \in E_{p,q}^s(\mathbb{R}^n)$ such that the composition $f \circ g_r$ does not belong to $B_{p,\infty}^r(\mathbb{R}^n)$.*

Remark 18 (i) A proof in case of smooth f can be found in [58, 59, 55]. However, by the same arguments one can deal with non-smooth functions.

(ii) If $1 + 1/p < s < n/p$, then $1 + 1/p < \varrho^* < s$. This indicates a certain loss of smoothness. Whether there exists a nonlinear function f such that $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset B_{p,\infty}^{\varrho^*}(\mathbb{R}^n)$ is still an open question.

(iii) Also the following observation is of some interest. We study the difference $d(s) := s - \varrho^*(s, p, n)$ for fixed n and p ($n > p + 1$). Obviously, $d(1 + 1/p) = d(n/p) = 0$ and $d(s) > 0$ if $1 + 1/p < s < n/p$. Moreover, the function $d(s)$ is concave on this interval, hence, it attains a maximal value $d(s_0)$ there. We have

$$s_0 := \frac{n}{p} + 1 - \sqrt{\frac{n-1}{p}} \quad \text{and} \quad d(s_0) = \left(\sqrt{\frac{n-1}{p}} - 1 \right)^2 + \frac{p-1}{\sqrt{p(n-1)}}$$

This shows that $d(s)$ has a bound depending on p and n , but it does not have an a priori bound for fixed p and independent of n .

We will use the following abbreviation:

$$\|f\|_{\dot{a}_p^r([-w,w])} := \left(\int_0^w |h|^{-rp-1} \int_{-w}^w |\Delta_h^{m+1} f(y)|^p dy dh \right)^{1/p} \quad (31)$$

with $m \in \mathbb{N}$ and $m+1 > r > 0$. The notation $\|f\|_{\dot{a}_p^r([-w,w])}$ reminds on a norm in a homogeneous Besov space. In some sense it is an incomplete one.

Theorem 13 *Suppose $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset E_{p,p}^r(\mathbb{R}^n)$ and $0 < r \leq s < n/p$. Let $f \in L_\infty(\mathbb{R})$. Then it follows that*

$$\sup_{w \geq 1} w^{-\gamma} \|f\|_{\dot{a}_p^r([-w,w])} < \infty \quad (32)$$

for all

$$\gamma > \gamma_0(s, n, p, r) := \frac{\frac{n}{p} + \frac{1}{p} \left(\frac{n}{p} - s \right) - r \left(\frac{n}{p} - s + 1 \right)}{\frac{n}{p} - s}. \quad (33)$$

Remark 19 (i) For the proof we refer to [61].

(ii) Let us comment on the condition (33). Fix n and p and consider $s \uparrow n/p$. Then the lower bound for γ tends to infinity, which means that the necessary condition (32) becomes less restrictive. This is connected with the fact that local singularities in spaces with $s \uparrow n/p$ become weaker and weaker, cf. (27) and (28). If we fix also s and consider $r \downarrow 0$, then the necessary condition (32) becomes again weaker (since the lower bound of γ increases). Clearly, that corresponds to the fact that the spaces $E_{p,p}^r(\mathbb{R}^n)$ become larger. A similar observation gives a converse result if $r \uparrow s$.

(iii) A first essential conclusion of Theorem 13 is obtained by observing that $\|f\|_{\dot{a}_p^r([-w,w])}$ is a non-decreasing function in w . So, whenever f is not a polynomial of low degree, $\|f\|_{\dot{a}_p^r([-w,w])}$ is bounded from below by a positive constant. Then Theorem 13 says (in the case that f is bounded) that $\gamma_0 \geq 0$ if, and only if, $r \leq \rho^*$, which also follows from Theorem 3. Hence, Theorem 13 represents an extension of Theorem 3.

Replacing the space $E_{p,p}^r(\mathbb{R}^n) = B_{p,p}^r(\mathbb{R}^n) = F_{p,p}^r(\mathbb{R}^n)$ by the Sobolev space $W_p^r(\mathbb{R}^n)$, $r \in \mathbb{N}$, we have necessary and sufficient conditions. Let

$$\mathcal{P}_m = \left\{ g : g(t) = \sum_{\ell=0}^m a_\ell t^\ell, \quad a_\ell \in \mathbb{R}, \quad \ell = 0, \dots, m \right\}, \quad m \in \mathbb{N},$$

and

$$A_{\gamma,p}(f) := \sup_{w \geq 1} w^{-\gamma} \left(\int_{-w}^w |f^{(m)}(y)|^p dy \right)^{1/p}, \quad (34)$$

which replaces (31).

Theorem 14 Suppose $0 \leq r \leq s < n/p$. Recall, ρ^* and γ_0 are defined in (30) and (33), respectively. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function if $r \geq 1$ and a continuous function if $r = 0$, but not an element of $\mathcal{P}_{r-1}$. Then $T_f(E_{p,p}^s(\mathbb{R}^n)) \subset W_p^r(\mathbb{R}^n)$ implies $r \leq \rho^*$, $f(0) = 0$, $f \in W_p^{r,loc}(\mathbb{R})$ and $A_{\gamma_0,p}(f) < \infty$.

Now we turn to the sufficient conditions. We concentrate on $r \geq 2$ (for $r = 0, 1$ we refer to [60]).

Theorem 15 Suppose $1 < p < \infty$, $2 \leq r \leq s < n/p$ and define again γ_0 by (33). Let $f(0) = 0$. If $A_{\gamma_0,p}(f) < \infty$, then $T_f(E_{p,p}^s(\mathbb{R}^n)) \subset W_p^r(\mathbb{R}^n)$ holds. Moreover, there exists some constant c such that

$$\|f \circ g\|_{W_p^r(\mathbb{R}^n)} \leq c A_{\gamma_0,p}(f) \left(\|g\|_{E_{p,p}^s(\mathbb{R}^n)} + \|g\|_{E_{p,p}^s(\mathbb{R}^n)}^{\gamma_0 + r - \frac{1}{p}} \right) \quad (35)$$

holds with c independent of f and g .

