博　士　論　文

On an algebro-geometric realization of the cohomology ring of conical symplectic resolutions

（錐的シンプレクティック特異点解消のコホモロジー環の代数幾何学的実現について）

疋田　辰之

京都大学大学院理学研究科

数学・数理解析専攻（数学系）

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On an algebro-geometric realization of the cohomology ring of conical symplectic resolutions

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Tatsuyuki Hikita

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Abstract

DeConcini-Procesi and Tanisaki show that the cohomology rings of Springer fibers of type A are isomorphic to the coordinate rings of scheme-theoretic intersections of some nilpotent orbit closures and a Cartan subalgebra. In this thesis, we give some variants of their result to other situations by reinterpreting them as certain isomorphisms between the cohomology rings of some symplectic varieties and the coordinate rings of some schemes coming from another symplectic varieties, which are known as symplectic duality. In particular, we show that the cohomology ring of the Hilbert scheme of n-points in the affine plane is isomorphic to the coordinate ring of the $\mathbb{G}_m$-fixed point scheme of the n-th symmetric product $S^n\mathbb{C}^2$ of $\mathbb{C}^2$ for a natural $\mathbb{G}_m$-action on $S^n\mathbb{C}^2$.

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Contents

CHAPTER 1

Introduction

One of the celebrated results in geometric representation theory is the localization theorem of Beilinson-Bernstein [**1**] and Brylinski-Kashiwara [**5**]. This theorem states that a certain category of $U(\mathfrak{g})$-modules is equivalent to a certain category of (twisted) $\mathcal{D}$-modules on the flag variety. This gives a connection between the representation theory of $\mathfrak{g}$ and the perverse sheaves on flag variety via the Riemann-Hilbert correspondence, which yields a proof of the Kazhdan-Lusztig conjecture on the character of irreducible highest weight representation. This theorem has some variants to other situations. One of such variant is given by the work of Gordon-Stafford [**13**] and Kashiwara-Rouquier [**25**] on a connection between the representation theory of rational Cherednik algebras (RCA) of type A and the quantizations of structure sheaves of Hilbert schemes of points in the affine plane. In some cases, this gives algebro-geometric realizations of associated graded modules of representations of RCA with respect to some filtrations [**14**], like filtered $\mathcal{D}$-modules on a smooth variety X give quasi-coherent sheaves on the cotangent bundle T^*X by taking its associated graded. For example, the ring of diagonal coinvariants DR_n, which is the associated graded of the unique finite dimensional representation $L_{-\frac{n+1}{n}}$ of RCA when the parameter equals to $-\frac{n+1}{n}$ [**12**], is isomorphic to the global section of some vector bundle (called the Procesi bundle) restricted to the punctual Hilbert scheme [**21**]. This connection between diagonal coinvariants and Hilbert schemes of points yields a celebrated result of Haiman [**21**] on the determination of bigraded characters of DR_n in terms of a certain operator (called the ∇ operator) on symmetric functions. This is one side of the story about geometric approach toward the combinatorics of DR_n.

Another celebrated result in geometric representation theory is the proof of the Deligne-Langlands conjecture by Kazhdan-Lusztig [**26**] on the classification of irreducible reresentations of affine Hecke algebras. The point is that affine Hecke algebras can be realized by the equivariant K-theory of Steinberg varieties with its product structure given by the convolution product. Then one can construct standard representations of affine Hecke algebras by considering the K-theory of Springer fibers. General formalism on the classification of irreducible representations of convolution algebras was developed by Chriss-Ginzburg [**6**] and applied in other situations such as quiver varieties [**36**]. Among them, Vasserot [**43**] gives an affine analogue of the above theory of Kazhdan-Lusztig and classified irreducible representations of double affine Hecke algebra (DAHA) in terms of the homology of affine Springer fibers (see also [**41**] or [**37**]). Since the representation theory of DAHA and the representation theory of RCA are similar, one can give another construction of some finite dimensional representations of RCA via the homology of affine Springer fibers. For type A, we can get another geometric realization of $L_{\frac{n+1}{n}} \cong L_{-\frac{n+1}{n}} \otimes \mathrm{sgn}$. In our previous work [**23**], we construct certain filtration on the homology of certain affine Springer fibers of type A, which gives a bigrading on its associated graded. We can give a combinatorial description of its bigraded character which is quite different from the formula obtained by the geometry of Hilbert schemes. In fact, the coincidence of these two formulas is known as the shuffle conjecture [**18**] in the combinatorics of diagonal coinvariants. Hence if we can prove the coincidence of bigradings coming from Hilbert scheme of points and affine Springer fibers, then the shuffle conjecture follows. Conversely, the shuffle conjecture suggests that there should be some relations between algebraic geometry of Hilbert of points and topology of affine Springer fibers.

The aim of this thesis is to find some other examples of this kind of phenomenon, that is, equalities between algebro-geometric objects and topological objects. One hint already exist in the work of DeConcini-Procesi [**8**] and Tanisaki [**40**]. They proved that the cohomology rings of Springer fibers of type A are isomorphic to the coordinate rings of scheme-theoretic intersections of certain nilpotent orbit closures and a Cartan subalgebra. Another hint is implicit in the work of Lusztig [**32**] on bases in equivariant K-theory of Springer fibers. It suggests that the cohomology rings of partial flag varieties are isomorphic to the coordinate rings of certain explicit closed subschemes of Slodowy slices. The nilpotent orbit closures appeared above can be resolved by cotangent bundles of partial flag varieties, and Slodowy slices can be resolved by Slodowy varieties which are homotopy equivalent to Springer fibers. That is, the cohomology rings of Springer fibers of type A can be written as the coordinate rings of some schemes related to partial flag varieties, and conversely, the cohomology rings of partial flag varieties can be written as the coordinate rings of some schemes related to Springer fibers of type A. We remark that the Slodowy varieties and the cotangent bundles of partial flag varieties appearing here are examples of symplectic duality in the sense of Braden-Licata-Proudfoot-Webster [**3**]. Main results of this thesis is to show that this phenomenon occurs in other examples of symplectic duality.

Let us explain the contents of this thesis. First two chapters are motivational, and discuss two geometric approaches toward the combinatorics of diagonal coinvariants. In chapter 2, we review some results of Haiman on the geometry of Hilbert schemes of points in the affine plane and combinatorics of diagonal coinvariants. In chapter 3, we review our previous work on the relation between affine Springer fibers of type A and the shuffle conjecture of Haglund-Haiman-Loehr-Remmel-Ulynaov. In chapter 4, we review some parts of symplectic duality and reformulate the result of DeConcini-Procesi and Tanisaki using symplectic duality. This reformulation leads to a proposal on the equality of the cohomology rings of conical symplectic resolutions and the coordinate rings of some schemes coming from dual conical symplectic resolutions. In later chapters, we illustrate this proposal by some other examples. In chapter 5, we check it for the cohomology rings of Spaltenstein varieties of type A. In chapter 6, we deal with hypertoric varieties. In chapter 7, we prove it for Hilbert schemes of points in the affine plane.

CHAPTER 2

Combinatorics of diagonal coinvariants

This chapter is a review of some results of Haiman on the combinatorics of diagonal coinvariants. This and the next chapters are included here for motivational purposes.

2.1. Symmeric functions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$, be a partition. We denote its size by $|\lambda| = \sum_i \lambda_i$ and its length by $\ell(\lambda) = l$. If $|\lambda| = n$, then we write $\lambda \vdash n$. We may also write $\lambda = (1^{\alpha_1} 2^{\alpha_2} \ldots)$ to indicate the partition with its α_i parts equal to i. The conjugate partition of λ is denoted by λ^T which is defined by $\lambda_i^T = \sum_{j \geq i} \alpha_j$. We set

$$n(\lambda) := \sum_i (i-1)\lambda_i.$$

The Young diagram of λ is the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq j < \lambda_{i+1}\}$. One pictures elements $(i,j) \in \mathbb{N} \times \mathbb{N}$ as boxes arranged with the i-axis vertical and the j-axis horizontal. A skew Young diagram λ/μ is the difference of Young diagrams λ and μ satisfying $\mu \subseteq \lambda$. For two partitions λ, $\mu \vdash n$, a semi-standard Young tableaux of shape λ and weight μ is a map $T : \lambda \to \mathbb{N}$ from the Young diagram of λ to $\mathbb{N}$ with its multiset of entries equal to $\{1^{\mu_1} 2^{\mu_2} \ldots\}$, which is weakly increasing on each row and strictly increasing on each column. We denote by $\mathrm{SSYT}(\lambda, \mu)$ the set of semi-standard Young tableaux of shape λ and weight μ. Its cardinality $K_{\lambda,\mu}$ is known as the Kostka number.

Let $\Lambda_{\mathbb{Q}} := \varprojlim_n \mathbb{Q}[z_1, \ldots, z_n]^{\mathfrak{S}_n}$ be the ring of symmetrc functions [**33**]. For a partition λ, we denote by $m_\lambda(z) \in \Lambda_{\mathbb{Q}}$ the monomial symmetric function associated with λ, that is, the sum of all distinct monomials in the z_i whose multiset of exponents equals the multiset of parts of λ. Let $e_n(z) = m_{(1^n)}(z)$ be the n-th elementary symmetric function and $p_n(z) = m_{(n)}(z)$ be the power-sum symmetric function of degree n. It is well-known that $\{p_n\}_{n\in\mathbb{N}}$ is algebraically independent and generates $\Lambda_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra. Let us write

$$p_\lambda(z) = \prod_i p_{\lambda_i}(z).$$

Let

$$s_\lambda(z) := \sum_{\mu\vdash n} K_{\lambda,\mu} m_\mu(z)$$

be the Schur function associated with λ. We write $\langle -,-\rangle$ for the Hall inner product on $\Lambda_{\mathbb{Q}}$ characterized by

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}.$$

It is well-known that irreducible representations of n-th symmetric group $\mathfrak{S}_n$ over $\mathbb{C}$ are parametrized by partitions of n. We denote by L_λ the irreducible representation of $\mathfrak{S}_n$ parametrized by $\lambda \vdash n$. For example, $L_{(n)}$ is the trivial representation and $L_{(1^n)}$ is the sign representation. Let V be a finite dimensional representation of $\mathfrak{S}_n$ over $\mathbb{C}$. We denote by

$$\mathcal{F}(V) := \sum_{\lambda\vdash n} \dim(\mathrm{Hom}_{\mathfrak{S}_n}(L_\lambda, V)) s_\lambda(z) \in \Lambda_{\mathbb{Q}}$$

its Frobenius characteristic. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ be a partition of n. We define the parabolic subgroup $\mathfrak{S}_\mu$ of $\mathfrak{S}$ by

$$\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_l}.$$

Lemma 2.1. For V a finite dimensional representation of $\mathfrak{S}_n$ over $\mathbb{C}$, we have

$$\mathcal{F}(V) = \sum_{\mu\vdash n} \dim(V^{\mathfrak{S}_\mu}) m_\mu(z).$$

Here, $V^{\mathfrak{S}_\mu}$ denotes the space of $\mathfrak{S}_\mu$-invariants in V.

For its proof, see for example [**23**].

2.2. Modified Macdonald polynomials

Let us recall the notion of plethystic substitution. Let $E(t_1, t_2, t_3, \ldots)$ be a formal series of rational functions in the parameters $t_1, t_2, \ldots$. We define the plethystic substitution $p_k[E]$ of E into p_k by

$$p_k[E] = E(t_1^k, t_2^k, \ldots).$$

For example, we have $p_k[Z] = p_k(z)$ for $Z = z_1 + z_2 + \cdots$ and

$$p_k[\frac{Z}{1-q}] = \sum_i \frac{z_i^k}{1-q^k} = \frac{p_k(z)}{1-q^k}.$$

For any symmetric function $f \in \Lambda_{\mathbb{Q}}$, we extend the definition of plethystic substitution $f[E]$ by first expressing f as a polynomial in p_k, say $f = \sum_\lambda c_\lambda p_\lambda$ for $c_\lambda \in \mathbb{Q}$, and then defining

$$f[E] = \sum_\lambda c_\lambda \prod_i p_{\lambda_i}[E].$$

Proposition 2.2 ([19]). For any $\mu \vdash n$, there exists $\tilde{H}_\mu(z; q, t) \in \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(q, t)$ such that

- $\tilde{H}_\mu(z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-q)] : \lambda \geq \mu\}$,
- $\tilde{H}_\mu(z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-q)] : \lambda \geq \mu^T\}$,
- $\langle \tilde{H}_\mu, s_{(n)} \rangle = 1$.

These conditions characterize $\tilde{H}_\mu(z; q, t)$ uniquely.

The $\tilde{H}_\mu$ are called modified Macdonald polynomials. The set of modified Macdonald polynomials form a basis of $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(q, t)$. We define a linear operator ∇ on $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(q, t)$ by

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^T)} \tilde{H}_\mu.$$

This is called the nabla operator.

