

A CATEGORIAL ANALYSIS OF LAMBDA CALCULUS MODELS

(Extended Abstract)

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The theory of models for type free lambda calculus was initiated by Dana Scott with the discovery of the D_∞ model. His construction of models consists of two parts: one is that from the category CL of continuous lattices to a reflexive domain, and another is that from the reflexive domain to a λ -model. Our question which triggered our research is the following:

Is it essential to use partial order relations or some topological properties in the second part of Scott's construction?

In this paper we investigate this problem and have the following two results.

- (1) Every λ -model is the induced groupoid of some reflection of a cartesian closed category.
- (2) The induced groupoid of any reflection of ξ -extensional cartesian closed categories can be made into a λ -model.

The first statement says that for every λ -model (even for a graph model), there is a categorial characterization similar to that of the D_∞ model. On the other hand, the second statement gives a sufficient condition for making the induced groupoid of the given reflection into a λ -model. So, we can solve the above problem negatively.

The readers may refer to [Bar81] and [Mac71] for the notions that are used in this paper without definitions.

Extensionality

In this chapter a characterization of the condition of weak extensionality is given in terms of the concepts of extensional subsets.

DEFINITION 1.1. Let $\mathcal{M} = (X, \cdot)$ be a groupoid and $S \subseteq X$. Then S is called extensional if

$$\forall a, b \in S (\forall c \in X. ac = bc) \rightarrow a = b.$$

DEFINITION 1.2. For a $p\lambda A$ $\mathcal{M} = (X, \cdot, \lambda^*)$, $F_{\mathcal{M}} = \{ (\lambda^*x. A)_\rho \mid A \in \mathcal{T}(\mathcal{M}), x \in \text{Vars}, \rho \in \text{Vals} \}$.

THEOREM 1.3. Let $\mathcal{M}$ be a $p\lambda A$. Then $\mathcal{M}$ is weakly extensional iff $F_{\mathcal{M}}$ is extensional.

From Lambda Models to Cartesian Closed Categories

2.1. Retracts of λ -models

In this section we introduce the notion of retracts of λ -models and prove that the set of all retracts forms a cartesian closed category (c. c. c.).

Let $\mathcal{M} = (X, \cdot, \lambda^*)$ be a fixed λ -model throughout this chapter, and we shall write F instead of $F_{\mathcal{M}}$.

PROPOSITION 2.1. F is extensional.

DEFINITION 2.2. For $a, b \in X$,

- (i) $a \circ b = (\lambda^*x. c_a(c_b x))_\rho$,
- (ii) $a \rightarrow b = (\lambda^*xy. c_b(x(c_a y)))_\rho$,
- (iii) $i = (\lambda^*x. x)_\rho$.

DEFINITION 2.3. (i) An element r of X is called a retract if $r \circ r = r$.

(ii) $\text{Ret} = \{ r \in X \mid r \text{ is a retract.} \}$.

In this chapter, $r, r_1, r_2, \dots$ denote arbitrary retracts.

DEFINITION 2.4. (i) An element a of X is called to have a type r , notation $a : r$, if $a = ra$.

(ii) $\text{RET}(r_1, r_2) = \{ \langle a, r_1, r_2 \rangle \mid a : r_1 \rightarrow r_2 \}$.

(iii) $A = \bigcup (r_1, r_2) \in \text{Ret} \times \text{Ret} \text{ RET}(r_1, r_2)$.

(iv) $\circ$ is a partial binary operator on A such that

(1) $\langle b, r_3, r_4 \rangle \circ \langle a, r_1, r_2 \rangle$ is defined iff $r_3 = r_2$,

(2) $\langle b, r_2, r_3 \rangle \circ \langle a, r_1, r_2 \rangle = \langle b \circ a, r_1, r_3 \rangle$.

(v) i is a function from Ret to A such that $i(r) = \langle r, r, r \rangle$.

(vi) $\text{RET} = (\text{Ret}, A, \circ, i)$.

THEOREM 2.5. The structure RET is a category.

In the rest of this chapter we shall write a and $b \circ a$ instead of $\langle a, r_1, r_2 \rangle$ and $\langle b, r_2, r_3 \rangle \circ \langle a, r_1, r_2 \rangle$, respectively if there occurs no confusion.

DEFINITION 2.6. (i) $1 = (\lambda^*x. i)_p$.

(ii) $!_r = \langle 1, r, 1 \rangle$.

(iii) $T \equiv \lambda^*xy. x$.

(iv) $F \equiv \lambda^*xy. y$.

