

Some Results on Jacobi Forms of Higher Degree

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Abstract

In this article, the author gives some of his results on Jacobi forms of higher degree without proof. The proof can be found in the references [Y1] and [Y2].

1 Jacobi Forms

First of all, we introduce the notations. We denote by Z , R and C the ring of integers, the field of real numbers and the field of complex numbers respectively. We denote by Z^+ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F . For any $M \in F^{(k,l)}$, tM denotes the transpose matrix of M . For $A \in F^{(k,l)}$, $\sigma(A)$ denotes the trace of A . For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n . For any positive integer $g \in Z^+$, we let

$$H_g := \{ Z \in C^{(g,g)} \mid Z = {}^tZ, \operatorname{Im} Z > 0 \}$$

the Siegel upper half plane of degree g . Let $Sp(g, R)$ and $Sp(g, Z)$ be the real symplectic group of degree g and the Siegel modular group of degree g respectively.

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Let

$$(1.1) \quad O_g(R^+) := \{ M \in R^{(2g, 2g)} \mid {}^t M J_g M = \nu J_g \text{ for some } \nu > 0 \}$$

be the group of *similitudes* of degree g , where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

Let $M \in O_g(R^+)$. If ${}^t M J_g M = \nu J_g$, we write $\nu = \nu(M)$. It is easy to see that $O_g(R^+)$ acts on H_g transitively by

$$M \cdot Z := (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$ and $Z \in H_g$.

For $l \in Z^+$, we define

$$(1.2) \quad O_g(l) := \{ M \in Z^{(2g, 2g)} \mid {}^t M J_g M = l J_g \}.$$

We observe that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(l)$ is equivalent to the conditions

$$(1.3) \quad {}^t A C = {}^t C A, \quad {}^t B D = {}^t D B, \quad {}^t A D - {}^t C B = l E_g$$

or

$$(1.4) \quad A {}^t B = B {}^t A, \quad C {}^t D = D {}^t C, \quad A {}^t D - B {}^t C = l E_g.$$

For two positive integers g and h , we consider the *Heisenberg group*

$$H_R^{(g, h)} := \{ [(\lambda, \mu), \kappa] \mid \lambda, \mu \in R^{(h, g)}, \kappa \in R^{(h, h)}, \kappa + \mu {}^t \lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] := [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda {}^t \mu' - \mu {}^t \lambda'].$$

We define the semidirect product of $O_g(R^+)$ and $H_R^{(g,h)}$

$$(1.5) \quad O_R^{(g,h)} =: O_g(R^+) \ltimes H_R^{(g,h)}$$

endowed with the following multiplication law

$$(1.6) \quad (M, [(\lambda, \mu), \kappa]) \cdot (M', [(\lambda', \mu'), \kappa']) \\ := (MM', [(\nu(M')^{-1}\tilde{\lambda} + \lambda', \nu(M')^{-1}\tilde{\mu} + \mu'), \nu(M')^{-1}\kappa + \kappa' + \nu(M')^{-1}(\tilde{\lambda}^t \mu' - \tilde{\mu}^t \lambda')]),$$

with $M, M' \in O_g(R^+)$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. Clearly the *Jacobi group* $G_R^{(g,h)} := Sp(g, R) \ltimes H_R^{(g,h)}$ is a normal subgroup of $O_R^{(g,h)}$. It is easy to see that $O_g(R^+)$ acts on $H_g \times C^{(h,g)}$ transitively by

$$(1.7) \quad (M, [(\lambda, \mu), \kappa]) \cdot (Z, W) := (M < Z >, \nu(W + \lambda Z + \mu)(CZ + D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$, $\nu = \nu(M)$, $(Z, W) \in H_g \times C^{(h,g)}$.

Let ρ be a rational representation of $GL(g, C)$ on a finite dimensional complex vector space V_ρ . Let $\mathcal{M} \in R^{(h,h)}$ be a symmetric half integral matrix of degree h . We define

$$(1.8) \quad (f|_{\rho, \mathcal{M}}[(M, [(\lambda, \mu), \kappa])])(Z, W) \\ := \exp\{-2\pi\nu i\sigma(\mathcal{M}[W + \lambda Z + \mu](CZ + D)^{-1}C)\} \\ \times \exp\{2\pi\nu i\rho(\mathcal{M}(\lambda Z^t \lambda + 2\lambda^t W + (\kappa + \mu^t \lambda)))\} \\ \times \sigma(CZ + D)^{-1} f(M < Z >, \nu(W + \lambda Z + \mu)(CZ + D)^{-1}),$$

where $\nu = \nu(M)$.