2.3. Diagonal coinvariants

Let $\mathbb{C}[\mathbf{x},\mathbf{y}] = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ be the ring of polynomial functions in $2n$ variables. We consider the diagonal action of $\mathfrak{S}_n$ on $\mathbb{C}[\mathbf{x},\mathbf{y}]$. Let $\mathfrak{m}$ be the homogeneous maximal ideal in $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\mathfrak{S}_n}$. The ring of diagonal coinvariants DR_n is defined by

$$DR_n = \mathbb{C}[\mathbf{x},\mathbf{y}]/\mathfrak{m}\mathbb{C}[\mathbf{x},\mathbf{y}].$$

The natural $\mathfrak{S}_n$-action on DR_n respects the bigrading

$$DR_n = \bigoplus_{i,j}(DR_n)_{i,j}$$

given by the x-grading and the y-grading. Let

$$\mathcal{F}(DR_n;q,t) := \sum_{i,j} t^i q^j \mathcal{F}((DR_n)_{i,j})$$

be the bigraded Frobenius series of DR_n. It turns out that $\mathcal{F}(DR_n;q,t)$ and its specializations are related to various combinatorial objects such as modified Macdonald polynomials, Dyck paths, parking functions, etc.

Let

$$C_n(q,t) := \langle e_n, \tilde{H}_\mu \rangle$$

be the bigraded Hilbert series of $\mathfrak{S}_n$-alternating part of DR_n. Then it follows from the next proposition that

$$C_n(1,1) = \frac{1}{n+1}\binom{2n}{n},$$

hence $C_n(q,t)$ gives a q,t-analog of the n-th Catalan number. It is well-known that the Catalan number is equal to the number of Dyck paths. Garsia-Haglund [**10**] gives a formula for the q,t-Catalan number $C_n(q,t)$ by a sum over Dyck paths with some weights given by certain statistics on Dyck paths.

Parking function is a map $f : \{1,\dots,n\} \to \{1,\dots,n\}$ satisfying $\#f^{-1}(\{1,\dots,k\}) \geq k$ for all $1 \leq k \leq n$. We denote by PF_n the set of parking functions and $\mathbb{C}\mathrm{PF}_n$ the $\mathbb{C}$-vector space spanned by PF_n. The weight of f is defined by $\mathrm{wt}(f) = \sum_{i=1}^{n}(f(i) - i)$. The symmetric group $\mathfrak{S}_n$ acts on PF_n or $\mathbb{C}\mathrm{PF}_n$ by permuting the domain of f and this action preserves the weight. Let $\mathbb{C}\mathrm{PF}_n = \oplus_d \mathbb{C}\mathrm{PF}_{n,d}$ be the grading given by weight.

Proposition 2.3 ([21]). We have

$$\mathcal{F}(DR_n; 1, t) = \sum_d t^d \mathcal{F}(\mathbb{C}\mathrm{PF}_{n,d} \otimes \mathrm{sgn}).$$

This proposition follows from the formula for $\mathcal{F}(DR_n; q, t)$ given in the next section. Since it is known that the number of parking functions is given by $(n+1)^{n-1}$, we have

$$\dim(DR_n) = (n+1)^{n-1}.$$

2.4. Hilbert schemes of points in the affine plane

In this section, we briefly review the work of Haiman on a relation between the combinatorics of diagonal coinvariants and the geometry of Hilbert schemes of points in the affine plane.

Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme parametrizing 0-dimensional subscheme of $\mathbb{C}^2$ of length n. Therefore, the set of $\mathbb{C}$-valued points of $\mathrm{Hilb}^n(\mathbb{C}^2)$ is given by

$$\{I \subset \mathbb{C}[x,y] \mid I : \text{ ideal}, \ \dim(\mathbb{C}[x,y]/I) = n\}.$$

Let $V(I) \subset \mathbb{C}^2$ be the closed subscheme of $\mathbb{C}^2$ defined by I. The multiplicity of a closed point $P \in V(I)$ is the length of the Artin local ring $(\mathbb{C}[x,y]/I)_P$. Let

$$\sigma : \mathrm{Hilb}^n(\mathbb{C}^2) \to S^n\mathbb{C}^2 = \mathbb{C}^{2n}/\mathfrak{S}_n$$

be the Hilbert-Chow morphism. On closed points, it is given by $\sigma(I) = \sum_i m_i P_i \in S^n\mathbb{C}^2$, where P_i runs over all closed points of $V(I)$ and m_i is the multiplicity of P_i. We set $Z_n = \sigma^{-1}(0)$.

Proposition 2.4. The Hilbert scheme of n-points in $\mathbb{C}^2$ $\mathrm{Hilb}^n(\mathbb{C}^2)$ is irreducible and nonsingular of dimension $2n$, and the Hilbert-Chow morphism $\sigma : \mathrm{Hilb}^n(\mathbb{C}^2) \to S^n\mathbb{C}^2$ is a resolution of singularities.

Proof. See for example [**35**]. □

Let X_n be the reduced closed subscheme of $\mathrm{Hilb}^n(\mathbb{C}^2) \times (\mathbb{C}^2)^n$ whose closed points are the tuples $(I, P_1, \ldots, P_n)$ satisfying $\sigma(I) = \sum_i P_i$. This is called the isospectral Hilbert scheme. We have a commutative diagram

$$\begin{array}{ccc} X_n & \to & \mathbb{C}^{2n} \\ \downarrow^{\rho} & & \downarrow \\ \mathrm{Hilb}^n(\mathbb{C}^2) & \xrightarrow{\sigma} & S^n\mathbb{C}^2. \end{array}$$

It is known that $P = \rho_* \mathcal{O}_{X_n}$ is a vector bundle of rank $n!$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$ with an $\mathfrak{S}_n$-action affording the regular representation on each fiber. This is called the Procesi bundle.

Theorem 2.5 ([21]). We have

$$H^i(Z_n, P) = \begin{cases} DR_n & i = 0, \\ 0 & i > 0. \end{cases}$$

The two-dimensional torus $\mathbb{T}^2 = \mathbb{G}_m^2$ acts on $\mathbb{C}^2$ by $(t, q) \cdot (x, y) = (t^{-1}x, q^{-1}y)$ for $(t, q) \in \mathbb{T}^2$ and $(x, y) \in \mathbb{C}^2$. This action induces a $\mathbb{T}^2$-action on $\mathrm{Hilb}^n(\mathbb{C}^2)$, Z_n, X_n and $\rho : X_n \to \mathrm{Hilb}^n(\mathbb{C}^2)$ is $\mathbb{T}^2$-equivariant. For $\mu \vdash n$, we associate a monomial ideal

$$I_\mu = \mathbb{C}\{x^i y^j \mid (i, j) \notin \mu\} \subset \mathbb{C}[x, y],$$

where we identified μ with its Young diagram. Then the $\mathbb{T}^2$-fixed points of $\mathrm{Hilb}^n(\mathbb{C}^2)$ are given by the ideals I_μ for $\mu \vdash n$. Let $P(I_\lambda)$ be the fiber of P at I_μ. Then $P(I_\mu)$ has a $\mathbb{T}^2$-action and hence is bigraded.

Theorem 2.6 ([20]). We have

$$\mathcal{F}(P(I_\mu); q, t) = \tilde{H}_\mu(z; q, t).$$

By Theorem 2.5, we can calculate the bigraded Frobenius series $\mathcal{F}(DR_n; q, t)$ by using the Atiyah-Bott type formula and a Koszul type resolution of $\mathcal{O}_{Z_n}$.

Theorem 2.7 ([21]). We have

$$\mathcal{F}(DR_n; q, t) = \sum_{\mu \vdash n} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})},$$

where $a(x)$ and $l(x)$ denote the number of points in μ strictly to the right of x and above x respectively, and

$$B_\mu(q,t) = \sum_{(i,j)\in\mu} t^i q^j,$$

$$\Pi_\mu(q,t) = \prod_{(i,j)\in\mu\setminus(0,0)} (1 - t^i q^j).$$

It is known (see for example [**17**]) that

$$e_n(z) = \sum_{\mu \vdash n} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in\mu}(q^{a(x)}-t^{1+l(x)})(t^{l(x)}-q^{1+a(x)})}.$$

Hence the above theorem can be stated as follows.

Corollary 2.8. We have

$$\mathcal{F}(DR_n; q, t) = \nabla e_n(z).$$

CHAPTER 3

Affine Springer fibers of type A

This chapter is a review of our previous work on a relation between the combinatorics of diagonal coinvariants and the geometry of affine Springer fibers of type A.

3.1. Affine Springer fibers

Let G be a simple simply connected algebraic group over $\mathbb{C}$. Let $\mathfrak{g}$ be its Lie algebra. We fix a Cartan and a Borel subgroup $T \subset B \subset G$ with their Lie algebras $\mathfrak{t}$ and $\mathfrak{b}$. Let W be the Weyl group of G. Let R be the set of roots and let $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$. For each $\alpha \in R$, we fix a nonzero element $e_\alpha \in \mathfrak{g}_\alpha$. Let $\check{Q}$ be the coroot lattice of G. Let $\check{\rho}$ be the half sum of positive coroots. Let $F = \mathbb{C}((\epsilon))$ be the field of formal Laurent series in ϵ and let $\mathcal{O} = \mathbb{C}[[\epsilon]]$ be the ring of formal power series in ϵ. Let $\mathrm{ev} : G(\mathcal{O}) \to G(\mathbb{C})$ be the map defined by taking ϵ to 0 and let $I := \mathrm{ev}^{-1}(B(\mathbb{C})) \subset G(F)$ be an Iwahori subgroup of $G(F)$. Let $\hat{\mathcal{B}} := G(F)/I$ be the affine flag variety and let $X := G(F)/G(\mathcal{O})$ be the affine Grassmannian.

Let $h = 1 + \langle \theta, \check{\rho} \rangle$ be the Coxeter number of G, where θ is the highest root and $\langle -, - \rangle$ is the natural pairing. We fix a positive integer r which is coprime to h. We write $r = mh + b$ with $1 \leq b < h$. We set

$$v_r := \epsilon^m \left(\epsilon \sum_{\substack{\alpha \in R \\ \langle \alpha, \check{\rho} \rangle = b - h}} e_\alpha + \sum_{\substack{\alpha \in R \\ \langle \alpha, \check{\rho} \rangle = b}} e_\alpha \right) \in \mathfrak{g}(\mathcal{O}).$$

This is a regular semisimple nil elliptic element in $\mathfrak{g}(F)$. In this paper, we only consider affine Springer fibers associated with v_r.

Definition 3.1 (Affine Springer fibers). Affine Springer fibers associated with v_r are defined by

$$\hat{\mathcal{B}}_{v_r} := \left\{ gI \in \hat{\mathcal{B}} \mid \mathrm{Ad}(g)^{-1}(v_r) \in \mathfrak{b} + \epsilon\mathfrak{g}(\mathcal{O}) \right\},$$

$$X_{v_r} := \left\{ gG(\mathcal{O}) \in X \mid \mathrm{Ad}(g)^{-1}(v_r) \in \mathfrak{g}(\mathcal{O}) \right\}.$$

Proposition 3.2. Affine Springer fibers $\hat{\mathcal{B}}_{v_r}$ and X_{v_r} have a structure of finite-dimensional algebraic varieties with finite number of irreducible components. The Borel-Moore homology group $H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C})$ of $\hat{\mathcal{B}}_{v_r}$ has a W-action and it is isomorphic as a W-module to $\check{Q}/r\check{Q}$.

Proof. The first part of the statement is a special case of [**27**], the second part is a special case of [**31**], and the third part is [**39**]. □

We remark that in type A, it is known that

$$\check{Q}/(n+1)\check{Q} \cong DR_n \otimes \mathrm{sgn}.$$

Our aim in this chapter is to understand the bigrading on DR_n in terms of the geometry of affine Springer fibers at least conjecturally.

3.2. Affine pavings

In this section, we review some results of Goresky-Kottwitz-MacPherson [**15**] on a paving of affine Springer fibers. Let $B \subset P \subset G$ be a standard parabolic subgroup with its Lie algebra $\mathfrak{p}$. Let $W_P \subset W$ be the Weyl group of a Levi subgroup of P. We take $y \in \check{Q} \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in R \\ \langle \alpha, y \rangle \geq 0}} \mathfrak{g}_\alpha$$

and $\langle\theta, y\rangle < 1$. We set

$$\hat{\mathfrak{g}}_y := \bigoplus_{\substack{\alpha\in R, m\in\mathbb{Z}\\ \langle\alpha,y\rangle+m\geq 0}} \mathfrak{g}_\alpha \epsilon^m.$$

There is a conected pro-algebraic subgroup $\hat{G}_y \subset G(F)$ of $G(F)$ with its Lie algebra $\hat{\mathfrak{g}}_y$ [**34**]. For example, if $y = \frac{1}{h}\check{\rho}$, then $\hat{G}_y = I$ and if $y = 0$, then $\hat{G}_y = G(\mathcal{O})$. We set $\mathcal{F}_y := G(F)/\hat{G}_y$ and

$$\mathcal{F}_y(v_r) := \{g\hat{G}_y \in \mathcal{F}_y \mid \mathrm{Ad}(g)^{-1}(v_r) \in \hat{\mathfrak{g}}_y\}.$$

This is a parahoric analogue of affine Springer fiber associated with v_r. Let us recall the Bruhat decomposition:

$$\mathcal{F}_y = \bigsqcup_{\check{\lambda}\in\check{Q}, w\in W/W_P} I\epsilon^{\check{\lambda}} w\hat{G}_y/\hat{G}_y.$$

Theorem 3.3. If $(\mathcal{F}_y(v_r) \cap I\epsilon^{\check{\lambda}} w\hat{G}_y/\hat{G}_y)$ is nonempty, then it is an affine space of dimension

$$\#\{(\alpha, k) \in R \times \mathbb{Z} \mid 0 \leq \langle \frac{1}{h}\check{\rho}, \alpha\rangle + k < \frac{r}{h}, \langle w\cdot y - \check{\lambda}, \alpha\rangle + k < 0\}.$$

Proof. This is a special case of [**15**]. □

Corollary 3.4. Odd degree part of the Borel-Moore homology of $\mathcal{F}_y(v_r)$ vanishes.