(v) $a \times b = (\lambda^*xy. y(c_a(xT))(c_b(xF)))_p$.

(vi) $p_{ab} = (\lambda^*x. c_a(xT))_p$.

(vii) $q_{ab} = (\lambda^*x. c_b(xF))_p$.

(viii) $\langle a, b \rangle = (\lambda^*xy. y(c_a x)(c_b x))_p$.

(ix) $e_{ab} = (\lambda^*x. c_b(xT(c_a(xF))))_p$.

(x) $a^+ = (\lambda^*xy. c_a(\lambda^*z. zxy))_p$.

We can show that $1, (r_1 \times r_2, p_{r_1 r_2}, q_{r_1 r_2})$ and $(r_1 \rightarrow r_2, e_{r_1 r_2})$ are a terminal, a product and an exponentiation of two retracts r_1 and r_2 , respectively.

THEOREM 2.7. The structure

$\text{RET} = (\text{Ret}, A, \circ, i, 1, !, \times, p, q, \langle \rangle, \rightarrow, e, +)$ is a c. c. c.

2.2. A Reflection of the Category of Retracts

In this section we show that the groupoid $(X, .)$ can be seemed as a induced groupoid of some reflection of the c. c. c. RET.

First, we define the notions of reflections and their induced groupoids.

DEFINITION 2.8. Let the structure

$\mathcal{C} = (O, A, ., i, 1, !, \times, p, q, \langle \rangle, \rightarrow, e, +)$ be a c. c. c.

(i) For $a \in \mathcal{C}$, $\tilde{a} = \mathcal{C}(1, a)$.

(ii) A triple (r, f, g) with $r \in \mathcal{C}$, $f \in \mathcal{C}(r \rightarrow r, r)$

and $g \in \mathcal{C}(r, r \rightarrow r)$ is called a reflection if

(1) $\text{Card}(\tilde{r}) > 1$,

(2) $g \circ f = i_{r \rightarrow r}$.

(iii) The induced groupoid of a reflection (r, f, g) is a groupoid $(\tilde{r}, *)$, where $a * b = e_{rr} \langle ga, b \rangle$ for each a and b in $\tilde{r}$.

PROPOSITION 2.9. A triple $(i, i \rightarrow i, i \rightarrow i)$ is a reflection.

DEFINITION 2.10. The function $\varphi: \tilde{i} \rightarrow X$ is defined by $\varphi(a) = a_i$ for each $a \in \tilde{i}$.

THEOREM 2.11. The function φ is an isomorphism.

Identifying all the isomorphic structures according to the custom in algebra, we can sum up the results of this chapter in the following.

COROLLARY 2.12. For a groupoid $\mathcal{M}$, if $\mathcal{M}$ can be made into a λ -model, then $\mathcal{M}$ is the induced groupoid of some reflection of a c. c. c.

The converse of this corollary will be investigated in the next chapter.

From Cartesian Closed Categories to Lambda Models

3.1. λ -theories and Cartesian Closed Categories

In this section we introduce the equational theory λ as a tool of studying syntactic aspects of c. c. c.

DEFINITION 3.1. (i) A class of primitive types P is a non-trivial class with an initial type $0 \in P$.

(ii) The class of types over a class of primitive types P , notation Typ_P or Typ , is a class inductively defined by

- (1) $P \subset \text{Typ}_P$,
- (2) $t_1, t_2 \in \text{Typ}_P \Rightarrow (t_1 \times t_2) \in \text{Typ}_P$,
- (3) $t_1, t_2 \in \text{Typ}_P \Rightarrow (t_1 \rightarrow t_2) \in \text{Typ}_P$.

In this section $t, t_1, \dots$ denote arbitrary types in Typ_P .

DEFINITION 3.2. (i) For each $t \in \text{Typ}$, Vars_t is a given countable set such that $t_1 = t_2 \Rightarrow \text{Vars}_{t_1} \cap \text{Vars}_{t_2} = \emptyset$.

(ii) $\text{Vars} = \bigcup_{t \in \text{Typ}} \text{Vars}_t$.