Remark 20 Both theorems are proved in [60]. However, the used arguments are as in [30, 8, 11, 54] (Thm. 14) and [9] (Thm. 15).

Various examples are treated in [59, 60, 61]. Here we concentrate on smooth periodic functions. If $f \not\equiv 0$ is periodic and smooth, then

$$\int_{-w}^w |f^{(n)}(y)|^p dy \asymp w, \quad w \geq 1.$$

Hence $A_{\gamma,p}(f) < \infty$ if, and only if, $\gamma \geq 1/p$. In such a situation Theorems 14 and 15 yield final results. Here another strange number occurs. Let

$$\varrho = \varrho(s, n, p) := \frac{\frac{n}{p}}{\frac{n}{p} - s + 1}. \quad (36)$$

Obviously, $\varrho < \varrho^*$ if $p < \infty$.

Corollary 2 Let $1 < p < \infty$ and let $2 \leq r \leq s < n/p$. Suppose that f is periodic, $f \in C^\infty(\mathbb{R})$, and $f \not\equiv 0$. Then $T_f(E_{p,p}^s(\mathbb{R}^n)) \subset W_p^r(\mathbb{R}^n)$ holds if, and only if, $f(0) = 0$ and $r \leq \varrho$.

This has a fractional counterpart, see [55, 5.3.6] and [61].

Theorem 16 Let $1 \leq p < \infty$ and $1 < s < n/p$. Suppose that f is periodic, $f(0) = 0$, $f \in C^\infty(\mathbb{R})$, and $f \not\equiv 0$. Then the following assertions are equivalent:

- (i) $T_f(F_{p,q}^s(\mathbb{R}^n)) \subset B_{p,\infty}^r(\mathbb{R}^n)$. (ii) $T_f(F_{p,\infty}^s(\mathbb{R}^n)) \subset F_{p,q}^r(\mathbb{R}^n)$. (iii) $r \leq \varrho$.

In [55, Thm. 5.3.6/3] we have been able also to prove the following associated inequality.

Theorem 17 *Let $f' \in C^\infty(\mathbb{R})$ and let $f(0) = 0$. Suppose $1 \leq q, r \leq \infty$ and $1 < s < n/p$. Let ϱ be as in (36). Then there exists a constant c such that*

$$\|f \circ g\|_{F_{p,r}^{\varrho}(\mathbb{R}^n)} \leq c (\|g\|_{F_{p,q}^s(\mathbb{R}^n)} + \|g\|_{F_{p,q}^s(\mathbb{R}^n)}^{\varrho}) \quad (37)$$

holds for all $g \in F_{p,q}^s(\mathbb{R}^n)$.

Remark 21 The proof of Thm. 17 relies on a specific estimate of products in Lizorkin-Triebel spaces, see [55, Thm. 4.6.2/5]. There is also counterpart for Besov spaces but less satisfactory.

4.4 The intermediate case (II)

Now we assume that $1 < s < 1 + 1/p$. Since $W_\infty^{1,loc}(\mathbb{R}) \not\subset E_{p,q}^{s,loc}(\mathbb{R})$ and $E_{p,q}^{s,loc}(\mathbb{R}) \not\subset W_\infty^{1,loc}(\mathbb{R})$, we have two independent necessary acting conditions, but we do not know if these two conditions are sufficient. Indeed, at this moment we do not have a conjecture how does the solution of *Problem 2* looks like. The best sufficient condition obtained so far is connected with a new class of functions which has been introduced for the first time by Bourdaud and Kateb.

We define $U_p^1(\mathbb{R})$ as the set of Lipschitz continuous functions f on $\mathbb{R}$ such that

$$\|f'\|_{U_p} := \sup_{t>0} t^{-1/p} \Omega_p(f', t) < +\infty, \quad (38)$$

see (16) for the definition of Ω_p .

Theorem 18 *Let $1 \leq p < +\infty$ and $0 < s < 1 + (1/p)$. If $f \in U_p^1(\mathbb{R})$ and $f(0) = 0$, then $T_f(B_{p,q}^s(\mathbb{R}^n)) \subset B_{p,q}^s(\mathbb{R}^n)$. Moreover, the inequality*

$$\|f \circ g\|_{B_{p,q}^s(\mathbb{R}^n)} \leq c (\|f'\|_\infty + \|f'\|_{U_p}) \|g\|_{B_{p,q}^s(\mathbb{R}^n)}$$

holds for all $g \in B_{p,q}^s(\mathbb{R}^n)$.

Remark 22 (i) A first proof of this theorem was found by Bourdaud and Kateb [16]. For $n = 1$, Kateb [38] improved Theorem 18 by obtaining the acting property under the condition $f' \in L_\infty \cap \dot{B}_{p,\infty}^{1/p}(\mathbb{R})$. Observe,

$$U_p^1(\mathbb{R}) \hookrightarrow \dot{W}_\infty^1 \cap \dot{B}_{p,\infty}^{1+1/p}(\mathbb{R})$$

and the embedding is proper, see [20]. An extension of this result of Kateb to the general n -dimensional case is still open.

- (ii) Whether Theorem 18 has a counterpart for Lizorkin-Triebel spaces is an open question.
- (iii) Of course, smooth periodic functions f do not act by composition on $E_{p,q}^s(\mathbb{R}^n)$, for $1 < s \leq 1 + 1/p$, see Theorem 16.