Lemma 1.1. Let $g_i = (M_i, [(\lambda_i, \mu_i), \kappa_i]) \in O_R^{(g,h)}$ ($i = 1, 2$). For any $f \in C^\infty(H_g \times C^{(h,g)}, V_\rho)$, we have

$$(1.9) \quad (f|_{\rho, \mathcal{M}}[g_1])|_{\rho, \nu(M_1)\mathcal{M}}[g_2] = f|_{\rho, \mathcal{M}}[g_1 g_2].$$

Definition 1.2. Let ρ and $\mathcal{M}$ be as above. Let

$$H_Z^{(g,h)} := \{ [(\lambda, \mu), \kappa] \in H_R^{(g,h)} \mid \lambda, \mu \in Z^{(h,g)}, \kappa \in Z^{(h,h)} \}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ is a holomorphic function $f \in C^\infty(H_g \times C^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_g^J := Sp(g, Z) \ltimes H_Z^{(g,h)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z, W) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in Z^{(g,h)}} C(T, R) \exp(2\pi i \sigma(TZ + RW))$$

with $c(T, R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \leq 2$, the condition (B) is superfluous by Koecher principle (see [Z] Lemma 1.6). We denote by $J_{\rho, \mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ . In the special case $V_\rho = C$, $\rho(A) = (\det A)^k$ ($k \in Z$, $A \in GL(g, C)$), we write $J_{k, \mathcal{M}}(\Gamma_g)$ instead of $J_{\rho, \mathcal{M}}(\Gamma_g)$ and call k the *weight* of a Jacobi form $f \in J_{k, \mathcal{M}}(\Gamma_g)$.

Ziegler ([Zi] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho, \mathcal{M}}(\Gamma_g)$ is finite dimensional.

2 Singular Jacobi Forms

In this section, we define the concept of singular Jacobi forms and characterize singular Jacobi forms.

Let $\mathcal{M}$ be a symmetric positive definite, half integral matrix of degree h . A Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ admits a Fourier expansion (see Definition

1.2 (B))

$$(2.1) \quad f(Z, W) = \sum_{T, R} c(T, R) e^{2\pi i \sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}, \quad Z \in H_g, \quad W \in C^{(h, g)}.$$

A Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that the Fourier coefficient $c(T, R)$ is zero unless $\det(4T - R\mathcal{M}^{-1}{}^tR) = 0$.

Example 2.1. Let $\mathcal{M} = {}^t\mathcal{M}$ be as above. Let $S \in Z^{(2k, 2k)}$ be a symmetric positive definite integral matrix of degree $2k$ and $c \in Z^{(2k, h)}$. We consider the theta series

$$(2.2) \quad \vartheta_{S, c}^{(g)}(Z, W) := \sum_{\lambda \in Z^{(2k, g)}} e^{\pi i \sigma(S[\lambda]Z + 2S\lambda^t(cW))}, \quad Z \in H_g, \quad W^{(h, g)}.$$

We assume that $2k < g + \text{rank}(\mathcal{M})$. Then $\vartheta_{S, c}(Z, W)$ is a singular Jacobi form in $J_{k, \mathcal{M}}(\Gamma_g)$, where $\mathcal{M} = \frac{1}{2} {}^t c \mathcal{M} c$. We note that if the Fourier coefficient $c(T, R)$ of $\vartheta_{S, c}^{(g)}$ is nonzero, there exists $\lambda \in Z^{(2k, g)}$ such that

$$\frac{1}{2} {}^t(\lambda, c) S(\lambda, c) = \begin{pmatrix} T & \frac{1}{2} R \\ \frac{1}{2} {}^t R & \mathcal{M} \end{pmatrix}.$$

Thus

$$\text{rank} \begin{pmatrix} T & \frac{1}{2} R \\ \frac{1}{2} {}^t R & \mathcal{M} \end{pmatrix} \leq 2k < g + \text{rank}(\mathcal{M}).$$

Therefore $\det(4T - R\mathcal{M}^{-1}{}^tR) = 0$.

The following natural question arises:

Problem: *Characterize the singular Jacobi forms.*

The author([Y1]) gives some answers for this problem. He characterizes singular Jacobi forms by the *differential equation* and the *weight* of the representation ρ .