DEFINITION 3.3. (i) The class of λ -terms over P , notation Γ_P or Γ , is a class expressed as a disjoint union of sets of the form $\Gamma_P = \bigcup \{ \Gamma_P(t_1, t_2) \mid t_1, t_2 \in \text{Typ}_P \}$, where $\Gamma_P(t_1, t_2)$ is the family of minimum sets satisfying the following nine conditions;

- (1) $\text{Vars}_t \subset \Gamma_P(0, t)$, (2) $I_t \in \Gamma_P(t, t)$, (3) $0_t \in \Gamma_P(t, 0)$,
 - (4) $P_{t_1, t_2} \in \Gamma_P(t_1 \times t_2, t_1)$, (5) $Q_{t_1, t_2} \in \Gamma_P(t_1 \times t_2, t_2)$,
 - (6) $E_{t_1, t_2} \in \Gamma_P((t_1 \rightarrow t_2) \times t_1, t_2)$,
 - (7) $A \in \Gamma_P(t_1, t_2)$ and $B \in \Gamma_P(t_2, t_3) \Rightarrow (BA) \in \Gamma_P(t_1, t_3)$,
 - (8) $A \in \Gamma_P(t_1, t_2)$ and $B \in \Gamma_P(t_1, t_3) \Rightarrow \langle A, B \rangle \in \Gamma_P(t_1, t_2 \times t_3)$,
 - (9) $A \in \Gamma_P(t_1 \times t_2, t_3) \Rightarrow A^+ \in \Gamma_P(t_1, t_2 \rightarrow t_3)$.
- (ii) $I_t, 0_t, P_{t_1, t_2}, Q_{t_1, t_2}$ and E_{t_1, t_2} are called constant

symbols.

(iii) Let $A, B \in \Gamma_P(t_1, t_2)$. Then a form $A = B$ is called a

γ -formula.

(iv) Let A and B be γ -terms. Then a notation $A \equiv B$ denotes syntactic equality.

$A, B, \dots$ denote arbitrary γ -terms, and all the subscripts which does not cause any ambiguity will be omitted.

DEFINITION 3.4. (i) The formal theory γ over P is an equational theory on Γ_P whose axiom schemes and deduction rules are the following;

- | | |
|---|--|
| (1) $A = A,$ | (2) $(AB)C = A(BC),$ |
| (3) $IA = A,$ | (4) $AI = A,$ |
| (5) $A = 0_t$ for $A \in \Gamma(t, 0),$ | (6) $P\langle A, B \rangle = A,$ |
| (7) $Q\langle A, B \rangle = B,$ | (8) $\langle PA, QA \rangle = A,$ |
| (9) $E\langle A^+P, Q \rangle = A,$ | (10) $(E\langle AP, Q \rangle)^+ = A,$ |
| (11) $\frac{A = B}{B = A},$ | (12) $\frac{A = B, B = C}{A = C},$ |
| (13) $\frac{A = C, B = D}{AB = CD},$ | (14) $\frac{A = C, B = D}{\langle A, B \rangle = \langle C, D \rangle},$ |
| (15) $\frac{A = B}{A^+ = B^+}.$ | |

(ii) A notation $\gamma \vdash A = B$ is defined as usual.

(iii) For $A \in \Gamma$, $FV(A) \subset \text{Vars}$ is the set of all variables occurred in A .

(iv) For $A \in \Gamma$, $x \in \text{Vars}$ and $B \in \Gamma(0, t)$, $A[x := B]$ is the γ -term obtained by substituting all the x in A by B .

DEFINITION 3.5. (i) For $x \in \text{Vars}_{t_1}$ and $A \in \Gamma(t_3, t_2)$, $kx. A \in \Gamma(t_3 \times t_1, t_2)$ is defined by

- (1) $kx. x \equiv 0,$

- (2) $kx. y \equiv yP$ if $x \neq y \in \text{Vars}$,
- (3) $kx. C \equiv CP$ if C is a constant symbol,
- (4) $kx. AB \equiv (kx. A) \langle kx. B, Q \rangle$,
- (5) $kx. \langle A, B \rangle \equiv \langle kx. A, kx. B \rangle$,
- (6) $kx. A^+ \equiv ((kx. A) \langle \langle PP, Q \rangle, QP \rangle)^+$.
- (ii) For $x \in \text{Vars}_{t_1}$ and $A \in \Gamma(t_3, t_2)$, $\lambda x. A \in \Gamma(t_3, t_1 \rightarrow t_2)$ is defined by $\lambda x. A \equiv (kx. A)^+$.

The following theorem is called the functional completeness theorem for the theory $\mathcal{Y}$.

THEOREM 3.6. For $A \in \Gamma(t_3, t_2)$, $x \in \text{Vars}_{t_1}$ and $B \in \Gamma(0, t_1)$,

- (i) $FV(\lambda x. A) = FV(A) - \{x\}$.
- (ii) $\mathcal{Y} \vdash E_{t_1, t_2} \langle \lambda x. A, B \rangle_{t_3} = A[x := B]$.