A second proof of Thm. 18 has been given in [20]. There we first investigated the limiting situation $s = 1 + 1/p$, which we will recall below, and afterwards we used nonlinear interpolation (more exactly, real interpolation of Lipschitz continuous operators) to derive the result for $0 < s < 1 + 1/p$.

Theorem 19 *Let $1 < p < \infty$. If $f \in U_p^1(\mathbb{R})$ and $f(0) = 0$, then $T_f(B_{p,1}^{1+1/p}(\mathbb{R}^n)) \subset B_{p,\infty}^{1+1/p}(\mathbb{R}^n)$. Moreover, the inequality*

$$\|f \circ g\|_{B_{p,\infty}^{1+1/p}(\mathbb{R}^n)} \leq c (\|f'\|_\infty + \|f'\|_{U_p}) \|g\|_{B_{p,1}^{1+1/p}(\mathbb{R}^n)}$$

holds for all $g \in B_{p,1}^{1+1/p}(\mathbb{R}^n)$.

There is a further result, rather close to the one-dimensional case of Thm. 19, we wish to mention. Recall, the Wiener classes $BV_p(\mathbb{R})$ have been introduced in Subsection 4.2.1. We will need the class of all primitives.

Definition 1 *Let $p \in [1, +\infty]$. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ belongs to $BV_p^1(\mathbb{R})$ if f is Lipschitz continuous and if its distributional derivative belongs to $BV_p(\mathbb{R})$.*

We endow $BV_p^1(\mathbb{R})$ with the norm

$$\|f\|_{BV_p^1(\mathbb{R})} := |f(0)| + \|f'\|_{BV_p(\mathbb{R})} \quad \forall f \in BV_p^1(\mathbb{R}),$$

which renders $BV_p^1(\mathbb{R})$ a Banach space. Concerning composition of functions belonging to $BV_p^1(\mathbb{R})$ we have proved in [20] the following satisfactory result.

Theorem 20 *Let $1 \leq p < \infty$. Then the following statements hold.*

- (i) *If $f, g \in BV_p^1(\mathbb{R})$, then $f \circ g \in BV_p^1(\mathbb{R})$, and*

$$\|f \circ g\|_{BV_p^1(\mathbb{R})} \leq \|f\|_{BV_p^1(\mathbb{R})} \left(1 + 2^{1/p} \|g\|_{BV_p^1(\mathbb{R})}\right).$$

- (ii) *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. Then the operator T_f maps $BV_p^1(\mathbb{R})$ to itself if, and only if, $f \in BV_p^1(\mathbb{R})$.*

Remark 23 Also the inclusion

$$BV_p^1(\mathbb{R}) \hookrightarrow U_p^1(\mathbb{R})$$

is proper, see [20] or [40].

The main example

There is one example which is of particular importance, mainly for historic reasons. We consider $f(t) := |t|$, $t \in \mathbb{R}$. This function belongs to the Besov space $B_{p,\infty}^{1+1/p,\text{loc}}(\mathbb{R})$ and is Lipschitz continuous, of course. It is also immediate that it belongs to $BV_p^1(\mathbb{R})$ and therefore to $U_p^1(\mathbb{R})$. By employing Thm. 19 and Thm. 18 we obtain the following corollary.

Corollary 3 *Let $1 < p < \infty$. The operator $g \mapsto |g|$ maps the Besov space $B_{p,1}^{1+1/p}(\mathbb{R}^n)$ into the Besov space $B_{p,\infty}^{1+1/p}(\mathbb{R}^n)$. Moreover, the inequality*

$$\| |g| \|_{B_{p,\infty}^{1+1/p}(\mathbb{R}^n)} \leq c \|g\|_{B_{p,1}^{1+1/p}(\mathbb{R}^n)}$$

holds for all $g \in B_{p,1}^{1+1/p}(\mathbb{R}^n)$.

Remark 24 The corollary can be derived also from a result of Savaré [57] who had investigated the mapping $g \mapsto |g|$ with respect to a Banach space $Z^{1+1/p,p}(\mathbb{R})$, where

$$B_{p,1}^{1+\frac{1}{p}}(\mathbb{R}) \hookrightarrow Z^{1+\frac{1}{p},p}(\mathbb{R}) \hookrightarrow B_{p,\infty}^{1+\frac{1}{p}}(\mathbb{R}).$$

For Besov spaces with $0 < s < 1 + 1/p$ we can argue by using nonlinear interpolation to obtain the boundedness of $g \mapsto |g|$ considered as a mapping of $B_{p,q}^s(\mathbb{R}^n)$ into itself. This method can not be applied to Lizorkin-Triebel spaces. By employing a different method the outcome is the following.

Theorem 21 *Let $1 \leq p, q \leq \infty$ ($1 < p < \infty$, $1 \leq q \leq \infty$ if $E = F$). In addition we assume $0 < s < 1 + 1/p$ ($s \neq 1$ if $p = 1$ in case $E = F$). Then the operator $g \mapsto |g|$ maps the space $E_{p,q}^s(\mathbb{R}^n)$ into itself. Moreover, the inequality*

$$\| |g| \|_{E_{p,q}^s(\mathbb{R}^n)} \leq c \|g\|_{E_{p,q}^s(\mathbb{R}^n)}$$

holds for all $g \in E_{p,q}^s(\mathbb{R}^n)$.

Remark 25 (i) Completely different methods have been used by Bourdaud, Meyer [22] and Oswald [51] to prove Thm. 21 with $E = B$. Whereas in the first reference the proof is based on Hardy's inequality the second reference is using spline techniques and Marchaud's inequality.

(ii) A first proof of Thm. 21 with $E = F$ has been given in [11], but with the extra condition $s \neq 1$. Here a similar method as in [22] is applied. A second proof has been published by Triebel [65, Thm. 25.8]. It relies on atomic decompositions and allows to deal with $s = 1$ if $1 < p < \infty$.

