

INEQUALITIES of ORDERS on LOCAL ANALYTIC ALGEBRAS

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1. Orders and generic rank

Let X be an analytic space over $k=\mathbb{R}$ or $\mathbb{C}$ and (O, m) $= (O_{X, \xi}, m_{\xi})$ the local ring at $\xi \in X$. O can be expressed as $O = k\{x\}/I$ for an algebra $k\{x\}$ of convergent power series in $x = (x_1, \dots, x_n)$ and an ideal $I \subset k\{x\}$. We define three kinds of orders for $f \in O$ as follows.

algebraic order: $v(f) := \sup\{p: f \in m^p\}$

reduced order ([L-T]): $\bar{v}(f) := \lim_{k \rightarrow \infty} v(f^k)/k$

analytic order along $A \subset |X|$: $\mu_{A, \xi}(f) := \sup\{p: \exists \alpha > 0, \exists \text{ nbd. } U \text{ of } \xi, \exists \text{ representative } \tilde{f} \text{ of } f \text{ over } U \text{ such that}$

$|f(x)| \leq \alpha |x - \xi|^p \text{ for } x \in A \cap U\}$

We know the following inequalities.

$\exists N \in \mathbb{N}: v(f) \leq \bar{v}(f) \leq v(f) + N \quad (f \in O) \quad ([L-T])$

if $A \supset B$, $\mu_{A, \xi}(f) \leq \mu_{B, \xi}(f) \quad (f \in O)$

Example 1. If $O = \mathbb{R}\{x, y\}$ and if $A = \{(x, y): |y| \leq |x|^p\} \quad (p \geq 1)$, then $v(y) = 1$ and $\mu_{A, 0}(y) = p$.

Example 2. If $O = \mathbb{R}\{x, y\}/(y^2 - x^3)$, then $v(y) = 1$ and $\bar{v}(y) = 3/2$.

Let $\phi: Y \rightarrow X$ be an analytic map such that $\phi(\eta) = \xi$. We define the generic rank $\text{grnk}_{\eta} \phi$ of ϕ at η as follows.

$\text{grnk}_{\eta} \phi := \varepsilon \cdot \inf\{\text{the topological dimension of } \phi(U): U \text{ is a nbd. of } \eta\} \quad (\varepsilon = 1 \text{ if } k = \mathbb{R} \text{ and } \varepsilon = 1/2 \text{ if } k = \mathbb{C}).$

2. The theorem of Lejeune and Teissier

The following is the most basic result on orders on

complex analytic algebras.

A. Lemma ([L-T], the original form is more general). Let X be a complex space reduced at ξ . Then, for $f \in \mathcal{O}$, the following conditions are equivalent.

$$(r) \quad \bar{v}(f) \geq p.$$

$$(l) \quad \mu_{|X|, \xi}(f) \geq p.$$

$$(i) \quad \exists s \in \mathbb{N}, \exists \sigma_1 \in m^{\text{pi}}_1: f^s - \sigma_1 f^{s-1} + \sigma_2 f^{s-2} - \dots + \sigma_s = 0.$$

(e) For $\forall$ (or $\exists$) proper surjective analytic map $\Pi: Y \rightarrow X$ such that Y is normal and $m_{\mathcal{O}_Y}$ is invertible, there exists a representative $\tilde{f}$ of f over U such that $\tilde{f} \in m^p_{\mathcal{O}_Y}(\Pi^{-1}(U))$.

(c) For any analytic map $\Phi: D \rightarrow X$ with $\Phi(0) = \xi$, we have $v_0(f \circ \Phi) \geq p \cdot \inf\{v_0(g \circ \Phi): g \in m\}$ (D is the unit open disc in $\mathbb{C}$).