Let $\mathcal{C} = (0, A, ., i, 1, !, X, p, q, \langle \rangle, \rightarrow, e, +)$ be a fixed c. c. c. in the rest of this chapter.

- DEFINITION 3.7. (i) $P = \{t\} \times \mathcal{O}$.
- (ii) We write t_a instead of (t, a) .
- (iii) $0 \equiv t_1$ (initial type).

PROPOSITION 3.8. P is a class of primitive types.

- DEFINITION 3.9. (i) $\text{Type}_{\mathcal{C}} = \text{Type}_P$.
- (ii) For $t \in \text{Type}_{\mathcal{C}}$, $\bar{t} \in \mathcal{C}$ is defined by
- (1) $\bar{t}_a = a$, (2) $\overline{t_1 \times t_2} = \bar{t}_1 \times \bar{t}_2$, (3) $\overline{t_1 \rightarrow t_2} = \bar{t}_1 \rightarrow \bar{t}_2$.

DEFINITION 3.10. (i) The class of extended $\mathcal{Y}$ -terms over $\mathcal{C}$, notation $\Gamma_{\mathcal{C}}$, is a class expressed as a disjoint union of sets of the form $\Gamma_{\mathcal{C}} = \bigcup \{ \Gamma_{\mathcal{C}}(t_1, t_2) \mid t_1, t_2 \in \text{Type}_{\mathcal{C}} \}$, where $\Gamma_{\mathcal{C}}(t_1, t_2)$ is the family of minimum sets satisfying the following;

- (1) $\text{Vars}_t \subset \Gamma_{\mathcal{C}}(0, t)$ (variables),
- (2) $f \in \mathcal{C}(\bar{t}_1, \bar{t}_2) \Rightarrow f \in \Gamma_{\mathcal{C}}(t_1, t_2)$ (constant symbols),

- (3) $A \in \Gamma_{\mathcal{C}}(t_1, t_2)$ and $B \in \Gamma_{\mathcal{C}}(t_2, t_3) \Rightarrow (BA) \in \Gamma_{\mathcal{C}}(t_1, t_3)$,
 (4) $A \in \Gamma_{\mathcal{C}}(t_1, t_2)$ and $B \in \Gamma_{\mathcal{C}}(t_1, t_3)$
 $\Rightarrow \langle A, B \rangle \in \Gamma_{\mathcal{C}}(t_1, t_2 \times t_3)$,
 (5) $A \in \Gamma_{\mathcal{C}}(t_1 \times t_2, t_3) \Rightarrow A^+ \in \Gamma_{\mathcal{C}}(t_1, t_2 \rightarrow t_3)$.
 (ii) $I_t, O_t, P_{t_1, t_2}, Q_{t_1, t_2}$ and E_{t_1, t_2} are names of $i_{\bar{t}}, !\bar{t}, p_{\bar{t}_1, \bar{t}_2},$
 $q_{\bar{t}_1, \bar{t}_2}$ and $e_{\bar{t}_1, \bar{t}_2}$, respectively.

DEFINITION 3.11. (i) A function $\rho : \text{Vars} \rightarrow \mathcal{A}$ is called a valuation in $\mathcal{C}$ if it satisfies $x \in \text{Vars}_t \Rightarrow \rho(x) \in \mathcal{C}(1, \bar{t})$.

(ii) For $A \in \Gamma_{\mathcal{C}}$ and a valuation ρ , the interpretation of A in $\mathcal{C}$ under ρ , notation $[A]_{\rho}^{\mathcal{C}}$ or $[A]_{\rho}$, is defined by

- (1) $[x]_{\rho}^{\mathcal{C}} = \rho(x)$ if $x \in \text{Vars}$, (2) $[c_f]_{\rho}^{\mathcal{C}} = f$,
 (3) $[AB]_{\rho}^{\mathcal{C}} = [A]_{\rho}^{\mathcal{C}}[B]_{\rho}^{\mathcal{C}}$, (4) $[\langle A, B \rangle]_{\rho}^{\mathcal{C}} = \langle [A]_{\rho}^{\mathcal{C}}, [B]_{\rho}^{\mathcal{C}} \rangle$,
 (5) $[A^+]_{\rho}^{\mathcal{C}} = ([A]_{\rho}^{\mathcal{C}})^+$.

DEFINITION 3.12. Let $A, B \in \Gamma_{\mathcal{C}}$ and ρ be a valuation.