Let us turn to the continuity of the mapping $g \mapsto |g|$. The Lipschitz continuity of $f(t) = |t|$ yields the continuity of $g \mapsto |g|$ considered as a mapping of $B_{p,q}^s(\mathbb{R}^n)$ into itself if $0 < s < 1$ and $1 \leq p, q \leq \infty$, see Thm. 7. Also the continuity with respect to $W_p^1(\mathbb{R}^n)$ is well-known, see Thm. 5. There is nothing known on the continuity in all other cases where we know $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset E_{p,q}^s(\mathbb{R}^n)$.

4.5 Optimal inequalities (II)

All estimates in Subsection 4.4 do not reflect the nonlinearity of T_f , they are all of the form $\leq c(f) \|g\|$. Hence, the norm of g enters with the power 1. This is in sharp contrast to the estimates given in case $s > 1 + 1/p$. However, by using essentially the same type of arguments as in Prop. 6, one can also prove the following, see [20] for all details.

Lemma 1 *Let $1 < p \leq +\infty$, and $s > 1 + (1/p)$. Let N be a norm on $\mathcal{D}(\mathbb{R}^n)$. Let E be a normed function space such that $\mathcal{D}(\mathbb{R}^n) \subseteq E \subseteq W_1^{1,loc}(\mathbb{R}^n)$ and such that there exists a positive constant A such that*

$$\sup_{h \neq 0} |h|^{1-s} \left(\int_{\mathbb{R}^n} \left| \frac{\partial g}{\partial x_i}(x+h) - \frac{\partial g}{\partial x_i}(x) \right|^p dx \right)^{1/p} \leq A \|g\|_E \quad \forall g \in E \quad (39)$$

for all $i = 1, \dots, n$.

If there exist a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a constant $B > 0$ such that T_f maps $\mathcal{D}(\mathbb{R}^n)$ into E , and such that the inequality

$$\|f \circ g\|_E \leq B(N(g) + 1) \quad \forall g \in \mathcal{D}(\mathbb{R}^n) \quad (40)$$

holds, then f must be an affine function.

Remark 26 (i) The spaces $E_{p,q}^t(\mathbb{R}^n)$, $t > 1 + 1/p$, satisfy the assumptions of Lemma 1 with $1 + 1/p < s < \min(t, 2)$.

(ii) A mapping $T : E \rightarrow E$ is called *sublinear*, if there exists a constant c such that

$$\|Tg\|_E \leq c(1 + \|g\|_E)$$

holds for all $g \in E$. Hence, by Lemma 1, a composition operator T_f , satisfying the acting property $T_f(E_{p,q}^s(\mathbb{R}^n)) \subset E_{p,q}^s(\mathbb{R}^n)$, can be sublinear only in case $s \leq 1 + 1/p$.

(iii) It is of a certain surprise that the boundary between sublinear and superlinear estimates is given by $s = 1 + 1/p$ and not simply $s = 1$ (think on a fractional version of the chain rule).

5 Composition operators defined on intersections

Adams and Frazier [2, 3] have been the first who have seen that it makes sense to consider composition operators on intersections of the type $H_p^s \cap W_{sp}^1(\mathbb{R}^n)$, where $H_p^s(\mathbb{R}^n) = F_{p,2}^s(\mathbb{R}^n)$ is a Bessel potential space. The main observation is the following: the Dahlberg phenomenon, see Prop. 3, disappears.

Theorem 22 *Let $1 < p, q < \infty$ and $s > 1$, $s \notin \mathbb{N}$. Let $m > s$ be a natural number and let $f \in C_b^m(\mathbb{R})$ s.t. $f(0) = 0$. Then T_f maps $F_{p,q}^s \cap W_{sp}^1(\mathbb{R}^n)$ into $F_{p,q}^s(\mathbb{R}^n)$. In either case the mapping is bounded and continuous.*

Remark 27 (i) For the first moment it is surprising that the relation of s to n/p does not have an influence. However, if $s > \max(1, n/p)$, then we have $F_{p,q}^s(\mathbb{R}^n) \hookrightarrow W_{sp}^1(\mathbb{R}^n)$, see (26), and we are back in the situation discussed in Subsection 4.2.4. If $1 < s < n/p$, then $F_{p,q}^s(\mathbb{R}^n) \not\subset W_{sp}^1(\mathbb{R}^n)$ and consequently $F_{p,q}^s \cap W_{sp}^1(\mathbb{R}^n)$ is strictly smaller than $F_{p,q}^s(\mathbb{R}^n)$. (ii) The theorem, as stated here, can be found in Brezis and Mironescu [26]. A different proof, but restricted to the case $p = q$, has been given by Maz'ya and Shaposnikova [44]. With some restrictions in q the boundedness of T_f is also proved in [55, 5.3.7]. For a similar result involving Besov spaces we also refer to [55, 5.3.7].