This theorem enables us to calculate the dimension of each graded piece of $H_*^{\mathrm{BM}}(\mathcal{F}_y(v_r), \mathbb{C})$. Since we have (see for example [**23**])

$$H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C})^{W_P} \cong H_*^{\mathrm{BM}}(\mathcal{F}_y(v_r), \mathbb{C}),$$

we can obtain some information on the W-module structure of $H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C})$. In type A, this information is enough to determine the W-module structure by Lemma 2.1.

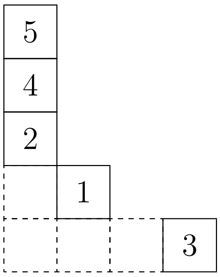

FIGURE 1. An example of semistandard Young tableau of skew shape $\lambda + (1^n)/\lambda$, where $n = 5$ and $\lambda = (3,1)$. In this case, d-inversions are the pairs $(1,2), (1,3), (2,3), (3,4)$. Hence dinv $= 4$.

3.3. Combinatorics

In this section, we briefly recall the conjectural combinatorial formula for $\mathcal{F}(DR_n; q, t)$ given by Haglund-Haiman-Loehr-Remmel-Ulyanov [**18**]. Let

$$\delta_n = (n-1, n-2, \dots, 1, 0)$$

be the staircase partition. Let $\lambda = (\lambda_1, \dots, \lambda_n) \subset \delta_n$ be a partition contained in δ_n. For every box $x = (i,j) \in \mathbb{N} \times \mathbb{N}$, we set $d(x) = i + j$. Let $T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)$ and consider two entries $x = (i,j), y = (i',j') \in \lambda + (1^n)/\lambda$. We say that the pair (x,y) forms a *d-inversion* if $T(x) < T(y)$ and either

- $d(y) = d(x)$ and $j > j'$ or
- $d(y) = d(x) + 1$ and $j < j'$

holds. We denote by $\mathrm{dinv}(T)$ the number of d-inversions of T.

We set

$$D_n(z; q, t) := \sum_{\lambda \subset \delta_n} \sum_{T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(T)} z^T,$$

where

$$z^T = \prod_{x \in \lambda + (1^n)/\lambda} z_{T(x)}.$$

Conjecture 3.5 (Shuffle conjecture). We have

$$\nabla e_n(z) = D_n(z; q, t).$$

This conjecture include Garsia-Haglund's formula for the q, t-Catalan number and Proposition 2.3 as a special case. Note that partitions contained in δ_n are in bijection with Dyck paths and standard Young tableaux on $\lambda + (1^n)/\lambda$ for $\lambda \subset \delta_n$ are in bijection with parking functions.

3.4. Main result

In the rest of this chapter, we assume $G = SL_n(\mathbb{C})$. Let $W_{\mathrm{aff}} = W \ltimes \check{Q}$ be the affine Weyl group and $s_0, s_1, \ldots, s_{n-1}$ be its simple reflections. We extend the index of s_i to all $i \in \mathbb{Z}$ by $s_{i+n} = s_i$. For $c \in \mathbb{N}$, we set $\mathrm{cyc}_c := s_{c-1}s_{c-2} \cdots s_1 s_0 \in W_{\mathrm{aff}}$. We set

$$X^{\leq c} := \bigsqcup_{\substack{w \in W_{\mathrm{aff}}/W \\ w \not\geq \mathrm{cyc}_{c+1}}} IwG(\mathcal{O})/G(\mathcal{O}) \subset X.$$

This is a closed subset of X. Let $\pi : \hat{\mathcal{B}}_{v_r} \to X_{v_r}$ be the natural projection. We set

$$\hat{\mathcal{B}}_{v_r}^{\leq c} := \pi^{-1}(X_{v_r} \cap X^{\leq c}) \subset \hat{\mathcal{B}}_{v_r}.$$

This gives a stratification of $\hat{\mathcal{B}}_{v_r}$. By Corollary 3.4, we have $F_c := H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}^{\leq c}, \mathbb{C}) \subset H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C})$ and

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_{\frac{(n-1)(r-1)}{2}} = H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C})$$

gives a filtration of $H_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C})$. This filtration is stable under the W-action.

Definition 3.6. We define a symmetric function $D_{(n,r)}(z; q, t) \in \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}[q, t]$ by

$$q^{\frac{(n-1)(r-1)}{2}} D_{(n,r)}(z; q^{-1}, t) = \sum_{i,j} t^i q^j \mathcal{F}(\mathrm{gr}_i H_{2j}^{\mathrm{BM}}(\hat{\mathcal{B}}_{v_r}, \mathbb{C}) \otimes \mathrm{sgn}).$$

Then our main result in [**23**] is the following.

Theorem 3.7. We have

$$D_n(z;q,t) = D_{(n,n+1)}(z;q,t).$$

Therefore, the symmetric functions $D_{(n,r)}$ generalizes the one defined by Haglund-Haiman-Loehr-Remmel-Ulyanov to any coprime (n,r). Left hand side of the shuffle conjecture is generalized to any coprime (n,r) by Gorsky-Negut [**16**]. Also, there is a generalization of shuffle conjectures for not necessarily coprime pair (n,r) by Bergeon-Garsia-Leven-Xin [**2**]. Since the left hand side of the shuffle conjecture can be understood by the algebraic geometry of Hilbert schemes of points and the right hand side can be understood by the topology of affine Springer fibers of type A, this conjecture suggests that there should be some relations between two very different geometries. Understanding this kind of relation in other examples is one motivation for the results in the remaining chapters.

We have introduced the stratification of X in an ad hoc way, but we have the following simple formula concerning this filtration. Note that the stratification also gives a filtration on $H_*^{\mathrm{BM}}(X,\mathbb{C})$.

Theorem 3.8. We have an equality

$$\sum_{i,j} t^i q^j \dim(\mathrm{gr}_i H_{2j}^{\mathrm{BM}}(X,\mathbb{C})) = \frac{1}{(1-qt)(1-q^2t)\cdots(1-q^{n-1}t)}.$$

Since a proof of this result does not appear in the literature, we give a proof of this theorem in the next section.

3.5. Proof of Theorem 3.8

As in [**23**], we take an auxiliary set $A = \mathbb{Z}_{\geq 0}^{n-1}$. For an element $(a_1, a_2, \ldots, a_{n-1})$ of A, we make the convention that $a_n = 0$ and $a_{i+n} = a_i + 1$ in order to extend the index to all integers. We set $a(a_i) = a_1 + a_2 + \cdots + a_{n-1}$. We often abbreviate $a(a_i)$ by a if there is no confusion. For $(a_i) \in A$, we set

$$\check{\lambda}(a_i) = (a_{1-a}, a_{2-a}, \ldots, a_{n-a}) \in \check{Q} \subset \mathbb{Z}^n.$$

In [**23**] Proposition 4.1, we show that this gives a bijection between A and $\check{Q}$.

By the length formula, the dimension of the cell $I\epsilon^{\check{\lambda}(a_i)}G(\mathcal{O})/G(\mathcal{O})$ corresponding to (a_i) is given by

$$\ell^{\mathrm{f}}(\check{\lambda}(a_i)) = \sum_{1 \le i < j \le n} |a_{i-a} - a_{j-a}| - \#\{(i,j) \mid 1 \le i < j \le n, a_{i-a} < a_{j-a}\}.$$

Lemma 3.9.

$$\ell^{\mathrm{f}}(\check{\lambda}(a_i)) = a + \sum_{1 \le i < j \le n-1} |a_i - a_j| - \#\{(i,j) \mid 1 \le i < j \le n-1, a_i < a_j\}$$

Proof. Let χ be the characteristic function. It suffices to note that for $0 \le i < j < n$,

$$\begin{aligned} |a_j - a_{n+i}| - \chi(a_j < a_{n+i}) &= |a_j - a_i - 1| - \chi(a_j \le a_i) \\ &= \begin{cases} a_i - a_j & \text{if } a_j \le a_i \\ a_j - a_i - 1 & \text{if } a_j > a_i \end{cases} \\ &= |a_i - a_j| - \chi(a_i < a_j) \end{aligned}$$

□

By this lemma and [**23**] Lemma 4.6, we have

$$\sum_{i,j} t^i q^j \dim(\mathrm{gr}_i H_{2j}^{\mathrm{BM}}(X, \mathbb{C})) = \sum_{(a_i) \in A} q^{a + \sum_{1 \le i < j \le n-1} |a_i - a_j| - \chi(a_i < a_j)} t^a.$$

We have to show that this is equal to

$$\sum_{b_1, \ldots, b_{n-1} \ge 0} q^{b_1 + 2b_2 + \cdots + (n-1)b_{n-1}} t^{b_1 + b_2 + \cdots + b_{n-1}}$$

We take $B = \mathbb{Z}_{\geq 0}^{n-1}$ to be the set of parameters for (b_i)'s. We will give an explicit bijection between A and B which preserves the two grading respectively. In order to do so, we divide the summation over A or B into $(n-1)!$ parts respectively.

For A, we divide it into subsets A_w for every $w \in \mathfrak{S}_{n-1}$, where

$$A_w = \left\{ (a_1, \dots, a_{n-1}) \in A \;\middle|\; \begin{array}{l} \text{for any } 1 \leq i < n, \\ a_{w(i)} \geq a_{w(i+1)} \text{ if } w(i) < w(i+1) \\ a_{w(i)} > a_{w(i+1)} \text{ if } w(i) > w(i+1) \end{array} \right\}.$$

It is easy to see that $A = \sqcup_{w \in \mathfrak{S}_{n-1}} A_w$. For $(a_i) \in A_w$, we have

$$\begin{aligned} \ell^{\mathrm{f}}(\check{\lambda}(a_i)) &= a + \sum_{1 \leq i < j < n} (a_{w(i)} - a_{w(j)}) - \ell(w) \\ &= \sum_{i=1}^{n-1} (n+1-2i) a_{w(i)} - \ell(w). \end{aligned}$$

For B, we divide it according to the values of $b_i \bmod n-i$. We set $C = \{(c_1, \dots, c_{n-1}) \mid 0 \leq c_i < n-i\}$. Note that $c_{n-1} = 0$ for every element in C. For $\mathbf{c} = (c_1, \dots, c_{n-1}) \in C$, we set

$$B_{\mathbf{c}} = \{(b_1, \dots, b_{n-1}) \in B \mid b_i \equiv c_i \bmod n-i\}.$$

Obviously, we have $B = \sqcup_{\mathbf{c} \in C} B_{\mathbf{c}}$.

We construct a bijection between $\mathfrak{S}_{n-1}$ and C. For $1 \leq l < n$, we take σ_l to be the cyclic permutation as follows:

$$\sigma_l(i) = \begin{cases} l & \text{if } i = 1 \\ i-1 & \text{if } 1 < i \leq l \\ i & \text{if } i > l \end{cases} \cdot$$

Let $\sigma : C \to \mathfrak{S}_{n-1}$ be the map defined by

$$\sigma(\mathbf{c}) = \sigma_{n-1}^{c_1}\sigma_{n-2}^{c_2}\cdots\sigma_2^{c_{n-2}}.$$

It is easy to see that σ is bijective.

For $w \in \mathfrak{S}_{n-1}$, we set $L(w) = \sum_{i:w(i)>w(i+1)} i(n-i) - \ell(w)$ and $M(w) = \sum_{i:w(i)>w(i+1)} i$.

Lemma 3.10. Let $1 \leq k < n-1$, $0 \leq c_k < n-k$, and $w \in \mathfrak{S}_{n-1}$. Assume that we have $w(n-k) > c_k$ and $w(l) = l$ for any $l > n-k$. Then we have

$$L(\sigma_{n-k}^{c_k} w) = kc_k + L(w)$$

$$M(\sigma_{n-k}^{c_k} w) = c_k + M(w).$$

Proof. By induction on c_k, it suffices to prove the assertion for $c_k = 1$. Let $1 \leq i < n-1$. If $w(i)$ and $w(i+1)$ are not equal to 1, it is easy to see that $w(i) > w(i+1)$ is equivalent to $\sigma_{n-k}w(i) > \sigma_{n-k}w(i+1)$. Hence the contribution of this i to $L(w)$ (resp. $M(w)$) and $L(\sigma_{n-k}w)$ (resp. $M(\sigma_{n-k}w)$) are the same.