Now we define a very important differential operator characterizing *singular Jacobi forms*. We let

$$(2.3) \quad \mathcal{P}_g := \{ Y \in R^{(g,g)} \mid Y = {}^t Y > 0 \}$$

be the open convex cone in the Euclidean space $R^{\frac{g(g+1)}{2}}$. We define the differential operator operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times R^{(h,g)}$ defined by

$$(2.4) \quad M_{g,h,\mathcal{M}} := \det(Y) \cdot \det \left(\frac{\partial}{\partial Y} + \frac{1}{8\pi} {}^t \left(\frac{\partial}{\partial V} \right) \mathcal{M}^{-1} \left(\frac{\partial}{\partial V} \right) \right),$$

where $\frac{\partial}{\partial Y} = \left(\frac{(1+\delta_{\mu\nu})}{2} \frac{\partial}{\partial y_{\mu\nu}} \right)$ and $\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}} \right)$.

Definition 2.2. An irreducible finite dimensional representation ρ of $GL(g, C)$ is determined uniquely by its highest weight $(\lambda_1, \dots, \lambda_g) \in Z^g$ with $\lambda_1 \leq \dots \leq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \dots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ .

Theorem A. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ . Then the following are equivalent:

- (1) f is a *singular* Jacobi forms.
- (2) f satisfies the *differential equation* $M_{g,h,\mathcal{M}}f = 0$.

Theorem B. Let $2\mathcal{M}$ be a symmetric positive definite, *unimodular* even matrix of degree h . Assume that ρ satisfies the following condition

$$(2.5) \quad \rho(A) = \rho(-A) \quad \text{for all } A \in GL(g, C).$$

Then any nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ is *singular* if and only if $2k(\rho) < g + \text{rank}(\mathcal{M})$. Here $k(\rho)$ denotes the *weight* of ρ .

Conjecture. For general ρ and $\mathcal{M}$ without the above assumptions on them, a nonvanishing Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is *singular* if and only if

$$2k(\rho) < g + \text{rank}(\mathcal{M}).$$

REMARKS. If $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ is a Jacobi form, we may write

$$(*) \quad f(Z, W) = \sum_{a \in \mathcal{N}} f_a(Z) \cdot \vartheta_{2\mathcal{M}, a, 0}(Z, W), \quad Z \in H_g, \quad W \in C^{(h, g)},$$

where $\{f_a : H_g \rightarrow V_\rho \mid a \in \mathcal{N}\}$ are uniquely determined holomorphic functions on H_g . A singular modular form of type ρ may be written as a finite sum of theta series $\vartheta_{S, P}(Z)$'s with pluriharmonic coefficients (cf. [F]). The following problem is quite interesting.

Problem. Describe the functions $\{f_a \mid a \in \mathcal{N}\}$ explicitly given by (*) when $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ is a *singular* Jacobi form.

3 The Siegel-Jacobi Operators

In this section, we investigate the Siegel-Jacobi operator and the action of Hecke operator on Jacobi forms. The Siegel-Jacobi operator

$$\Psi_{g, r} : J_{\rho, \mathcal{M}}(\Gamma_g) \mapsto J_{\rho(r), \mathcal{M}}(\Gamma_r)$$

is defined by

$$(\Psi_{g, r} f)(Z, W) := \lim_{t \rightarrow \infty} f \left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W, 0) \right), \quad f \in J_{\rho, \mathcal{M}}(\Gamma_g),$$

$Z \in H_r$, $W \in C^{(h, r)}$ and $J_{\rho, \mathcal{M}}(\Gamma_g)$ denotes the space of all Jacobi forms of index $\mathcal{M}$ with respect to an irreducible rational finite dimensional representation ρ of $GL(g, C)$. We note that the above limit always exists because a Jacobi form f admits a Fourier expansion converging uniformly on any set of the form

$$\{(Z, W) \in H_g \times C^{(h, g)} \mid \text{Im } Z \geq Y_0 > 0, W \in K \subset C^{(h, g)} \text{ compact}\}.$$

Here the representation $\rho^{(r)}$ of $GL(r, C)$ is defined as follows. Let $V_\rho^{(r)}$ be the subspace of V_ρ generated by $\{f(Z, W) \mid f \in J_{\rho, \mathcal{M}}(\Gamma_g), (Z, W) \in H_g \times C^{(h, g)}\}$. Then $V_\rho^{(r)}$ is invariant under

$$\left\{ \begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} : g \in GL(r, C) \right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $GL(r, C)$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(g)v := \rho \left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) v, \quad g \in GL(r, C), \quad v \in V_\rho^{(r)}.$$