Of course, one may ask for larger or simply different subspaces of $F_{p,q}^s(\mathbb{R}^n)$ such that all functions $f \in C_b^m(\mathbb{R})$ generate a composition operator T_f s.t. this subspace is mapped into $F_{p,q}^s(\mathbb{R}^n)$ by T_f . In [26] one can find a simple argument which explains that $F_{p,q}^s \cap W_{sp}^1(\mathbb{R}^n)$ is nearly optimal. Let $g \in F_{p,q}^s(\mathbb{R}^n)$. Since $f_1(t) = \cos t - 1$ and $f_2(t) = \sin t$ are admissible functions it follows

$$(\cos g - 1 + i \sin g) \in F_{p,q}^s \cap L_\infty(\mathbb{R}^n).$$

Since

$$F_{p,q}^s \cap L_\infty(\mathbb{R}^n) \hookrightarrow F_{sp,2}^1(\mathbb{R}^n) = W_{ps}^1(\mathbb{R}^n),$$

see [26] (but traced there to Oru), we obtain $e^{ig} - 1 \in W_{ps}^1(\mathbb{R}^n)$. This implies

$$i e^{ig} \frac{\partial g}{\partial x_\ell} \in L_{ps}(\mathbb{R}^n) \quad \text{for all } \ell = 1, \dots, n.$$

Thus $g \in \dot{W}_{ps}^1(\mathbb{R}^n)$, the homogeneous Sobolev space. This space consists of all regular real-valued distributions s.t. all derivatives of the first order belong to $L_{ps}(\mathbb{R}^n)$ and is endowed with the seminorm

$$\|g\|_{\dot{W}_{ps}^1(\mathbb{R}^n)} := \sum_{|\alpha|=1} \|g^{(\alpha)}\|_{L_{ps}(\mathbb{R}^n)}.$$

Hence, the optimal subspace is contained in $F_{p,q}^s \cap \dot{W}_{sp}^1(\mathbb{R}^n)$. In case of Bessel potential and Slobodeckij spaces, Thm. 22 can be improved to become optimal with this respect. Recall, $F_{p,2}^s(\mathbb{R}^n) = H_p^s(\mathbb{R}^n)$ in the sense of equivalent norms.

Theorem 23 *Let $1 < p < \infty$ and $s > 1$, $s \notin \mathbb{N}$. Let $m > s$ be a natural number and let $f \in C_b^m(\mathbb{R})$ s.t. $f(0) = 0$. Then T_f maps $F_{p,2}^s \cap \dot{W}_{sp}^1(\mathbb{R}^n)$ into $F_{p,2}^s(\mathbb{R}^n)$. Furthermore, there exists a constant c_f such that*

$$\|f \circ g\|_{F_{p,2}^s(\mathbb{R}^n)} \leq c_f \left(\|g\|_{F_{p,2}^s(\mathbb{R}^n)} + \|g\|_{\dot{W}_{ps}^1(\mathbb{R}^n)}^s \right)$$

holds for all $g \in F_{p,2}^s \cap \dot{W}_{sp}^1(\mathbb{R}^n)$.

Remark 28 Thm. 23 has been proved by Adams and Frazier in [3].

Now we turn to Slobodeckij spaces. Recall, if $s > 0$ is not an natural number, then $F_{p,p}^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n)$ holds in the sense of equivalent norms.

Theorem 24 *Let $1 \leq p < \infty$ and $s > 1$, $s \notin \mathbb{N}$. Let $m > s$ be a natural number and let $f \in C_b^m(\mathbb{R})$ s.t. $f(0) = 0$. Then T_f maps $F_{p,p}^s \cap \dot{W}_{sp}^1(\mathbb{R}^n)$ into $F_{p,p}^s(\mathbb{R}^n)$. Furthermore, there exists a constant c such that*

$$\|f \circ g\|_{F_{p,p}^s(\mathbb{R}^n)} \leq c \|f\|_{C_b^m(\mathbb{R})} \left(\|g\|_{F_{p,p}^s(\mathbb{R}^n)} + \|g\|_{\dot{W}_{ps}^1(\mathbb{R}^n)}^s \right)$$

holds for all $g \in F_{p,p}^s \cap \dot{W}_{sp}^1(\mathbb{R}^n)$ and all $f \in C_b^m(\mathbb{R})$ s.t. $f(0) = 0$. In either case the mapping T_f is continuous.

Remark 29 (i) Thm. 24 has been proved by Maz'ya and Shaposnikova in [44].

(ii) Open problem: prove Theorems 22, 23, 24 for the maximal range of s, p, q and under minimal regularity conditions on f . Up to now, only the case of Sobolev spaces has a complete answer, given by the following statements, see [14, Thm. 1 and 2].

Theorem 25 *Let m be an integer ≥ 2 , $1 \leq p < +\infty$ — with the exception of $m = n$ and $p = 1$ — and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function.*

(i) T_f takes $W_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$ to itself if, and only if, $f(0) = 0$ and $f' \in W_{p,unif}^{m-1}(\mathbb{R})$.

(ii) The inequality

$$\|f \circ g\|_{W_p^m(\mathbb{R}^n)} \leq c C_f \left(\|g\|_{\dot{W}_{mp}^1(\mathbb{R}^n)}^m + \|g\|_{W_p^m(\mathbb{R}^n)} \right) \quad (41)$$

holds for all f s.t. $f(0) = 0$ and $f' \in W_{p,unif}^{m-1}(\mathbb{R})$, and all $g \in W_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, where C_f is defined by (8).

Theorem 26 *Let m be an integer ≥ 2 , $1 \leq p < +\infty$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. Then T_f takes $W_p^m \cap L_\infty(\mathbb{R}^n)$ to itself if, and only if, $f \in W_p^{m,loc}(\mathbb{R})$ and $f(0) = 0$.*

Remark 30 The exceptional case of Thm. 25 is covered by Thm. 26 since $W_1^n \cap \dot{W}_n^1(\mathbb{R}^n)$ is embedded into $L_\infty(\mathbb{R}^n)$.

6 Composition operators on vector-valued spaces

We turn to the vector-valued situation but restrict ourselves to Sobolev spaces.