If $w(i) = 1$, then we have by assumption $1 \leq i < n-k$. Hence we have $w(i) < w(i+1)$ and $\sigma_{n-k}w(i) = n-k > \sigma_{n-k}w(i+1)$. If $i > 1$, then we also have $w(i-1) > w(i)$ and $\sigma_{n-k}w(i-1) < \sigma_{n-k}w(i)$. Therefore, we have $M(\sigma_{n-k}w) - M(w) = i - (i-1) = 1$ and

$$L(\sigma_{n-k}w) - L(w) = i(n-i) - (i-1)(n-i+1) - \ell(\sigma_{n-k}w) + \ell(w)$$

$$= n + 1 - 2i - \ell(\sigma_{n-k}w) + \ell(w)$$

Concerning the length, all $j < i$ contribute to $\ell(w)$ and not to $\ell(\sigma_{n-k}w)$, and all $i < j \leq n-k$ contribute to $\ell(\sigma_{n-k}w)$ and not to $\ell(w)$. It is easy to see that the contributions of pairs $j < j'$ with $j, j' \neq i$ to the lengths are the same. Hence we have $\ell(\sigma_{n-k}w) - \ell(w) =$

$(n-k-i)-(i-1) = n+1-2i-k$. Therefore, we have $L(\sigma_{n-k}w) - L(w) = n+1-2i-(n+1-2i-k) = k$. This proves the lemma. $\square$

Corollary 3.11. We have $L(\sigma(\mathbf{c})) = c_1 + 2c_2 + \cdots + (n-2)c_{n-2}$ and $M(\sigma(\mathbf{c})) = c_1 + c_2 + \cdots + c_{n-2}$.

Let $w = \sigma(\mathbf{c})$. We give a bijection between A_w and $B_{\mathbf{c}}$. We do this inductively by setting $a_{w(n-1)} = \frac{b_1 - c_1}{n-1}$ and

$$a_{w(n-i-1)} = \begin{cases} a_{w(n-i)} + \frac{b_{i+1}-c_{i+1}}{n-i-1} & \text{if } w(n-i-1) < w(n-i) \\ a_{w(n-i)} + \frac{b_{i+1}-c_{i+1}}{n-i-1} + 1 & \text{if } w(n-i-1) > w(n-i) \end{cases}$$

for $1 \leq i < n-1$. Then it is easy to see that this gives a bijection between $B_{\mathbf{c}}$ and A_w. Let us compare the two grading. For the q-grading, we have

$$\begin{aligned} \ell^{\mathrm{f}}(\check{\lambda}(a_i)) &= \sum_{i=1}^{n-1}(-n+1+2i)a_{w(n-i)} - \ell(w) \\ &= \sum_{j=1}^{n-1} \frac{b_j - c_j}{n-j}\left(\sum_{i=j}^{n-1}(-n+1+2i)\right) + \sum_{j:w(j)>w(j+1)}\left(\sum_{i=1}^{j}(n+1-2i)\right) - \ell(w) \\ &= \sum_{j=1}^{n-1} j(b_j - c_j) + \sum_{j:w(j)>w(j+1)} j(n-j) - \ell(w) \\ &= \sum_{j=1}^{n-1} j(b_j - c_j) + L(w) \\ &= \sum_{j=1}^{n-1} jb_j. \end{aligned}$$

For the t-grading, we have

$$\begin{aligned}\sum_{i=1}^{n-1} a_i &= \sum_{j=1}^{n-1}\left(\sum_{i=j}^{n-1}\frac{b_j - c_j}{n-j}\right) + \sum_{j:w(j)>w(j+1)}\sum_{i=1}^{j} 1\\ &= \sum_{j=1}^{n-1}(b_j - c_j) + \sum_{j:w(j)>w(j+1)} j\\ &= \sum_{j=1}^{n-1}(b_j - c_j) + M(w)\\ &= \sum_{j=1}^{n-1} b_j.\end{aligned}$$

This proves Theorem 3.8.

CHAPTER 4

Symplectic duality

In this chapter, we give a general proposal for a generalization of DeConcini-Procesi and Tanisaki's theorem. This formulation involves symplectic duality in the sense of Braden-Licata-Proudfoot-Webster [**3**].

4.1. Symplectic duality

Let $\mathfrak{M}$ be a smooth algebraic variety over $\mathbb{C}$ with an algebraic symplectic form ω and an action of $\mathbb{S} = \mathbb{G}_m$ such that $s^*\omega = s^l\omega$ for some integer $l \geq 1$ and $s \in \mathbb{S}$. This collection of data is called conical symplectic resolution if

- the canonical morphism $\mathfrak{M} \to \mathfrak{M}_0 := \mathrm{Spec}(\mathbb{C}[\mathfrak{M}])$ is projective and birational,
- $\mathbb{S}$ acts on $\mathbb{C}[\mathfrak{M}]$ with only non-negative weights and the space of $\mathbb{S}$-invariants $\mathbb{C}[\mathfrak{M}]^{\mathbb{S}}$ is one-dimensional, consisting only of constant functions.

We also assume that there exists a Hamiltonian action of $\mathbb{T} = \mathbb{G}_m$ on $\mathfrak{M}$ commuting with the $\mathbb{S}$-action such that $\mathfrak{M}^{\mathbb{T}}$ is finite. By using $\mathbb{S}$-equivariant quantization of $\mathfrak{M}$ and $\mathbb{T}$-actions, we can associate category $\mathcal{O}$ for $(\mathfrak{M}, \mathbb{T})$. For example, if $\mathfrak{M}$ is the cotangent bundle of a flag variety, this category $\mathcal{O}$ is equivalent to the (principal block of the) usual BGG category $\mathcal{O}$. Symplectic duality is a duality between $(\mathfrak{M}, \mathbb{T})$ and another pair $(\mathfrak{M}^!, \mathbb{T}^!)$ of conical symplectic resolution $\mathfrak{M}^!$ with a Hamiltonian $\mathbb{T}^!$-action. It is expected for example that the category $\mathcal{O}$ for $(\mathfrak{M}, \mathbb{T})$ and the category $\mathcal{O}$ for $(\mathfrak{M}^!, \mathbb{T}^!)$ are Koszul dual to each other.

Let us give some examples of symplectic duality. The cotangent bundle of the flag variety of a reductive group G is symplectic dual to the cotangent bundle of the flag variety of its Langlands dual ${}^LG$. The Slodowy varieties of type A and the cotangent bundles of partial flag varieties are symplectic dual (see section 4.2). More generally, S3-varieties of type A are symplectic dual to another S3-varieties of type A (Chapter 5). Smooth hypertoric varieties are symplectic dual to hypertoric varieties associated with Gale dual hyperplane arrangement (Chapter 6). Hilbert scheme of points in the affine plane is self-dual (Chapter 7). For more examples and properties of symplectic duality, see [**3**] section 10.

4.2. Proposal

In [**8**] and [**40**], DeConcini-Procesi and Tanisaki show that the cohomology rings of Springer fibers of type A are isomorphic to the coordinate rings of scheme-theoretic intersections of some nilpotent orbit closures and a Cartan subalgebra. The purpose of this thesis is to generalize their result to wider situations by reinterpreting them as a certain isomorphism between the cohomology rings of some symplectic varieties and the coordinate rings of some schemes coming from another symplectic varieties.

First we recall the result of DeConcini-Procesi and Tanisaki. Let $G = \mathrm{GL}_n(\mathbb{C})$. We fix a Borel subgroup $B \subset G$ and a Cartan subgroup $T \subset B$. We take a parabolic subgroup $B \subset P \subset G$ and its Levi subgroup L. We denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{p}$, and $\mathfrak{l}$ the Lie algebras of G, B, T, P, and L respectively. Let $\mathfrak{n}$ and $\mathfrak{n}_P$ be the nilpotent radicals of $\mathfrak{b}$ and $\mathfrak{p}$. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone of $\mathfrak{g}$ and let $\mathcal{N}_P = \mathrm{Ad}(G) \cdot \mathfrak{n}_P \subset \mathcal{N}$ a closed subvariety of $\mathcal{N}$. If $\lambda \vdash n$ is the partition of n corresponding to P, then $\mathcal{N}_P$ is the closure of the nilpotent orbit whose Jordan block is of type λ^T.

We take a regular nilpotent element e in $\mathfrak{l}$. Consider the Springer resolution

$$T^*(G/B) \cong \{(gB, X) \in G/B \times \mathfrak{g} \mid \mathrm{Ad}(g)^{-1}(X) \in \mathfrak{n}\} \xrightarrow{\mu} \mathcal{N}$$

given by $\mu(gB, X) = X$. We call $\mathcal{B}_e := \mu^{-1}(e)$ the Springer fiber associated with e.

Theorem 4.1 ([8],[40]). There is an isomorphism of graded algebras

$$H^*(\mathcal{B}_e, \mathbb{C}) \cong \mathbb{C}[\mathcal{N}_P \cap \mathfrak{t}].$$

Here, $\mathcal{N}_P \cap \mathfrak{t}$ is the scheme-theoretic intersection of $\mathcal{N}_P$ and $\mathfrak{t}$ in $\mathfrak{g}$ and the grading on $\mathbb{C}[\mathcal{N}_P \cap \mathfrak{t}]$ comes from the $\mathbb{G}_m$-action on $\mathcal{N}_P \cap \mathfrak{t}$ induced by the scaling action $t \cdot X = t^{-2}X$ for $t \in \mathbb{G}_m$ and $X \in \mathfrak{g}$.

We summarize the above theorem in the following diagram:

$$\begin{array}{ccccc} T^*(G/P) & \hookleftarrow & G/P = \mu_P^{-1}(0) & & \\ \big\downarrow{\scriptstyle \mu_P} & & & & \\ \mathcal{N}_P & \hookleftarrow & \mathcal{N}_P \cap \mathfrak{t} & \cong & \mathrm{Spec} H^*(\mathcal{B}_e, \mathbb{C}). \end{array} \tag{4.1}$$

Here, μ_P is the parabolic analogue of Springer resolution

$$T^*(G/P) \cong \{(gP, X) \in G/P \times \mathfrak{g} \mid \mathrm{Ad}(g)^{-1}(X) \in \mathfrak{n}_P\} \xrightarrow{\mu_P} \mathcal{N}_P$$

given by $\mu_P(gP, X) = X$. For type A, μ_P always gives an resolution of singularities of $\mathcal{N}_P$ and $\mathcal{N}_P$ is normal ([**29**]). Hence $\mathcal{N}_P$ can be understood as the affinization of $T^*(G/P)$. We also note that $T^*(G/P)$ is homotopy equivalent to G/P.

On the other hand, we will give an algebro-geometric realization of $H^*(G/P, \mathbb{C})$ in the next chapter. Let us take a $\mathfrak{sl}_2$-triple $\{e, h, f\}$ containing e. Let $Z_{\mathfrak{g}}(f)$ (resp. $Z_{\mathfrak{l}}(f)$) be the centralizer of f in $\mathfrak{g}$ (resp. $\mathfrak{l}$). Consider the Slodowy slice $S_e := \mathcal{N} \cap (e + Z_{\mathfrak{g}}(f))$. There is a $\mathbb{G}_m$-action on S_e given by $t \cdot X = t^{-2}\mathrm{Ad}(t^h)X$ for $t \in \mathbb{G}_m$ and $X \in S_e$.

Proposition 4.2 (Theorem 5.1 for $P = B$, $Q = P$). There is a graded algebra isomorphism

$$H^*(G/P, \mathbb{C}) \cong \mathbb{C}[S_e \cap (e + Z_{\mathfrak{l}}(f))].$$

Here, $S_e \cap (e + Z_{\mathfrak{l}}(f))$ is the scheme-theoretic intersection of S_e and $e + Z_{\mathfrak{l}}(f)$ in $\mathfrak{g}$ and the grading on $\mathbb{C}[S_e \cap (e + Z_{\mathfrak{l}}(f))]$ comes from the $\mathbb{G}_m$-action on S_e above.

We summarize this proposition in the following diagram:

$$\begin{array}{ccl} \tilde{S}_e & \hookleftarrow & \mathcal{B}_e = \mu_e^{-1}(e) \\ \downarrow{\scriptstyle \mu_e} & & \\ S_e & \hookleftarrow & S_e \cap (e + Z_{\mathfrak{l}}(f)) \quad \cong \mathrm{Spec} H^*(G/P, \mathbb{C}). \end{array} \tag{4.2}$$

Here, $\tilde{S}_e := \mu^{-1}(S_e)$ is the Slodowy variety and μ_e is the restriction of μ to $\tilde{S}_e$. It is known that μ_e gives a resolution of singularities of S_e and S_e is the affinization of $\tilde{S}_e$. Moreover, $\tilde{S}_e$ is homotopy equivalent to $\mathcal{B}_e$.

Note that there is some similarity between the above two diagrams (4.1) and (4.2), where the roles of $\mathcal{B}_e$ and G/P are exchanged to each other. Recall that $T^*(G/P)$ and $\tilde{S}_e$ are related to each other by symplectic duality. The aim of this thesis is to generalize the above theorem of DeConcini-Procesi and Tanisaki to other cases of symplectic duality. For this purpose, we have to understand the scheme-theoretic intersections $\mathcal{N}_P \cap \mathfrak{t}$ and $S_e \cap (e + Z_{\mathfrak{l}}(f))$ more intrinsically.