In the Siegel case, we have the so-called Siegel Φ -operator

$$\Phi = \Phi_{g, g-1} : [\Gamma_g, k] \longrightarrow [\Gamma_{g-1}, k]$$

defined by

$$(\Phi f)(Z) := \lim_{t \rightarrow \infty} f \left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} \right), \quad f \in [\Gamma_g, k], \quad Z \in H_{g-1},$$

where $[\Gamma_g, k]$ denotes the vector space of all Siegel modular forms on H_g of weight k .

Here $[\Gamma_g, k]$ denotes the vector space of all Siegel modular forms on H_g of weight k .

The following properties of Φ are known :

(S1) If $k > 2g$ and k is even, Φ is surjective.

(S2) If $2k < g$, then Φ is injective.

(S3) If $2k + 1 < g$, then Φ is bijective.

H. Maass([M1]) proved the statement (1) using Poincaré series. E. Freitag ([F2]) proved the statements (2) and (3) using the theory of singular modular forms.

The author([Y2]) proves the following theorems:

Theorem C. Let $2\mathcal{M} \in Z^{(h,h)}$ be a positive definite, unimodular symmetric even matrix of degree h . We assume that ρ satisfies the condition (3.1):

$$(3.1) \quad \rho(A) = \rho(-A) \quad \text{for all } A \in GL(g, C).$$

We also assume that ρ satisfies the condition $2k(\rho) < g + \text{rank}(\mathcal{M})$. Then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})$$

is injective. Here $k(\rho)$ denotes the *weight* of ρ .

Theorem D. Let $2\mathcal{M} \in Z^{(h,h)}$ be as above in Theorem A. Assume that ρ satisfies the condition (3.1) and $2k(\rho) + 1 < g + \text{rank}(\mathcal{M})$. Then The Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})$$

is an isomorphism.

Theorem E. Let $2\mathcal{M} \in Z^{(h,h)}$ be as above in Theorem A. Assume that $2k > 4g + \text{rank}(\mathcal{M})$ and $k \equiv 0 \pmod{2}$. Then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

is surjective.

The proof of the above theorems is based on the important Shimura correspondence, the theory of singular modular forms and the result of H. Maass.

We recall

$$O_g(l) := \{ M \in Z^{(2g, 2g)} \mid {}^t M J_g M = l J_g \}.$$

$O_g(l)$ is decomposed into finitely many double cosets *mod* Γ_g , i.e.,

$$(3.2) \quad O_g(l) = \cup_{j=1}^m \Gamma_g g_j \Gamma_g \quad (\text{disjoint union}).$$

We define

$$(3.3) \quad T(l) := \sum_{j=1}^m \Gamma_g g_j \Gamma_g \in \mathcal{H}^{(g)}, \quad \text{the Hecke algebra.}$$

Let $M \in O_g(l)$. For a Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$, we define

$$(3.4) \quad f|_{\rho, \mathcal{M}}(\Gamma_g M \Gamma_g) := l^{gk(\rho) - \frac{g(g+1)}{2}} \sum_i f|_{\rho, \mathcal{M}}[(M_i, [(0, 0), 0])],$$

where $\Gamma_g M \Gamma_g = \cup_i \Gamma_g M_i$ (finite disjoint union) and $k(\rho)$ denotes the weight of ρ .

Theorem F. Let $M \in O_g(l)$ and $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$. Then

$$f|_{\rho, \mathcal{M}}(\Gamma_g M \Gamma_g) \in J_{\rho, l\mathcal{M}}(\Gamma_g).$$

For a prime p , we define

$$(3.5) \quad O_{g,p} := \cup_{l=0}^{\infty} O_g(p^l).$$

Let $\check{\mathcal{L}}_{g,p}$ be the $\mathbb{C}$ -module generated by all left cosets $\Gamma_g M$, $M \in O_{g,p}$ and $\check{\mathcal{H}}_{g,p}$ the $\mathbb{C}$ -module generated by all double cosets $\Gamma_g M \Gamma_g$, $M \in O_{g,p}$. Then $\check{\mathcal{H}}_{g,p}$ is a commutative associative algebra. Since $j(\check{\mathcal{H}}_{g,p}) \subset \check{\mathcal{L}}_{g,p}$, we have a monomorphism $j : \check{\mathcal{H}}_{g,p} \longrightarrow \check{\mathcal{L}}_{g,p}$.