Let $k \in \mathbb{N}$, $k \geq 2$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$. We study the associated composition operator

$$T_f(g) := f \circ g, \quad g = (g_1, \dots, g_k)$$

under the assumption $g \in W_p^m(\mathbb{R}^n, \mathbb{R}^k)$. To describe a sufficient condition for the acting property we need a further class of functions. We need the following simple mappings: for $1 \leq j \leq k$ we define

$$\sigma_j f(x_1, \dots, x_k) := f(x_1, \dots, x_{j-1}, x_k, x_{j+1}, \dots, x_{k-1}, x_j), \quad x \in \mathbb{R}^n.$$

Let $1 \leq p < \infty$ and $k > 1$. Then the space $E_p(\mathbb{R}^k)$ is the collection of all measurable functions f s.t.

$$\|f\|_{E_p(\mathbb{R}^k)} := \sum_{j=1}^k \left(\sup_{a \in \mathbb{R}} \int_a^{a+1} \|\sigma_j f(\cdot, t)\|_{L_\infty(\mathbb{R}^{k-1})}^p dt \right)^{1/p} < \infty.$$

By $\dot{W}_{E_p}^m(\mathbb{R}^k)$ we denote the homogeneous Sobolev space built on E_p , i.e., the semi-norm is generated by

$$\|f\|_{\dot{W}_{E_p}^m(\mathbb{R}^k)} := \sum_{|\alpha|=m} \|f^{(\alpha)}\|_{E_p(\mathbb{R}^k)}.$$

Then we have the following partial generalization of Thm. 25 :

Theorem 27 *Let $1 < p < \infty$ and $m, k \geq 2$. Let $f \in \dot{W}_\infty^1 \cap \dot{W}_{E_p}^m(\mathbb{R}^k)$ and suppose $f(0) = 0$. Then T_f maps $W_p^m \cap \dot{W}_{pm}^1(\mathbb{R}^n, \mathbb{R}^k)$ into $W_p^m(\mathbb{R}^n)$. Moreover, there exists a constant c s.t.*

$$\|f \circ g\|_{W_p^m(\mathbb{R}^n)} \leq c \left(\|f\|_{\dot{W}_{E_p}^m(\mathbb{R}^k)} + \|f\|_{\dot{W}_\infty^1(\mathbb{R}^k)} \right) \left(\|g\|_{W_p^m(\mathbb{R}^n)} + \|g\|_{\dot{W}_{mp}^1(\mathbb{R}^n)}^m \right), \quad (42)$$

holds for all $f \in \dot{W}_\infty^1 \cap \dot{W}_{E_p}^m(\mathbb{R}^k)$ and all $g \in W_p^m \cap \dot{W}_{pm}^1(\mathbb{R}^n, \mathbb{R}^k)$.

Remark 31 (i) This theorem has been proved in [14, Thm. 3]. The estimate (42) is not found there, however, it can be derived as in the scalar case.

(ii) For $T_f(W_p^m \cap \dot{W}_{pm}^1(\mathbb{R}^n, \mathbb{R}^k)) \subset W_p^m(\mathbb{R}^n)$ it is necessary that f is locally Lipschitz continuous, see [4] and [14].

Let us have a short look on Thm. 2. A naive but natural conjecture concerning an extension to the vector-valued case would consist in the following:

Let $f \in \dot{W}_\infty^1 \cap W_p^{m,loc}(\mathbb{R}^k)$ such that $f(0) = 0$. Then T_f maps $W_p^m \cap \dot{W}_{pm}^1(\mathbb{R}^n, \mathbb{R}^k)$ into $W_p^m(\mathbb{R}^n)$.

We do not believe that this statement holds true. The space $\dot{W}_\infty^1 \cap \dot{W}_{E_p}^m(\mathbb{R}^k)$ is much smaller than $\dot{W}_\infty^1 \cap W_p^{m,loc}(\mathbb{R}^k)$. This can be easily seen by studying tensor products of functions. Let

$$f(x) := f_1(x_1) \cdot \dots \cdot f_k(x_k), \quad x = (x_1, \dots, x_k) \in \mathbb{R}^k.$$

If $f \in \dot{W}_\infty^1 \cap \dot{W}_{E_p}^m(\mathbb{R}^k)$, then each of the functions f_j has to belong to $W_\infty^{m,loc}(\mathbb{R})$, at least if all components of f are nontrivial or more exactly, are not polynomials. To guarantee $f \in \dot{W}_\infty^1 \cap W_p^{m,loc}(\mathbb{R}^k)$ it is sufficient to have $f_j \in W_p^{m,loc}(\mathbb{R})$ for all j .

In our understanding the extension to the vector-valued case will be not a straightforward generalization. We expect some new phenomena.

Remark 32 The extension of Thm. 27 to fractional order of smoothness is completely open.

7 On Problem 3

Recall, *Problem 3* consists in characterizing those function spaces E s.t. the acting condition $T_f(E) \subset E$ is equivalent to boundedness of $T_f : E \rightarrow E$ and is also equivalent to the continuity of $T_f : E \rightarrow E$. With other words, if one has established the acting property then one gets boundedness and continuity for free. If a space E has this property then we will write $E \in P_3$, otherwise $E \notin P_3$.

Below we have made a list of more or less classical function spaces and fixed their relation to P_3 . In addition we have given some references, sometimes inside our survey, sometimes not. We will use the convention $-\infty < a < b < \infty$.