Let us recall the notion of fixed point scheme (see [**9**]). Let H be an algebraic group over $\mathbb{C}$ and let X be a scheme over $\mathbb{C}$ with H-action. Consider the contravariant functor h_X^H from the category of $\mathbb{C}$-schemes to the category of sets given by

$$h_X^H(Y) = (\text{the set of } H\text{-equivariant morphisms } Y \to X),$$

where Y is a $\mathbb{C}$-scheme equipped with trivial H-action. This functor is known to be representable by a closed subscheme X^H of X, which is called the H-fixed point scheme of X. For $X = \mathrm{Spec}(A)$, the ideal of definition of H-fixed point scheme X^H in X is generated by all $h \cdot f - f$ for $h \in H(\mathbb{C})$ and $f \in A$.

Let us consider the adjoint action of T on $\mathcal{N}_P$. One can easily see that the scheme-theoretic intersection $\mathcal{N}_P \cap \mathfrak{t}$ is isomorphic to the T-fixed point scheme $\mathcal{N}_P^T$ of T (or $\mathbb{G}_m$-fixed point scheme for some generic subgroup $\mathbb{G}_m \subset T$). Similarly, the scheme-theoretic intersection $S_e \cap (e + Z_{\mathfrak{l}}(f))$ can be understood as the $Z(L)$-fixed point scheme $S_e^{Z(L)}$ for the adjoint action of the center $Z(L)$ of L on S_e.

Therefore, the above results can be considered as examples of the phenomenon that the cohomology ring of a conical symplectic resolution is isomorphic to the coordinate ring of a $\mathbb{G}_m$-fixed point scheme of the affinization of symplectic dual conical symplectic resolution. Main result of this thesis show that this phenomenon occurs for Hilbert scheme of points in the affine plane.

Let us explain the main result of this thesis. Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme of n-points in the affine plane. The affinization of $\mathrm{Hilb}^n(\mathbb{C}^2)$ is given by n-th symmetric product $S^n\mathbb{C}^2$ of $\mathbb{C}^2$ and the Hilbert-Chow morphism $\mathrm{Hilb}^n(\mathbb{C}^2) \to S^n\mathbb{C}^2$ gives a resolution of singularities. Let us consider the action of $\mathbb{T} = \mathbb{G}_m$ on $S^n\mathbb{C}^2$ induced by its action on $\mathbb{C}^2$ given by $t \cdot (x, y) = (t^{-1}x, ty)$ for $t \in \mathbb{T}$, $(x, y) \in \mathbb{C}^2$. Since $\mathrm{Hilb}^n(\mathbb{C}^2)$ is symplectic dual to itself, we come to the following statement by applying the above consideration to this case:

Theorem 4.3. There is an isomorphism of graded algebras

$$H^*(\mathrm{Hilb}^n(\mathbb{C}^2), \mathbb{C}) \cong \mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}].$$

Here, the grading on $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$ comes from the $\mathbb{G}_m$-action induced by the its action on $\mathbb{C}^2$ given by $s \cdot (x, y) = (s^{-1}x, s^{-1}y)$ for $s \in \mathbb{G}_m$, $(x, y) \in \mathbb{C}^2$.

This theorem is proved in Chapter 7. We also prove in Chapter 5 and 6 that the above phenomenon also occurs for the case of S3-varieties and hypertoric varieties. It would be interesting to find some conditions under which this kind of phenomenon can be expected to hold for more general symplectic dual pair of conical symplectic resolutions.

CHAPTER 5

Spaltenstein varieties

In this chapter, we check the proposal in Chapter 4 in the case of S3-varieties of type A.

5.1. Main result

Let $G = \mathrm{GL}_n$ and $\mathfrak{g} = \mathfrak{gl}_n$. We fix a Cartan subalgebra $\mathfrak{t}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$. Let $\mathfrak{b} \subset \mathfrak{p}, \mathfrak{q} \subset \mathfrak{g}$ be two standard parabolic subalgebras and P, Q be the parabolic subgroups of G with Lie algebras $\mathfrak{p}$, $\mathfrak{q}$. We denote a Levi and the nilpotent part of $\mathfrak{p}$ (resp. $\mathfrak{q}$) by $\mathfrak{l}_P$ and $\mathfrak{n}_P$ (resp. $\mathfrak{l}_Q$ and $\mathfrak{n}_Q$). Let L_Q be the Levi subgroup of G with its Lie algebra $\mathfrak{l}_Q$. We take a regular nilpotent element e_P of $\mathfrak{l}_P$. We also take a regular nilpotent element e_Q of $\mathfrak{l}_Q$ and fix a $\mathfrak{sl}_2$-triple $\{e_Q, h_Q, f_Q\}$. We denote the centralizer of f_Q in $\mathfrak{l}_Q$ (resp. in $\mathfrak{g}$) by $Z_{\mathfrak{l}_Q}(f_Q)$ (resp. $Z_{\mathfrak{g}}(f_Q)$). Let $\mathcal{N}_P = \mathrm{Ad}(G) \cdot \mathfrak{n}_P$ and consider the scheme-theoretic intersection $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{l}_Q}(f_Q))$ of $\mathcal{N}_P$ and $e_Q + Z_{\mathfrak{l}_Q}(f_Q)$ in $\mathfrak{g}$. Then there is a $\mathbb{G}_m$-action on $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{l}_Q}(f_Q))$ induced from the $\mathbb{G}_m$-action on $\mathfrak{g}$ given by $t \cdot X = t^{-2}\mathrm{Ad}(t^{h_Q})X$ for $t \in \mathbb{G}_m$ and $X \in \mathfrak{g}$. Let $\mathcal{X}_{e_P}^Q = \{gQ \in G/Q \mid \mathrm{Ad}(g)^{-1}e_P \in \mathfrak{n}_Q\}$ be the Spaltenstein variety associated to e_P and Q.

Theorem 5.1. There is a graded algebra isomorphism

$$H^*(\mathcal{X}_{e_P}^Q, \mathbb{C}) \cong \mathbb{C}[\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{l}_Q}(f_Q))].$$

Here, the grading on $\mathbb{C}[\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{l}_Q}(f_Q))]$ comes from the $\mathbb{G}_m$-action above.

If $Q = B$ and $e_Q = 0$, then $\mathcal{X}_{e_P}^Q$ coincides with the Springer fiber $\mathcal{B}_{e_P}$ and the above description of its cohomology ring reduces to Theorem 4.1.

Remark 5.2. Spaltenstein variety $\mathcal{X}_{e_P}^Q$ is contained in and homotopy equivalent to some S3-variety. Affinization of the symplectic dual S3-variety is given by $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{g}}(f_Q))$. Let $Z(L_Q)$ be the center of L_Q. Then $Z(L_Q)$ acts on $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{g}}(f_Q))$ by the adjoint action. One can easily prove that the scheme-theoretic intersection $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{l}_Q}(f_Q))$ is isomorphic to the $Z(L_Q)$-fixed point scheme $(\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{g}}(f_Q)))^{Z(L_Q)}$ of $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{g}}(f_Q))$.

For the proof of Theorem 5.1, we use the presentation of the cohomology ring $H^*(\mathcal{X}_{e_P}^Q, \mathbb{C})$ by Brundan-Ostrik [**4**] and the defining equations of $\mathcal{N}_P$ in $\mathfrak{g}$ which was conjectured by Tanisaki [**40**] and proved by Weyman [**44**]. We first recall some result from [**4**].

5.2. Cohomology rings of Spaltenstein varieties

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ be the transpose of the partition corresponding to P and $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i \geq 0$, the composition of n corresponding to Q. Then e_P is the nilpotent matrix whose Jordan block is of type λ^T, and $\mathcal{N}_P$ is the closure of the nilpotent orbit whose Jordan block is of type λ.

Let $R := \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in n-variables. We define its grading by $\deg(x_i) = 2$. Let $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_n}$ be the parabolic subgroup of n-th symmetric group $\mathfrak{S}_n$. For $1 \leq i \leq l$ and $r \in \mathbb{Z}_{\geq 0}$, we denote by $e_r(\mu; i)$ the r-th elementary symmeric polynomial in the variables $\{x_k \mid \mu_1 + \cdots + \mu_{i-1} + 1 \leq k \leq \mu_1 + \cdots + \mu_i\}$. We also set $e_0(\mu; i) = 1$. Then the algebra of $\mathfrak{S}_\mu$-invariant polynomials $R_\mu := R^{\mathfrak{S}_\mu}$ is freely generated by $\{e_r(\mu; i) \mid 1 \leq i \leq n, 1 \leq r \leq \mu_i\}$.

For $m \geq 1$, $1 \leq i_1 < \cdots < i_m \leq n$ and $r \geq 0$, let

$$e_r(\mu; i_1, \ldots, i_m) := \sum_{r_1 + \cdots + r_m = r} e_{r_1}(\mu; i_1) \cdots e_{r_m}(\mu; i_m).$$

Let I_μ^λ be the ideal of R_μ generated by

$$\left\{ e_r(\mu; i_1, \ldots, i_m) \;\middle|\; \begin{array}{l} m \geq 1, 1 \leq i_1 < \cdots < i_m \leq n, \\ r > \mu_{i_1} + \cdots + \mu_{i_m} - \lambda_{a+1} - \cdots - \lambda_n \\ \text{where } a := \#\{i \mid \mu_i > 0, i \neq i_1, \ldots, i_m\} \end{array} \right\}.$$

Theorem 5.3 ([4]). There is an isomorphism of graded algebras

$$H^*(\mathcal{X}_{e_P}^Q, \mathbb{C}) \cong R_\mu / I_\mu^\lambda.$$

5.3. Defining equations of nilpotent orbit closures

Next we recall the defining equations of nilpotent orbit closures of $\mathfrak{g}$. We have $\mathbb{C}[\mathfrak{g}] = \mathbb{C}[x_{ij}]$, where x_{ij} $(1 \leq i, j \leq n)$ is the (i,j)-th coordinate of matrices. Let $\{g_u^\lambda\}_u$ be the set of coefficients of t^k in s-minors of $(tI - (x_{ij}))$ with $s = 1, \ldots, n$ and $k < \lambda_{n-s+1} + \lambda_{n-s+2} + \cdots + \lambda_n$.

Theorem 5.4 ([44]). The defining ideal of $\mathcal{N}_P$ in $\mathfrak{g}$ is generated by $\{g_u^\lambda\}_u$.

5.4. Proof of Theorem 5.1

First we prepare some notation. We define $\mu_i \times \mu_i$-matrices E_i and F_i as follows:

$$E_i = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$F_i = \begin{pmatrix} 0 & \mu_i - 1 & 0 & \cdots & 0 \\ 0 & 0 & 2(\mu_i - 2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & \mu_i - 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

We set $Z_i(x^{(i)}) = Z_i(x_1^{(i)}, \ldots, x_{\mu_i}^{(i)}) := E_i + x_1^{(i)} I + x_2^{(i)} F_i + x_3^{(i)} F_i^2 + \cdots + x_{\mu_i}^{(i)} F_i^{\mu_i - 1}$. Then we may assume

$$e_Q = \begin{pmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_n \end{pmatrix}$$

and

$$f_Q = \begin{pmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_n \end{pmatrix}.$$

Any element of $e_Q + Z_{\mathfrak{l}_Q}(f_Q)$ can be written as

$$Z(x) := \begin{pmatrix} Z_1(x^{(1)}) & & 0 \\ & \ddots & \\ 0 & & Z_n(x^{(n)}) \end{pmatrix}$$

for some $x_j^{(i)}$'s. We regard $x_j^{(i)}$'s as coordinates on $e_Q + Z_{\mathfrak{l}_Q}(f_Q)$ and define $\tilde{e}_r(\mu; i) \in \mathbb{C}[e_Q + Z_{\mathfrak{l}_Q}(f_Q)]$ for $1 \le r \le \mu_i$ by

$$\det(tI - Z_i(x^{(i)})) = t^{\mu_i} - \tilde{e}_1(\mu; i) t^{\mu_i - 1} + \cdots + (-1)^{\mu_i} \tilde{e}_{\mu_i}(\mu; i).$$

Then $\tilde{e}_r(\mu; i)$ is homogeneous of degree $2r$ and $\{\tilde{e}_r(\mu; i) \mid 1 \le i \le n, 1 \le r \le \mu_i\}$ freely generate the ring $\mathbb{C}[e_Q + Z_{\mathfrak{l}_Q}(f_Q)]$. We denote by $\psi : R_\mu \xrightarrow{\sim} \mathbb{C}[e_Q + Z_{\mathfrak{l}_Q}(f_Q)]$ the graded algebra isomorphism given by $\psi(e_r(\mu; i)) = \tilde{e}_r(\mu; i)$. We denote by $\tilde{e}_r(\mu; i_1, \ldots, i_m)$ the image of $e_r(\mu; i_1, \ldots, i_m)$ under ψ.