In a left coset $\Gamma_g M$, $M \in O_{g,p}$, we can choose a representative M of the form

$$(3.6) \quad M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^t A D = p^{k_0} E_g, \quad {}^t B D = {}^t D B,$$

$$(3.7) \quad A = \begin{pmatrix} a & \alpha \\ 0 & A^* \end{pmatrix}, \quad B = \begin{pmatrix} b & {}^t\beta_1 \\ \beta_2 & B^* \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix},$$

where $\alpha, \beta_1, \beta_2, \delta \in Z^{g-1}$. Then we have

$$(3.8) \quad M^* := \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{g-1,p}.$$

For any integer $r \in Z$, we define

$$(3.9) \quad (\Gamma_g M)^* := \frac{1}{d^r} \Gamma_{g-1} M^*.$$

If $\Gamma_g M \Gamma_g = \cup_{j=1}^m \Gamma_g M_j$ (*disjoint union*), $M, M_j \in O_{g,p}$, then we define in a natural way

$$(3.10) \quad (\Gamma_g M \Gamma_g)^* = \frac{1}{d^r} \sum_{j=1}^m \Gamma_{g-1} M_j^*.$$

We extend the above map (3.9) linearly on $\check{\mathcal{H}}_{g,p}$ and then we obtain an algebra homomorphism

$$(3.11) \quad \check{\mathcal{H}}_{g,p} \longrightarrow \check{\mathcal{H}}_{g-1,p}$$

$$T \longmapsto T^*.$$

It is known that the above map is a surjective map ([ZH] Theorem 2).

Theorem G. Suppose we have

(a) a rational finite dimensional representation

$$\rho : GL(g, C) \longrightarrow GL(V_\rho),$$

(b) a rational finite dimensional representation

$$\rho_0 : GL(g-1, C) \longrightarrow GL(V_{\rho_0})$$

(c) a linear map $R : V_\rho \longrightarrow V_{\rho_0}$ satisfying the following properties (1) and (2):

- (1) $R \circ \rho \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \rho_0(A) \circ R$ for all $A \in GL(g-1, C)$.
- (2) $R \circ \rho \begin{pmatrix} a & 0 \\ 0 & E_{g-1} \end{pmatrix} = a^r R$ for some $a \in Z$.

Then for any $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ and $T \in \check{\mathcal{H}}_{g,p}$, we have

$$(R \circ \Psi_{g,g-1})(f|T) = R(\Psi_{g,g-1}f)|T^*,$$

where T^* is an element in $\check{\mathcal{H}}_{g-1,p}$ defined by (3.11).

Corollary. The Siegel-Jacobi operator is compatible with the action of $T \mapsto T^*$. Precisely, we have the following commutative diagram:

$$\begin{array}{ccc} J_{\rho, \mathcal{M}}(\Gamma_g) & \xrightarrow{\psi_{g,g-1}} & J_{\rho^{(g-1)}, \mathcal{N}}(\Gamma_{g-1}) \\ \downarrow T & & \downarrow T^* \\ J_{\rho, \mathcal{N}}(\Gamma_g) & \xrightarrow{\psi_{g,g-1}} & J_{\rho^{(g-1)}, \mathcal{N}}(\Gamma_{g-1}) \end{array} .$$

Here $\mathcal{N}$ is a certain symmetric half integral semipositive matrix of degree h .

Definition 3.2. Let $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then we have a Fourier expansion given by (B) in Definition 1.2. A Jacobi form f is called a *cuspidal form* if $c(T, R) \neq 0$ implies $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} > 0$. We denote by $J_{\rho, \mathcal{M}}^{cusp}(\Gamma_g)$ the vector space of all cuspidal forms in $J_{\rho, \mathcal{M}}(\Gamma_g)$.

Theorem H. Let $1 \leq r \leq g$. Assume $k(\rho) > g + r + \text{rank}(\mathcal{M}) + 1$ and $k(\rho)$ even. Then

$$J_{\rho, \mathcal{M}}^{cusp}(\Gamma_r) \subset \Psi_{g,r}(J_{\rho, \mathcal{M}}(\Gamma_g)).$$

4 Final Remarks

In this section we give some open problems which should be investigated and give some remarks.