Function space E	$E \in P_3$	References
$L_p(\mathbb{R}^n)$, $1 \leq p < \infty$	yes	[6, 3.4, 3.7]
$L_\infty(\mathbb{R}^n)$	no	[6, 3.7]
$C_b^m(\mathbb{R}^n)$, $m \in \mathbb{N}$	yes	obvious
$C^s(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$	no	[31], [35], [48]
$W_p^1(\mathbb{R}^n)$, $1 \leq p < \infty$	yes	Thm. 5(i)
$W_p^m(\mathbb{R}^n)$, $1 < p < \infty$, $m \in \mathbb{N}$, $m > n/p$	yes	Thm. 5(ii)
$W_p^m(\mathbb{R}^n)$, $1 \leq p < \infty$, $m \in \mathbb{N}$, $1 + 1/p < m < n/p$	yes	Prop. 3
$F_{p,q}^s(\mathbb{R})$, $1 < p < \infty$, $1 \leq q < \infty$, $1 + 1/p < s$	yes	Cor. 1
$F_{p,\infty}^s(\mathbb{R})$, $1 < p < \infty$, $1 + 1/p < s$	no	Cor. 1
$F_{p,q}^s(\mathbb{R}^n)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, $1 + 1/p < s < n/p$	yes	Prop. 3
$F_{p,q}^{1+1/p}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \leq q \leq \infty$	yes	Prop. 4
$B_{p,q}^s(\mathbb{R})$, $1 < p < \infty$, $p \leq q < \infty$, $1 + 1/p < s$	yes	Cor. 1
$B_{p,\infty}^s(\mathbb{R})$, $1 < p < \infty$, $1 + 1/p < s$	no	Cor. 1
$B_{p,q}^s(\mathbb{R}^n)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, $1 + 1/p < s < n/p$	yes	Prop. 4
$B_{p,q}^{1+1/p}(\mathbb{R}^n)$, $1 \leq p < \infty$, $1 < q \leq \infty$	yes	Prop. 3
$BMO(\mathbb{R}^n)$	no	[19]
$VMO(\mathbb{R}^n)$	no	[19]
$CMO(\mathbb{R}^n)$	yes	[19]
$bmo(\mathbb{R}^n)$	no	[19]
$vmo(\mathbb{R}^n)$	yes	[19]
$cmo(\mathbb{R}^n)$	yes	[19]
$AC[a, b]$	yes	[5]
$BV_p[a, b]$	yes	[5]
$Lip_\alpha[a, b]$, $0 < \alpha \leq 1$	no	[31], [35], [5]
$A(\mathbb{T})$	yes	[39, 8.6], [15],

Definitions of all these function spaces will be given in the Appendix below.

8 Concluding remarks

As it becomes clear by the long list of open problems, in our opinion the theory of composition operators in function spaces of fractional order of smoothness (like Besov and Lizorkin-Triebel spaces) is just at its beginning. Even worst is the situation with respect to Nemytskij operators. There are nearly no final results in the general case up to our knowledge. A few information can be found in [55, 5.5.4]. A bit better is the situation when we restrict us to operators of the type

$$N(g)(x) := f(x, g(x)), \quad x \in \mathbb{R}^n, \quad g \in E.$$

These special Nemytskij operators are studied in the monograph by Appell and Zabrejko [6], see also the recent survey [5]. However, the knowledge concentrates on either spaces with smoothness 0 (Lebesgue spaces, Orlicz spaces), or smoothness 1 (first order Sobolev-Orlicz spaces) or on spaces with $p = \infty$ (Hölder spaces).

9 Appendix

Here we recall the definition of the function spaces used in this survey.

9.1 Besov and Lizorkin-Triebel spaces

To introduce the (inhomogeneous) Besov-Triebel-Lizorkin spaces we make use of the characterizations via differences and derivatives.

Definition 2 *Let $s > 0$, $1 \leq q \leq \infty$, and let $M \in \mathbb{N}$ be such that $M \leq s < M + 1$.*

(i) *Suppose $1 \leq p < \infty$. Then the Lizorkin-Triebel space $F_{p,q}^s(\mathbb{R}^n)$ is the collection of all real-valued functions $f \in L_p(\mathbb{R}^n)$ s.t.*

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)} := \|f\|_{L_p(\mathbb{R}^n)} + \left\| \left(\int_0^1 t^{-sq} \left(\frac{1}{t^n} \int_{|h|<t} |\Delta_h^{M+1} f| dh \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)} < \infty.$$

(ii) *Suppose $1 \leq p \leq \infty$. Then the Besov space $B_{p,q}^s(\mathbb{R}^n)$ is the collection of all real-valued functions $f \in L_p(\mathbb{R}^n)$ s.t.*

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} := \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^{M+1} f\|_{L_p(\mathbb{R}^n)}^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty.$$

Remark 33 The spaces $F_{p,q}^s(\mathbb{R}^n)$, $B_{p,q}^s(\mathbb{R}^n)$ are Banach spaces. Nowadays there exists a rich literature on this subject. We refer to Frazier and Jawerth [33], Besov, Il’jin, and Nikol’skij [7], Nikol’skij [50], Peetre [53] and Triebel [63, 64, 66].

In some cases, an alternative equivalent norm in Besov spaces can be obtained as follows. We concentrate on $n = 1$. Using the functional $\Omega_p(f, t)$ defined in (16), we have the following result, see e.g. [64, Thm. 3.5.3, p. 194]:

Proposition 8 *Let $1/p < s < 1$. Then a real-valued function f belongs to $B_{p,q}^s(\mathbb{R})$ if, and only if,*

$$\|f\|_{L_p(\mathbb{R})} + \left(\int_0^\infty \left(\frac{\Omega_p(f, t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} < +\infty. \quad (43)$$

Moreover, the above expression generates an equivalent norm on $B_{p,q}^s(\mathbb{R})$.

Remark 34 The condition $s > 1/p$ cannot be avoided. Indeed, (43) implies that f is locally bounded, a property which is not shared by all Besov functions for $s < 1/p$.

In Subsection 4.2.1 we have also used homogeneous Besov-Triebel-Lizorkin spaces. Here is a definition.