3. The main theorem

The following inequalities are well-known.

$$(1) \quad v(fg) \geq v(f) + v(g) \quad (f, g \in \mathcal{O}).$$

$$(2) \quad v_\eta(f \circ \Phi) \geq v_\xi(f) \quad (f \in \mathcal{O}) \quad \text{for any analytic map } \Phi: (Y, \eta) \rightarrow (X, \xi).$$

$$(3) \quad \mu_{A, \xi}(f) \geq v(f) \quad (f \in \mathcal{O}) \quad \text{for any } A \subset |X|.$$

B. Theorem. Let X be a complex space reduced and irreducible at ξ (i.e. $\mathcal{O}$ is an integral domain) or a real analytic space whose complexification is reduced and irreducible at ξ (i.e. $\mathcal{O} \otimes_{\mathbb{R}} \mathbb{C}$ is an integral domain). Then we have the following.

$$(1) \quad \exists a_1 \geq 1, \exists b_1 \geq 0: v(fg) \leq a_1(v(f) + v(g)) + b_1 \quad (f, g \in \mathcal{O}).$$

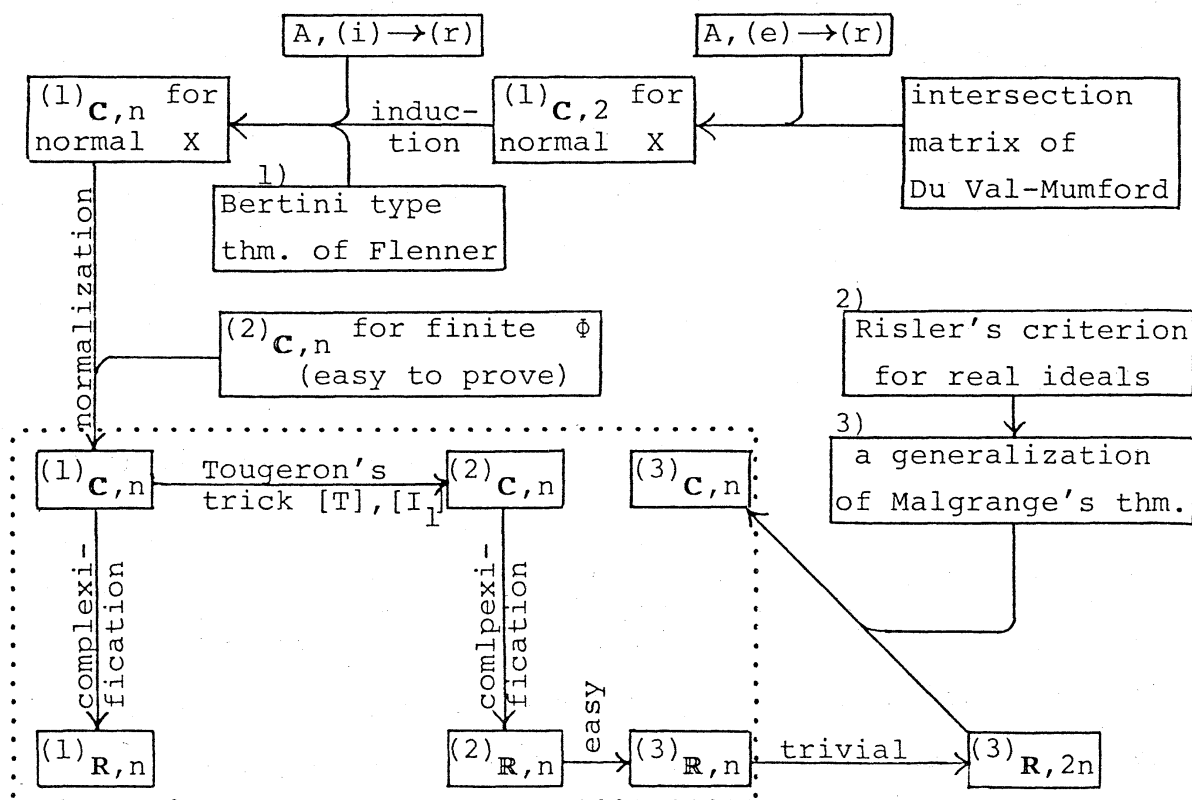
$$(2) \quad \text{If } \text{grnk}_\eta \Phi = \dim X_\xi, \exists a_2 \geq 1, \exists b_2 \geq 0: v(f \circ \Phi) \leq a_2 v(f) + b_2 \quad (f \in \mathcal{O}).$$

$$(3) \quad \text{If } A \text{ is an open subanalytic set adherent to } \xi, \exists a_3 \geq 1,$$

$$\left\{ \exists b_3 \geq 0; \mu_{A, \xi}(f) \leq a_3 v(f) + b_3 \quad (f \in \mathcal{O}) \right\}.$$