In order to prove the assertion, it suffices to show that the defining ideal $\tilde{I}_\mu^\lambda$ of $\mathcal{N}_P \cap (e_Q + Z_{\mathfrak{l}_Q}(f_Q))$ in $\mathbb{C}[e_Q + Z_{\mathfrak{l}_Q}(f_Q)]$ coincides with $\psi(I_\mu^\lambda)$. By Theorem 5.4, $\tilde{I}_\mu^\lambda$ is generated by coefficients of t^k of various s-minors of $tI - Z(x)$ with $k < \lambda_{n-s+1} + \cdots + \lambda_n$. If we remove the first row and the last column of $Z_i(x^{(i)})$, then we obtain a upper triangular matrix with diagonal entries 1. Hence for $s < \mu_i$, there is an s-minor of $tI - Z_i(x^{(i)})$ which equals to ± 1.

Let $l = \#\{i \mid \mu_i > 0\}$. Consider an s-minor of $tI - Z(x)$. Note that nonzero s-minor of $tI - Z(x)$ is certain product of s_i-minors of $tI - Z_i(x^{(i)})$ with $s_1 + \cdots + s_n = s$. We set $m = l - (n - s)$. Then we have $\#\{i \mid s_i = \mu_i > 0\} \ge m$.

First we assume $m \le 0$. Then there is an s-minor of $tI - Z(x)$ which equals to ± 1. If $\lambda_{n-s+1} + \cdots + \lambda_n = 0$, then s-minors of $tI - Z(x)$ do not contribute to $\tilde{I}_\mu^\lambda$. If $\lambda_{n-s+1} + \cdots + \lambda_n \ge 1$, then we have $1 \in \tilde{I}_\mu^\lambda$. On the other hand, there exists i such that $\mu_i = 0$ by the assumption $m \le 0$ and $1 = e_0(\mu; i) \in I_\mu^\lambda$ since we have

$$\begin{aligned}
\mu_i - \lambda_{l+1} - \cdots - \lambda_n &= -\lambda_{n-s+m+1} - \cdots - \lambda_n \\
&\le -\lambda_{n-s+1} - \cdots - \lambda_n \\
&< 0.
\end{aligned}$$

Hence we have $\tilde{I}_\mu^\lambda = \psi(I_\mu^\lambda)$ in this case.

Next we consider the case of $m \geq 1$. Let $1 \leq i_1 < \cdots < i_m \leq l$ be some labels satisfying $s_i = \mu_i$. Then this s-minor of $tI - Z(x)$ is a product of some polynomial and

$$\det(tI - Z_{i_1}) \cdots \det(tI - Z_{i_m}) = (t^{\mu_{i_1}} - \tilde{e}_1(\mu; i_1)t^{\mu_{i_1}-1} + \cdots + (-1)^{\mu_{i_1}}\tilde{e}_{\mu_{i_1}}(\mu; i_1)) \cdot$$

$$\cdots (t^{\mu_{i_m}} - \tilde{e}_1(\mu; i_m)t^{\mu_{i_m}-1} + \cdots + (-1)^{\mu_{i_m}}\tilde{e}_{\mu_{i_m}}(\mu; i_m))$$

$$= \sum_{r=0}^{\mu_{i_1}+\cdots+\mu_{i_m}} (-1)^r \tilde{e}_r(\mu; i_1, \ldots, i_m) t^{\mu_{i_1}+\cdots+\mu_{i_m}-r}.$$

Hence the coefficients of t^k with $k < \lambda_{n-s+1} + \cdots + \lambda_n$ are contained in the ideal generated by $\{\tilde{e}_r(\mu; i_1, \ldots, i_m) \mid r > \mu_{i_1} + \cdots + \mu_{i_m} - \lambda_{l-m+1} - \cdots - \lambda_n\}$. Conversely, if we choose $s_i = \mu_i - 1$ for $i \neq i_1, \ldots, i_m$ and s_i-minors of $tI - Z_i$ which are ± 1, then $\pm\tilde{e}_r(\mu; i_1, \ldots, i_m)$ appears as a coefficient of t^k for some s-minor of $(tI - Z(x))$ with $k < \lambda_{n-s+1} + \cdots + \lambda_n$. This proves $\tilde{I}^\lambda_\mu = \psi(I^\lambda_\mu)$.

CHAPTER 6

Hypertoric varieties

In this chapter, we check our proposal for hypertoric varieties.

6.1. Hypertoric varieties

We briefly recall the definition and some properties of hypertoric varieties following [**38**]. Let $T^n = (\mathbb{G}_m)^n$ be the n-dimensional complex torus and $\mathfrak{t}^n$ its Lie algebra with a full lattice $\mathfrak{t}^n_{\mathbb{Z}}$ and its basis $\{\varepsilon_i\}$. Let $\mathfrak{t}^d$ be a complex vector space of dimension d with a full lattice $\mathfrak{t}^d_{\mathbb{Z}}$. Let $\{a_1, \ldots, a_n\} \subset \mathfrak{t}^d_{\mathbb{Z}}$ be a collection of nonzero vectors which spans $\mathfrak{t}^d_{\mathbb{Z}}$. Let $a : \mathfrak{t}^n \to \mathfrak{t}^d$ be the linear map defined by $a(\varepsilon_i) = a_i$ and let $\mathfrak{t}^k$ be the kernel of a with a full lattice $\mathfrak{t}^k_{\mathbb{Z}}$. Then we have the following exact sequences

$$0 \to \mathfrak{t}^k \xrightarrow{\iota} \mathfrak{t}^n \xrightarrow{a} \mathfrak{t}^d \to 0$$

and

$$0 \to \mathfrak{t}^k_{\mathbb{Z}} \to \mathfrak{t}^n_{\mathbb{Z}} \to \mathfrak{t}^d_{\mathbb{Z}} \to 0.$$

This gives an exact sequence of tori

$$0 \to T^k \to T^n \to T^d \to 0.$$

T^n acts on $T^*\mathbb{C}^n$ symplectically by

$$(\lambda_1, \ldots, \lambda_n) \cdot (z_1, \ldots, z_n, w_1, \ldots, w_n) = (\lambda_1 z_1, \ldots, \lambda_n z_n, \lambda_1^{-1} w_1, \ldots, \lambda_n^{-1} w_n)$$

for $(\lambda_1, \ldots, \lambda_n) \in T^n$ and $(z_1, \ldots, z_n, w_1, \ldots, w_n) \in T^*\mathbb{C}^n$. The moment map $\mu_n : T^*\mathbb{C}^n \to (\mathfrak{t}^n)^*$ for this action is given by $\mu_n(z_1, \ldots, z_n, w_1, \ldots, w_n) = (z_1 w_1, \ldots, z_n w_n)$. Then the moment map for the action of T^k on $T^*\mathbb{C}^n$ is given by $\mu = \iota^* \circ \mu_n$. For $\alpha \in (\mathfrak{t}^k_{\mathbb{Z}})^*$ a character of T^k, we define the hypertoric variety associated to $\mathcal{A} = \{a_1, \ldots, a_n\}$ and α by

$$\mathfrak{M}_\alpha(\mathcal{A}) = \mu^{-1}(0) /\!\!/_\alpha T^k.$$

Here the quotient above is the GIT quotient with respect to α. For $r = (r_1, \ldots, r_n) \in (\mathfrak{t}^n)^*$ a lift of α along ι^* and $i = 1, \ldots, n$, we set

$$H_i = \{x \in (\mathfrak{t}^d)^*_{\mathbb{R}} \mid x \cdot a_i + r_i = 0\}.$$

Hyperplane arrangement $\{H_1, \ldots, H_n\}$ is called simple if every subset of m hyperplanes with nonempty intersection intersects in codimension m and $\mathcal{A}$ is called unimodular if every collection of d linearly independent vectors $\{a_{i_1}, \ldots, a_{i_d}\}$ spans $\mathfrak{t}^d_{\mathbb{Z}}$ over $\mathbb{Z}$.

Proposition 6.1. Hypertoric variety $\mathfrak{M}_\alpha(\mathcal{A})$ associated with $\mathcal{A}$ and α is smooth if and only if $\{H_1, \ldots, H_n\}$ is simple and $\mathcal{A}$ is unimodular.

The torus T^d naturally acts on $\mathfrak{M}_\alpha(\mathcal{A})$ and preserves the symplectic form.

6.2. Main result

We set $\check{\mathfrak{t}}^n = (\mathfrak{t}^n)^*$, $\check{\mathfrak{t}}^d = (\mathfrak{t}^d)^*$, and $\check{\mathfrak{t}}^k = (\mathfrak{t}^k)^*$. We denote by $\check{T}^n$, $\check{T}^d$, and $\check{T}^k$ the dual tori of T^n, T^d, and T^k. Set $b = \iota^*$ and set $b_i \in \check{\mathfrak{t}}^k_{\mathbb{Z}}$ to be the image of the standard basis of $\check{\mathfrak{t}}^n_{\mathbb{Z}}$ under b. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be the Gale dual configuration of $\mathcal{A}$. Let us fix a basis of $\mathfrak{t}^d_{\mathbb{Z}} \cong \mathbb{Z}^d$ and its dual basis $\check{\mathfrak{t}}^d_{\mathbb{Z}} \cong \mathbb{Z}^d$. Let us write $a_i = (a_{i1}, \ldots, a_{id})$ using this basis. Then the moment map $\check{\mu} : T^*\mathbb{C}^n \to \mathfrak{t}^d$ for the $\check{T}^d$-action on $T^*\mathbb{C}^n$ is given by

$$\check{\mu}(z_1, \ldots, z_n, w_1, \ldots, w_n) = \left(\sum_i a_{i1} z_i w_i, \ldots, \sum_i a_{id} z_i w_i \right). \tag{6.1}$$

The action of $(\lambda_1, \ldots, \lambda_d) \in \check{T}^d$ on $\mathbb{C}[T^*\mathbb{C}^n]$ is given by

$$(\lambda_1, \ldots, \lambda_d) \cdot z_i = \lambda_1^{a_{i1}} \cdots \lambda_d^{a_{id}} z_i,$$

$$(\lambda_1, \ldots, \lambda_d) \cdot w_i = \lambda_1^{-a_{i1}} \cdots \lambda_d^{-a_{id}} w_i.$$

We consider the hypertoric variety $\mathfrak{M}_0(\mathcal{B}) = \mathrm{Spec}(\mathbb{C}[\check{\mu}^{-1}(0)]^{\check{T}^d})$ associated to $\mathcal{B}$ and $0 \in (\mathfrak{t}^d)^*_{\mathbb{Z}}$. The torus $\check{T}^k = \check{T}^n/\check{T}^d$ naturally acts on $\mathfrak{M}_0(\mathcal{B})$ and there is another $\mathbb{G}_m$ action on $\mathfrak{M}_0(\mathcal{B})$ induced from its action on $T^*\mathbb{C}^n$ given by $t \cdot (z, w) = (t^{-1}z, t^{-1}w)$. This $\mathbb{G}_m$-action induces an $\mathbb{G}_m$-action on the fixed point scheme $\mathfrak{M}_0(\mathcal{B})^{\check{T}^k}$. The aim of this chapter is to prove the following.

Theorem 6.2. If $\mathfrak{M}_\alpha(\mathcal{A})$ is smooth, then there is an isomorphism of graded algebras

$$H^*(\mathfrak{M}_\alpha(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}[\mathfrak{M}_0(\mathcal{B})^{\check{T}^k}].$$

Here, grading on $\mathbb{C}[\mathfrak{M}_0(\mathcal{B})^{\check{T}^k}]$ comes from the $\mathbb{G}_m$-action above.

6.3. Cohomology rings of hypertoric varieties

Let $\Delta_\mathcal{A}$ be the matroid complex associated to $\mathcal{A}$, that is, the simplicial complex consisting of all sets $S \subset \{1, \ldots, n\}$ such that $\{a_i \mid i \in S\}$ are linearly independent. Let

$$\mathcal{SR}(\Delta_\mathcal{A}) := \mathbb{C}[e_1, \ldots, e_n] / \left(\prod_{i \in S} e_i \mid S \notin \Delta_\mathcal{A} \right)$$

be the Stanley-Reisner ring of $\Delta_\mathcal{A}$. We define its grading by setting $\deg(e_i) = 2$. Then the cohomology ring of $\mathfrak{M}_\alpha(\mathcal{A})$ can be described as follows.