Definition 3 *Let $s > 0$, $1 \leq q \leq \infty$, and let $M \in \mathbb{N}$ be such that $M \leq s < M + 1$.*

(i) *Suppose $1 \leq p < \infty$. Then the homogeneous Lizorkin-Triebel space $\dot{F}_{p,q}^s(\mathbb{R}^n)$ is the collection of all regular real-valued distributions f s.t.*

$$\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} := \left\| \left(\int_0^\infty t^{-sq} \left(\frac{1}{t^n} \int_{|h|<t} |\Delta_h^{M+1} f| dh \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)} < \infty.$$

(ii) *Suppose $1 \leq p \leq \infty$. Then the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is the collection of all regular real-valued distributions f s.t.*

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^{M+1} f\|_{L_p(\mathbb{R}^n)}^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty.$$

Remark 35 Of course, we have $f = g$ in $\dot{E}_{p,q}^s(\mathbb{R}^n)$ if $f - g$ is a polynomial of degree $\leq M$. Since we only deal with functions $f \in \dot{E}_{p,q}^s \cap L_\infty(\mathbb{R})$, satisfying $f(0) = 0$, this does not matter.

9.2 BMO-type spaces

$BMO(\mathbb{R}^n)$ is the set of real-valued locally integrable functions g on $\mathbb{R}^n$ such that

$$\|g\|_{BMO} := \sup_Q f_Q \left| g - f_Q g \right| < +\infty,$$

where the supremum is taken on all cubes Q with sides parallel to the coordinate axes and where

$$f_Q g$$

denotes the mean value of the function g on Q . The quotient space of $BMO(\mathbb{R}^n)$, endowed with the above seminorm, by the subspace of constant functions, is a Banach space. Since the operator T_f is clearly not defined on the quotient space, we prefer to consider $BMO(\mathbb{R}^n)$ as a Banach space of ‘true’ functions with the following norm:

$$\|g\|_* := \|g\|_{BMO} + f_{Q_0} |g| \quad \forall g \in BMO(\mathbb{R}^n),$$

where Q_0 is the unit cube $[-1/2, +1/2]^n$. We denote by $bmo(\mathbb{R}^n)$ the linear subspace of $BMO(\mathbb{R}^n)$ consisting of those functions g which satisfy also the following condition

$$\sup_{|Q| \geq 1} f_Q |g| < +\infty,$$

where $|Q|$ denotes the Lebesgue measure of Q or, equivalently,

$$\sup_{|Q|=1} f_Q |g| < +\infty,$$

see [19, Lem. 7]. It turns out that $bmo(\mathbb{R}^n)$ is a Banach space for the norm

$$\|g\|_{bmo} := \|g\|_{BMO} + \sup_{|Q|=1} f_Q |g| \quad \forall g \in bmo(\mathbb{R}^n).$$

We denote by $cmo(\mathbb{R}^n)$ the closure of $\mathcal{D}(\mathbb{R}^n)$ in $bmo(\mathbb{R}^n)$, and we endow $cmo(\mathbb{R}^n)$ with the norm of $bmo(\mathbb{R}^n)$. Similarly, we denote by $CMO(\mathbb{R}^n)$ the closure of $\mathcal{D}(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, and we endow $CMO(\mathbb{R}^n)$ with the norm of $BMO(\mathbb{R}^n)$.

According to Sarason [56], a function g of $BMO(\mathbb{R}^n)$ which satisfies the limiting condition

$$\lim_{a \rightarrow 0} \left(\sup_{|Q| \leq a} f_Q \left| g - f_Q g \right| \right) = 0 \quad (44)$$

is said to be of vanishing mean oscillation. The subspace of $BMO(\mathbb{R}^n)$ consisting of the functions of vanishing mean oscillation is denoted $VMO(\mathbb{R}^n)$, and we endow $VMO(\mathbb{R}^n)$

with the norm of $BMO(\mathbb{R}^n)$. We note that the space $VMO(\mathbb{R}^n)$ considered by Coifman and Weiss [29] is different from that considered by Sarason, and it coincides with our $CMO(\mathbb{R}^n)$. As it is well known, $VMO(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n)$. For example, the function $\log|x|$ belongs to $BMO(\mathbb{R}^n)$, but not to $VMO(\mathbb{R}^n)$, see e.g. Stein [62, Ch. IV, §. I.1.2], and Brezis and Nirenberg [27, p. 211]. We set

$$vmo(\mathbb{R}^n) := VMO(\mathbb{R}^n) \cap bmo(\mathbb{R}^n),$$

and we endow the space $vmo(\mathbb{R}^n)$ with the norm of $bmo(\mathbb{R}^n)$.

For the convenience of the reader, we display all the subspaces of $BMO(\mathbb{R}^n)$ we have introduced in the following diagram:

$$\begin{array}{ccc} bmo(\mathbb{R}^n) & \subsetneq & BMO(\mathbb{R}^n) \\ \cup \mathfrak{H} & & \cup \mathfrak{H} \\ vmo(\mathbb{R}^n) & \subsetneq & VMO(\mathbb{R}^n) \\ \cup \mathfrak{H} & & \cup \mathfrak{H} \\ cmo(\mathbb{R}^n) & \subsetneq & CMO(\mathbb{R}^n) \end{array}$$

where all inclusions are proper and continuous.

9.3 Some further classical function spaces

A definition of the Wiener class $BV_p(\mathbb{R})$ has been given in Subsection 4.2.1. The space $BV_p[a, b]$ is obtained by restricting the intervals $[a_k, b_k]$ to subintervals of $[a, b]$.

By $AC[a, b]$ we denote the collection of all absolutely continuous functions on $[a, b]$ endowed with the norm

$$\|f\|_{AC[a,b]} := |f(a)| + \int_a^b |f'(t)| dt.$$

Finally, by $A(\mathbb{T})$ we denote the Wiener algebra on the torus, i.e., the set of all continuous, 2π -periodic functions f s.t.

$$\|f\|_{A(\mathbb{T})} := \sum_{k=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} f(t) e^{ikt} dt \right| < \infty.$$

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