If $\mathcal{O}$ is not an integral domain, a_1 does not exist ($\inf a_1 = \infty$). So we may consider $1/\inf a_1$ as the distance of $\mathcal{O}$ from non-integral domains. If $X=k$, $\inf a_2$ coincides with the reduced order of function ϕ . Hence $\inf a_2$ may be seen as a kind of order of analytic map ϕ . $1/\inf a_3$ expresses something about the size of A_ξ .

If $n := \dim X_\xi = 1$, $B, (1)$ follows from $A, (e) \rightarrow (r)$ immediately. The rest is proved in the following way ((1) $_{\mathbb{C}, n}$ means the assertion (1) for n -dimensional complex space, etc.).



1) Let $(\mathcal{O}, m)$ be an analytically irreducible local k -algebra and let $\varphi_1, \dots, \varphi_p \in m$. Flenner has given a sufficient

condition under which a general linear combination of φ_i generate a prime ideal of $\hat{O}$ (even of $\hat{O}$).

2) In the real case Nullstellensatz does not hold for radicals but for real ideals (real radicals). Risler has proved that, if $I \subset \mathbb{R}\{x\}$ is a prime ideal, I is a real ideal iff $\dim \mathbb{R}\{x\}/I = (\text{topological dimension of } V(I) \text{ (the real analytic germ defined by } I))$.

3) A germ X_ξ of a complex analytic set can be canonically considered as a germ X_ξ^r of a real analytic set. Malgrange has proved that, if X_ξ is irreducible, X_ξ^r is also so together with its complexification $\tilde{X}_\xi$. We use the ringed space version of this theorem.

4. p-th power in $\mathbb{C}\{x\}$.

Let p be a prime number and suppose that u is not a p-th power in $\mathbb{C}\{x\}$ ($x = (x_1, \dots, x_n)$). If we apply $B, (1)$ to $O = \mathbb{C}\{x, y\}/(u - (-y)^p)$, we have the following.

C. Theorem. There exist $a \geq p$, $b \geq v(u)$, $b' \geq 0$ depending only on n, p, u such that $av(f) + b \geq v(f^p u - g^p)$, $av(g) + b' \geq v(f^p u - g^p)$ ($f, g \in \hat{O}$).

We can consider $\theta(p, u) := p / \inf a$ as a distance of u from p-th powers in $\mathbb{C}\{x\}$.

5. Some problems and remarks.

1) Let X be a complex space reduced and irreducible everywhere. Is a_1 in B locally bounded? cf. $[R_2]$, p.259.

2) Let $\phi: Y \rightarrow X$ be an analytic map such that X is reduced

and irreducible at $\xi = \Phi(\eta)$. Does existence of a_1 imply Gabrielov's regularity: $\text{grnk}_\eta \Phi = \dim X_\xi$? cf. [B].

3) Do B , (1) and C hold for more general rings and ideals?

Example 3. Let A be a finitely generated ring over $\mathbb{C}$ and $\mathfrak{m}$ one of its maximal ideal. Suppose that the completion $\hat{A}_\mathfrak{m}$ of the localization $A_\mathfrak{m}$ is an integral domain. Then B , (1) implies a similar inequality for A .

4) Are $\inf a_1$ and $\inf b_1$ attained? Are they rational numbers?

5) If g is fixed, B , (1) holds with $a_1=1$ by the theorem of Artin-Rees. If we do not care the linearity in the inequalities, B , (1) and C follows from Artin's strong approximation theorem ([A]).

(The detailed proofs of the results will be given in [I₂]).

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