Let

$$G_R^{(g,h)} := Sp(g, R) \ltimes H_R^{(g,h)}$$

be the *Jacobi group* of degree g . Let $\Gamma_g^J := Sp(g, Z) \ltimes H_Z^{(g,h)}$ be the discrete subgroup of $G_R^{(g,h)}$. For the case $g = h = 1$, the spectral theory for $L^2(\Gamma_1^J \backslash G_R^{(1,1)})$ had been investigated almost completely in [B1] and [B-B]. For general g and h , the spectral theory for $L^2(\Gamma_g^J \backslash G_R^{(g,h)})$ is not known yet.

Problem 1. Decompose the Hilbert space $L^2(\Gamma_g^J \backslash G_R^{(g,h)})$ into irreducible components of the Jacobi group $G_R^{(g,h)}$ for general g and h . In particular, classify all the irreducible unitary or admissible representations of the Jacobi group $G_R^{(g,h)}$ and establish the *Duality Theorem* for the Jacobi group $G_R^{(g,h)}$.

Problem 2. Give the *dimension formulae* for the vector space $J_{\rho, \mathcal{M}}(\Gamma_g)$ of Jacobi forms.

Problem 3. Construct Jacobi forms. Concerning this problem, discuss the *vanishing theorem* on the vector space $J_{\rho, \mathcal{M}}(\Gamma_g)$ of Jacobi forms.

Problem 4. Develop the theory of L-functions for the Jacobi group $G_R^{(g,h)}$. There are several attempts to establish L-functions in the context of the Jacobi group by Japanese mathematicians A. Murase and T. Sugano using so-called the Whittaker-Shintani functions.

Problem 5. Give applications of Jacobi forms, for example in algebraic geometry and physics. In fact, Jacobi forms have found some applications

in proving non-vanishing theorems for L-functions of modular forms [BFH], in the theory of Heeger points [GKS], in the theory of elliptic genera [Za] and in the string theory [C].

By a certain lifting, we may regard Jacobi forms as smooth functions on the Jacobi group $G_R^{(g,h)}$ which are invariant under the action of the discrete subgroup Γ_g^J and satisfy the differential equations and a certain growth condition.

Problem 6. Develop the theory of *automorphic forms* on the Jacobi group $G_R^{(g,h)}$. We observe that the Jacobi group is *not reductive*.

Finally for historical remarks on Jacobi forms, we refer to [B2].

References

- [B1] R. Berndt, The Continuous Part of $L^2(\Gamma^J \backslash G^J)$ for the Jacobi Group G^J , Abh. Math. Sem. Univ. Hamburg **60**, 225-248 (1990).
- [B2] R. Berndt, On Automorphic Forms for the Jacobi Group, Hamburg Univ., Heft 17 (1991).
- [B-B] R. Berndt and S. Boecherer, Jacobi Forms and Discrete Series Representations of the Jacobi Group, Math. Z. **204**, 13-44 (1990).
- [BFH] D. Bump, S. Friedberg and J. Hoffstein, Nonvanishing Theorems for L-functions of Modular Forms and their Derivatives.
- [C] J. L. Cardy, Operator Content of Two-dimensional Conformally Invariant Theories, Nuclear Physics B **270**, 186-204 (1986).

- [E-Z] Eichler, M. and Zagier, D., The Theory of Jacobi forms, Progress in math., **55**, Birkhäuser, Boston-Basel-Stuttgart (1985)
- [F] E. Freitag, Thetareihen mit harmonischen Koeffizienten zur Siegelschen Modulgruppe, Math. Ann. **254**, 27-51 (1980)
- [GKZ] B. Gross, W. Kohnen and D. Zagier, Heeger Points and the Derivative of L-series II. Math. Ann. **278**, 497-562 (1987).
- [Y1] J.-H. Yang, The Differential Operators and Singular Jacobi Forms, preprint (1991).
- [Y2] J.-H. Yang, The Siegel-Jacobi Operator, preprint (1992).
- [Y-S] J.-H. Yang and Jin Woo Son, A note on Jacobi forms of higher degree, Jour. Korean Math. Soc., 341-358 (1991)
- [Za] D. Zagier, Note on the Landweber-Stong Elliptic Genus, Springer Lecture Notes **1326**, 216-224.
- [Zi] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg **59**, 191-224 (1989)

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