- (i) $\mathcal{C}, \rho \models A = B$ iff $[A]_{\rho}^{\mathcal{C}} = [B]_{\rho}^{\mathcal{C}}$.
 (ii) $\mathcal{C} \models A = B$ iff $\mathcal{C}, \rho \models A = B$ for every valuation ρ .

DEFINITION 3.13. The extended γ -theory over $\mathcal{C}$, notation $\gamma(\mathcal{C})$, is the extension of the theory γ obtained by validating the axiomschemas and rules also for terms in $\Gamma_{\mathcal{C}}$.

The following is the key theorem to understand the relation between γ -theories and c. c. c.

THEOREM 3.14. For extended γ -terms A and B ,

$$\gamma(\mathcal{C}) \vdash A = B \Rightarrow \mathcal{C} \models A = B.$$

DEFINITION 3.15. Notations $FV(A)$, $A[x := B]$, $\mathbf{k}x. A$ and $\lambda x. A$ are extended for terms in $\Gamma_{\mathcal{C}}$ reasonably.

Theorem 3.6 remains valid for the theory $\gamma(\mathcal{C})$.

THEOREM 3.16. For $A \in \Gamma_{\mathcal{C}}(t_3, t_2)$, $x \in \text{Vars}_{t_1}$ and $B \in \Gamma_{\mathcal{C}}(0, t_1)$, $\mathcal{C} \models E_{t_1, t_2} \langle \lambda x. A, B O_{t_3} \rangle = A[x := B]$.

This theorem is a generalization of the result of Lambek

[Lam74].

DEFINITION 3.17. $\mathcal{C}$ is ξ -extensional if
for all $A, B \in \Gamma_{\mathcal{C}}$ and $x \in \text{Vars}$, $\mathcal{C} \models A = B \Rightarrow \mathcal{C} \models \lambda x. A = \lambda x. B$.

3.2. Reflections of Cartesian Closed Categories

In this section we show that the induced groupoid of any reflection of c. c. c. can be made into a $p\lambda A$.

Let (r, f, g) be a fixed reflection of $\mathcal{C}$, and $(\tilde{r}, *)$ be the induced groupoid of (r, f, g) .

DEFINITION 3.18. (i) $t \equiv t_r$.

(ii) $\mathcal{R} = \Gamma_{\mathcal{C}}(0, t)$.

(iii) $F \equiv c_f$.

(iv) $G \equiv c_g$.

(v) For $A, B \in \mathcal{R}$, $A * B \equiv E_{tt}(GA, B)$.

In this section $A, B, \dots$ denote arbitrary extended λ -terms in $\mathcal{R}$, and $x, y, \dots$ denote arbitrary variables in Vars_t .

DEFINITION 3.19. (i) $\lambda^{\circ}x. A \equiv F(\lambda x. A)$.

(ii) $\lambda^{\circ}x_1 \dots x_n. A \equiv \lambda^{\circ}x_1. (\lambda^{\circ}x_2. (\dots (\lambda^{\circ}x_n. A) \dots))$.

(iii) $k = [\lambda^{\circ}xy. x]_p$.

(iv) $s = [\lambda^{\circ}xyz. x * z * (y * z)]_p$.

(v) For $A \in \mathcal{T}(\tilde{r}, *)$ and $x \in \text{Vars}_t$,

$\lambda^{\circ}x. A \in \mathcal{T}(\tilde{r}, *)$ is defined by

(1) $\lambda^{\circ}x. x \equiv c_{skk}$, (2) $\lambda^{\circ}x. y \equiv c_k y$ if $y \neq x$.

(3) $\lambda^{\circ}x. c_f \equiv c_k c_f$, (4) $\lambda^{\circ}x. AB \equiv c_s(\lambda^{\circ}x. A)(\lambda^{\circ}x. B)$.

In the rest of this section, $\mathcal{M}$ denotes $(\tilde{r}, *, \lambda^{\circ})$.

THEOREM 3.20. $\mathcal{M}$ is a $p\lambda A$.

For making $\mathcal{M}$ into a λ -model, it is sufficient that $F_{\mathcal{M}}$ is extensional by 1.3. We shall prove that the condition of ξ -extensionality is sufficient to make $F_{\mathcal{M}}$ extensional in the

next theorem.

THEOREM 3.21. The induced groupoid of any reflection of ξ -extensional c. c. c. can be made into a λ -model.

This theorem is a partial result concerning with the converse of 2.12.

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