Theorem 6.3 ([22],[28]). If $\mathfrak{M}_\alpha(\mathcal{A})$ is smooth (or has at worst orbifold singularities), then there is an isomorphism of graded algebras

$$H^*(\mathfrak{M}_\alpha(\mathcal{A}), \mathbb{C}) \cong \mathcal{SR}(\Delta_\mathcal{A}) / \left(\sum_{i=1}^n a_{ij} e_i \mid j = 1, \ldots, d \right).$$

6.4. Proof of Theorem 6.2

For $x \in \mathbb{Z}$, we write $[x]_+ := \max(x, 0)$. We set $u_i := z_i w_i$ for $1 \le i \le n$ and $v_{\vec{m}} = \prod_i z_i^{[m_i]_+} w_i^{[-m_i]_+}$ for $\vec{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ with $\sum_i m_i a_i = 0$. Then we have $u_i,\ v_{\vec{m}} \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]^{\check{T}^d} = \mathbb{C}[T^*\mathbb{C}^n/\check{T}^d]$. If a monomial $\prod_i z_i^{c_i} w_i^{c'_i}$ is contained in $\mathbb{C}[T^*\mathbb{C}^n/\check{T}^d]$, then we have $\sum_i (c_i - c'_i) a_i = 0$. Hence we can write $\prod_i z_i^{c_i} w_i^{c'_i} = v_{\vec{m}} \prod_i u_i^{\min(c_i, c'_i)}$ by setting $m_i = c_i - c'_i$. Therefore, $\mathbb{C}[T^*\mathbb{C}^n/\check{T}^d]$ is generated by $\{v_{\vec{m}} \mid \sum_i m_i a_i = 0\}$ as a $\mathbb{C}[u_1, \dots, u_n]$-module and the defining ideal of the $\check{T}^k$-fixed point scheme $(T^*\mathbb{C}^n/\check{T}^d)^{\check{T}^k}$ in $T^*\mathbb{C}^n/\check{T}^d$ is generated by $v_{\vec{m}}$'s for $\vec{m} \neq 0$. It follows that $\mathbb{C}[(T^*\mathbb{C}^n/\check{T}^d)^{\check{T}^k}]$ is generated by $u_1, \dots, u_n$ as a $\mathbb{C}$-algebra.

Let $S \subset \{1, \dots, n\}$ be a circuit of $\Delta_{\mathcal{A}}$, i.e. minimal among the subsets of $\{1, \dots, n\}$ which is not in $\Delta_{\mathcal{A}}$. There is a relation $\sum_{i \in S} p_i a_i = 0$, where all $p_i \in \mathbb{Z}$ are nonzero. For any $i_0 \in S$, $a_{i_0} = -\sum_{i \in S \setminus \{i_0\}} \frac{p_i}{p_{i_0}} a_i$ and $\{a_i\}_{i \in S \setminus \{i_0\}}$ is linearly independent. Hence from the unimodularity of $\mathcal{A}$, we have $\frac{p_i}{p_{i_0}} \in \mathbb{Z}$. Therefore, we can take $p_i = \pm 1$ for all $i \in S$. We set $p_i = 0$ for $i \notin S$ and $\vec{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$. Then we have $\prod_{i \in S} u_i = v_{\vec{p}} v_{-\vec{p}}$ in $\mathbb{C}[T^*\mathbb{C}^n/\check{T}^d]$ and hence $\prod_{i \in S} u_i = 0$ in $\mathbb{C}[(T^*\mathbb{C}^n/\check{T}^d)^{\check{T}^k}]$. On the other hand, if a monomial $\prod_i u_i^{q_i}$ in u_i is zero in $\mathbb{C}[(T^*\mathbb{C}^n/\check{T}^d)^{\check{T}^k}]$, then there exists $\vec{c}, \vec{m}, \vec{m}' \in \mathbb{Z}^n$ with $\sum_i m_i a_i = \sum_i m'_i a_i = 0$ and $\vec{m} \neq 0$ such that

$$\prod_i u_i^{q_i} = v_{\vec{m}} v_{\vec{m}'} \prod_i u_i^{c_i}.$$

Hence the subset $\{i \mid q_i \neq 0\} \subset \{1, \dots, n\}$ contains $\{i \mid m_i \neq 0\}$ which is not in $\Delta_{\mathcal{A}}$. Therefore, $\prod_i u_i^{q_i}$ is contained in the ideal of $\mathbb{C}[u_1 \dots, u_n]$ generated by $\prod_{i \in S} u_i$ for $S \notin \Delta_{\mathcal{A}}$. It follows that we have an isomorphism of graded algebras

$$\mathbb{C}[(T^*\mathbb{C}^n/\check{T}^d)^{\check{T}^k}] \cong \mathbb{C}[u_1, \dots, u_n] / \left(\prod_{i \in S} u_i \mid S \notin \Delta_{\mathcal{A}} \right) \cong \mathcal{SR}(\Delta_{\mathcal{A}})$$

by sending u_i to e_i. By (6.1), we have

$$\mathbb{C}[\mathfrak{M}_0(\mathcal{B})] \cong \mathbb{C}[T^*\mathbb{C}^n/\check{T}^d]/\left(\sum_i a_{ij}u_i \mid j = 1, \dots, d\right).$$

It follows that

$$\begin{aligned}\mathbb{C}[\mathfrak{M}_0(\mathcal{B})^{\check{T}^k}] &\cong \mathbb{C}[(T^*\mathbb{C}^n/\check{T}^d)^{\check{T}^k}]/\left(\sum_i a_{ij}u_i \mid j = 1, \dots, d\right) \\ &\cong \mathcal{SR}(\Delta_{\mathcal{A}})/\left(\sum_{i=1}^n a_{ij}e_i \mid j = 1, \dots, d\right).\end{aligned}$$

By Theorem 6.3, this implies Theorem 6.2.

CHAPTER 7

Hilbert schemes of points in the affine plane

In this chapter, we prove Theorem 4.3.

7.1. Cohomology rings of Hilbert schemes of points

In [**30**] and [**42**], Lehn-Sorger and Vasserot independently give an description of the cohomology ring of $\mathrm{Hilb}^n(\mathbb{C}^2)$ as the center of the group ring of the symmetric group $\mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$. In this section, we recall some results of [**30**] and [**42**].

For a partition $\hat{\lambda}$ of n, we denote by $\mathfrak{C}_{\hat{\lambda}} \subset \mathfrak{S}_n$ the conjugacy class of $\mathfrak{S}_n$ consisting of permutation whose cycle type is $\hat{\lambda}$. If $\hat{\lambda} = (1^{\hat{\alpha}_1} 2^{\hat{\alpha}_2} \ldots)$, then we have

$$\#\mathfrak{C}_{\hat{\lambda}} = \frac{n!}{\prod_i i^{\hat{\alpha}_i} \hat{\alpha}_i!}.$$

We define the characteristic function $\chi_{\hat{\lambda}} \in \mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$ by

$$\chi_{\hat{\lambda}} = \sum_{\sigma \in \mathfrak{C}_{\hat{\lambda}}} \sigma.$$

Then $\{\chi_{\hat{\lambda}}\}_{\hat{\lambda} \vdash n}$ forms an basis of $\mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$. For $\sigma \in \mathfrak{S}_n$ of cycle type $\hat{\lambda}$, we define its degree by $\deg(\sigma) = 2(n - \ell(\hat{\lambda}))$. Let $\mathbb{C}[\mathfrak{S}_n]_d$ be the subspace of $\mathbb{C}[\mathfrak{S}_n]$ spanned by σ with $\deg(\sigma) = d$ and

$$F^d\mathbb{C}[\mathfrak{S}_n] := \bigoplus_{d' \leq d} \mathbb{C}[\mathfrak{S}_n]_{d'}.$$

Then $F^d\mathbb{C}[\mathfrak{S}_n]$ defines a filtration on $\mathbb{C}[\mathfrak{S}_n]$ compatible with the product. The induced product on $\mathrm{gr}^F \mathbb{C}[\mathfrak{S}_n]$ is called the cup product and denoted by $\cup$. Since $\chi_{\hat{\lambda}}$ is homogeneous, $\mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$ inherits from $\mathbb{C}[\mathfrak{S}_n]$ the gradation, filtration, and the cup product.

Theorem 7.1 ([30],[42]). There is an isomorphism of graded algebras

$$H^*(\mathrm{Hilb}^n(\mathbb{C}^2),\mathbb{C}) \cong \mathrm{gr}^F\, \mathcal{Z}(\mathbb{C}[\mathfrak{S}_n]).$$

We prove Theorem 4.3 by identifying $\mathrm{gr}^F\, \mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$ and $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$. For later use, we give a formula for the cup product with $\chi_{(k+1,1^{n-k-1})}$ for $1 \leq k \leq n-1$. For two partitions $\hat{\mu} = (1^{\hat{\beta}_1}2^{\hat{\beta}_2}\ldots)$ and $\hat{\nu} = (1^{\hat{\gamma}_1}2^{\hat{\gamma}_2}\ldots)$, we write $\hat{\mu} \preceq \hat{\nu}$ if $\hat{\beta}_i \leq \hat{\gamma}_i$ for any i. Note that this partial order is not related to the usual partial order on the set of partitions.

Lemma 7.2. For $\hat{\lambda} = (1^{\hat{\alpha}_1}2^{\hat{\alpha}_2}\ldots) \vdash n$, we have

$$\chi_{(k+1,1^{n-k-1})} \cup \chi_{\hat{\lambda}} = \sum_{\substack{\hat{\nu}=(1^{\hat{\gamma}_1}2^{\hat{\gamma}_2}\ldots) \\ \ell(\hat{\nu})=k+1,\hat{\nu}\preceq\hat{\lambda}}} \frac{k!|\hat{\nu}|(\hat{\alpha}_{|\hat{\nu}|}+1)}{\prod_i \hat{\gamma}_i!}\chi_{\hat{\lambda}_{\hat{\nu}}}.$$

Here, $\hat{\nu}$ runs over partitions with $\ell(\hat{\nu}) = k+1$ and $\hat{\nu} \preceq \hat{\lambda}$, and $\hat{\lambda}_{\hat{\nu}} = (1^{\hat{\beta}_1}2^{\hat{\beta}_2}\ldots) \vdash n$ is defined by

$$\hat{\beta}_i = \begin{cases} \hat{\alpha}_i - \hat{\gamma}_i & \text{if } i \neq |\hat{\nu}| \\ \hat{\alpha}_{|\hat{\nu}|}+1 & \text{if } i = |\hat{\nu}|. \end{cases}$$

Proof. For an element $\sigma \in \mathfrak{C}_{(k+1,1^{n-k-1})}$ (say $\sigma = (123\ldots k+1)$) and $\tau \in \mathfrak{C}_{\hat{\lambda}}$, the equality $\deg(\sigma\tau) = \deg(\sigma) + \deg(\tau)$ holds if and only if $1, 2, \ldots, k+1$ are contained in different cycles for the disjoint cycle decomposition of τ. If the number of elements of $\{1, 2, \ldots, k+1\}$ which are contained in cycle of length i is $\hat{\gamma}_i$, then the cycle type of $\sigma\tau$ is given by $\hat{\lambda}_{\hat{\nu}}$. For a fixed $\sigma \in \mathfrak{C}_{(k+1,1^{n-k-1})}$, one can see that the number of $\tau \in \mathfrak{C}_{\hat{\lambda}}$ such that the cycle type of $\sigma\tau$ equals to $\hat{\lambda}_{\hat{\nu}}$ is given by

$$\frac{(k+1)!(n-k-1)!}{\prod_i i^{\hat{\alpha}_i-\hat{\gamma}_i}\hat{\gamma}_i!(\hat{\alpha}_i-\hat{\gamma}_i)!}.$$

Hence we have

$$\begin{aligned}\chi_{(k+1,1^{n-k-1})} \cup \chi_{\hat{\lambda}} &= \sum_{\substack{\hat{\nu}=(1^{\hat{\gamma}_1}2^{\hat{\gamma}_2}\dots) \\ \ell(\hat{\nu})=k+1, \hat{\gamma}_i \leq \hat{\alpha}_i}} \frac{(k+1)!(n-k-1)!}{\prod_i i^{\hat{\alpha}_i-\hat{\gamma}_i}\hat{\gamma}_i!(\hat{\alpha}_i-\hat{\gamma}_i)!} \frac{\#\mathfrak{C}_{(k+1,1^{n-k-1})}}{\#\mathfrak{C}_{\hat{\lambda}_{\hat{\nu}}}} \chi_{\hat{\lambda}_{\hat{\nu}}} \\ &= \sum_{\hat{\nu}} \frac{(k+1)!(n-k-1)!}{\prod_i i^{\hat{\alpha}_i-\hat{\gamma}_i}\hat{\gamma}_i!(\hat{\alpha}_i-\hat{\gamma}_i)!} \frac{n!}{(k+1)(n-k-1)!} \frac{\prod_i i^{\hat{\beta}_i}\hat{\beta}_i!}{n!} \chi_{\hat{\lambda}_{\hat{\nu}}} \\ &= \sum_{\hat{\nu}} \frac{k!|\hat{\nu}|(\hat{\alpha}_{|\hat{\nu}|}+1)}{\prod_i \hat{\gamma}_i!} \chi_{\hat{\lambda}_{\hat{\nu}}},\end{aligned}$$

which completes the proof. □

7.2. MacMahon symmetric functions

In this section, we study the ring structure of $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$. First we prepare some notations on symmetric functions in two set of variables (called MacMahon symmetric functions in [**11**]). For an element $(a,b) \in \mathbb{N}\times\mathbb{N}$, an unordered sequence of vectors $\Lambda = (a_1,b_1)(a_2,b_2)\dots(a_l,b_l)$ is called a bipartite partition of (a,b) if $(a_i,b_i) \in \mathbb{N}\times\mathbb{N}\backslash\{(0,0)\}$ for any i and $\sum_{i=1}^{l} a_i = a$, $\sum_{i=1}^{l} b_i = b$. We set $\ell(\Lambda) = l$ and $|\Lambda| = (a,b)$. We have a natural surjection

$$\mathbb{C}[S^{n+1}\mathbb{C}^2] = \mathbb{C}[x_1,\dots,x_{n+1},y_1,\dots,y_{n+1}]^{\mathfrak{S}_{n+1}} \twoheadrightarrow \mathbb{C}[S^n\mathbb{C}^2] = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$$

induced by $x_i, y_i \mapsto x_i, y_i$ for $1 \leq i \leq n$ and $x_{n+1}, y_{n+1} \mapsto 0$. We consider the projective limit

$$S := \varprojlim_n \mathbb{C}[S^n\mathbb{C}^2]$$

with respect to these surjections. This is the ring of MacMahon symmetric functions.

For a bipartite partition $\Lambda = (a_1,b_1)\dots(a_l,b_l)$, we define the monomial symmetric function $m_\Lambda \in S$ by symmetrization of the monomial $x_1^{a_1}y_1^{b_1}\cdots x_l^{a_l}y_l^{b_l}$ with coefficients 0 or

1. For example, we have

$$m_{(1,1)(1,1)} = \sum_{i<j} x_i y_i x_j y_j.$$

It is clear that the monomial symmetric functions form an basis of S. Just as power sum symmetric functions freely generate the ring of symmetric functions, $m_{(a,b)}$'s generate S as a $\mathbb{C}$-algebra and they are algebraically independent ([**7**]). Hence we have

$$S \cong \mathbb{C}[m_{(a,b)} \mid (a,b) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}].$$

The kernel of the natural surjection $S \twoheadrightarrow \mathbb{C}[S^n\mathbb{C}^2]$ is generated by $\{m_\Lambda \mid \ell(\Lambda) > n\}$ as an ideal or a vector space. If $|\Lambda| = (a,b)$, then the weight of m_Λ with respect to the $\mathbb{T}$-action induced by $t \cdot (x,y) = (t^{-1}x, ty)$ for $(x,y) \in \mathbb{C}^2$ is $a - b$. Hence the ideal of definition for the $\mathbb{T}$-fixed point scheme $(S^n\mathbb{C}^2)^{\mathbb{T}}$ in $\mathbb{C}[S^n\mathbb{C}^2]$ is generated by $\{m_{(a,b)} \mid a \neq b\}$. Therefore, we have the following.

Lemma 7.3.

$$\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}] \cong S/\left(m_\Lambda, m_{(a,b)} \mid \ell(\Lambda) > n, a \neq b\right).$$

In particular, $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$ is generated as a $\mathbb{C}$-algebra by $m_{(a,a)}$'s.

For $(a,b) \in \mathbb{N}\times\mathbb{N}\setminus\{(0,0)\}$ and $\Lambda = (a_1,b_1)\ldots(a_l,b_l)$, we denote by $(a,b)\Lambda$ the bipartite partition $(a,b)(a_1,b_1)\ldots(a_l,b_l)$. If $(a,b) = (a_i,b_i)$ for some i, we denote by $\Lambda \setminus (a,b)$ the bipartite partition $(a_1,b_1)\ldots(a_{i-1},b_{i-1})(a_{i+1},b_{i+1})\ldots(a_l,b_l)$. We have $\ell(\Lambda \setminus (a,b)) = \ell(\Lambda) - 1$ and $|\Lambda \setminus (a,b)| = |\Lambda| - (a,b)$.

Lemma 7.4. Let Λ be a bipartite partition. For any $(i,j) \in \mathbb{N}\times\mathbb{N}\setminus\{(0,0)\}$, we denote by $c_{(i,j)}$ the multiplicity of (i,j) in Λ. Then for $(a,b) \in \mathbb{N}\times\mathbb{N}\setminus\{(0,0)\}$, we have

$$m_{(a,b)}m_\Lambda = (c_{(a,b)}+1)m_{(a,b)\Lambda} + \sum_{\substack{(i,j)\\ c_{(i,j)}>0}} (c_{(a+i,b+j)}+1)m_{(a+i,b+j)\Lambda\setminus(i,j)}.$$

Proof. For $c = c_{(a+i,b+j)}+1$, we consider the monomial

$$x_1^{a+i}y_1^{b+j}\cdots x_c^{a+i}y_c^{b+j} \times (\text{monomial in } x_i \text{ and } y_i \text{ for } i > c)$$

in $m_{(a+i,b+j)\Lambda\setminus(i,j)}$ and its coefficient in the expansion of $m_{(a,b)}m_\Lambda$. In the expansion, this monomial appears as

$$x_1^a y_1^b \cdot x_1^i y_1^j x_2^{a+i} y_2^{b+j} \cdots x_c^{a+i} y_c^{b+j} \cdots,$$

$$x_2^a y_2^b \cdot x_1^{a+i} y_1^{b+j} x_2^i y_2^j x_3^{a+i} y_3^{b+j} \cdots x_c^{a+i} y_c^{b+j} \cdots,$$

$$\cdots\cdots$$

$$x_c^a y_c^b \cdot x_1^{a+i} y_1^{b+j} x_2^{a+i} y_2^{b+j} \cdots x_{c-1}^{a+i} y_{c-1}^{b+j} x_c^i y_c^j \cdots.$$

Hence the coefficient of $m_{(a+i,b+j)\Lambda\setminus(i,j)}$ in $m_{(a,b)}m_\Lambda$ is given by $c = c_{(a+i,b+j)}+1$. The coefficient of $m_{(a,b)\Lambda}$ can be understood in the same way. $\square$

We denote by $\bar{m}_\Lambda$ the image of m_Λ under $S \twoheadrightarrow \bar{S} := S/(m_{(a,b)} \mid a \neq b)$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we denote by $(\lambda, 0)$ the bipartite partition $(\lambda_1,0)(\lambda_2,0)\ldots(\lambda_l,0)$.

Lemma 7.5. $\{\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}\}$ forms a basis of $\bar{S}$.

Proof. We first prove that the monomial symmetric functions of the form $\bar{m}_{(a,a)(b,b)(c,c)\ldots}$ span $\bar{S}$. If Λ does not contain (a,b) with $a \neq b$, then $\bar{m}_\Lambda$ is already of the form $\bar{m}_{(a,a)(b,b)(c,c)\ldots}$. If Λ contains (a,b) with $a \neq b$, then we can expand $\bar{m}_\Lambda$ in terms of $\bar{m}_\Phi$ with $\ell(\Phi) = \ell(\Lambda) - 1$

by Lemma 7.4 and $\bar{m}_{(a,b)} = 0$. By induction on the length of Λ, we get an expansion of $\bar{m}_\Lambda$ in terms of monomial symmetric functions of the form $\bar{m}_{(a,a)(b,b)(c,c)\dots}$.

We next prove that we can expand $\bar{m}_{(a,a)(b,b)(c,c)\dots}$ in terms of $\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}$'s. More generally, we show that monomial symmetric functions of the form $\bar{m}_{(a_1,b_1)\dots(a_l,b_l)(0,1)^m}$ with $a_i \geq b_i$ for any i can be written as a linear combination of $\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}$'s. We prove this claim by induction on $d = \sum_i b_i$. If $d = 0$, then there is nothing to prove.

Assume $d > 0$ and the claim holds for smaller d. Then at least one of b_i is positive. We can assume $b_1 > 0$. By Lemma 7.4, we have

$$
\begin{aligned}
m_{(a_1,b_1-1)}m_{(a_2,b_2)\dots(a_l,b_l)(0,1)^{m+1}} = c_0 m_{(a_1,b_1-1)(a_2,b_2)\dots(a_l,b_l)(0,1)^{m+1}} + c_1 m_{(a_1,b_1)\dots(a_l,b_l)(0,1)^m} \\
+ \sum_{i=2}^{l} c_i m_{(a_2,b_2)\dots(a_1+a_i,b_1+b_i-1)\dots(a_l,b_l)(0,1)^{m+1}}
\end{aligned}
$$

for some $c_0, c_1, \dots, c_l \in \mathbb{Z}$ with $c_1 \neq 0$. Hence by $\bar{m}_{(a_1,b_1-1)} = 0$ and the induction hypothesis, we can expand $\bar{m}_{(a_1,b_1)\dots(a_l,b_l)(0,1)^m}$ in terms of $\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}$'s.

Since we have $\bar{S} \cong \mathbb{C}[m_{(a,a)} \mid a \in \mathbb{Z}_{>0}]$, the dimension of the degree $2k$-component of $\bar{S}$ is given by the number of partitions of k. Hence $\{\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} \mid |\lambda| = k\}$ is linearly independent. This proves the lemma. □

In order to simplify some formulas, we understand that $1/x! = 0$ for $x < 0$ in the below.

Lemma 7.6. For $k \geq l > 0$ and $\lambda = (1^{\alpha_1}2^{\alpha_2}\dots)$, we have

$$
\bar{m}_{(k,l)(\lambda,0)(0,1)^{|\lambda|+k-l}} = \sum_{\mu=(1^{\beta_1}2^{\beta_2}\dots)} \frac{(-1)^l l!(\beta_{|\lambda|-|\mu|+k}+1)}{(l-\ell(\lambda)+\ell(\mu))!\prod_i(\alpha_i-\beta_i)!} \bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}.
$$

Here, μ runs over all the partitions. By the above convention, only partitions satisfying $\ell(\lambda) - \ell(\mu) \leq l$ and $\mu \preceq \lambda$ contribute.

Proof. We prove this formula by induction on l. Assume $l = 1$. By using Lemma 7.4 for $(a, b) = (k, 0)$ and $\bar{m}_{(k,0)} = 0$, we have

$$\bar{m}_{(k,1)(\lambda,0)(0,1)^{|\lambda|+k-1}} = -(\alpha_k + 1)\bar{m}_{(k,0)(\lambda,0)(0,1)^{|\lambda|+k}} - \sum_{i:\alpha_i>0} (\alpha_{k+i} + 1)\bar{m}_{(k+i,0)(\lambda\setminus i,0)(0,1)^{|\lambda|+k}}.$$

The first term is equal to the contribution of $\mu = \lambda$ and the second term is equal to the contribution of $\mu = \lambda \setminus i$.

Assume $l \geq 2$ and the formula holds for smaller l. By using Lemma 7.4 for $(a, b) = (k, l - 1)$ and the induction hypothesis, we have

$$\begin{aligned}\bar{m}_{(k,l)(\lambda,0)(0,1)^{|\lambda|+k-l}} &= -\bar{m}_{(k,l-1)(\lambda,0)(0,1)^{|\lambda|+k-l+1}} - \sum_{j:\alpha_j>0} \bar{m}_{(k+j,l-1)(\lambda\setminus j,0)(0,1)^{|\lambda|+k-l+1}} \\ &= \sum_{\mu=(1^{\beta_1}2^{\beta_2}\dots)} \bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}} \left\{ \frac{(-1)^l(l-1)!(\beta_{|\lambda|-|\mu|+k}+1)}{(l-\ell(\lambda)+\ell(\mu)-1)!\prod_i(\alpha_i-\beta_i)!} \right. \\ &\qquad \left. + \sum_j \frac{(-1)^l(l-1)!(\beta_{|\lambda|-|\mu|+k}+1)}{(l-\ell(\lambda)+\ell(\mu))!(\alpha_j-\beta_j-1)!\prod_{i\neq j}(\alpha_i-\beta_i)!} \right\} \\ &= \sum_{\mu} \frac{(-1)^l l!(\beta_{|\lambda|-|\mu|+k}+1)}{(l-\ell(\lambda)+\ell(\mu))!\prod_i(\alpha_i-\beta_i)!} \bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}.\end{aligned}$$

Here, in the last equality, we used the formula

$$\frac{(n_0 + \cdots + n_a)!}{n_0! \cdots n_a!} = \sum_{i=0}^{a} \frac{(n_0 + \cdots n_a - 1)!}{(n_j - 1)! \prod_{i\neq j} n_i!} \tag{7.1}$$

for $n_0, \dots, n_a \in \mathbb{Z}_{\geq 0}$ with $n_0 + \cdots + n_a > 0$. We applied this formula for $n_0 = l - \ell(\lambda) + \ell(\mu)$ and $n_i = \alpha_i - \beta_i$ for $i \geq 1$. □

For $\mu = (1^{\beta_1}2^{\beta_2}\dots)$, $\nu = (1^{\gamma_1}2^{\gamma_2}\dots)$, and $x \in \mathbb{Z}_{\geq|\nu|}$, we set

$$f^{\mu}_{\nu}(x) = \frac{(x - |\nu|)!(x - |\mu| + 1)}{(x - |\nu| - \ell(\nu) + \ell(\mu) + 1)! \prod_i(\gamma_i - \beta_i)!}.$$

Remark that we have $f^{\mu}_{\nu}(x) = 0$ if $\mu \not\leq \nu$ and $f^{\nu}_{\nu}(x) = 1$. We denote the partition $(1^{\beta_1} \dots (j-1)^{\beta_{j-1}} j^{\beta_j+1} (j+1)^{\beta_{j+1}} \dots)$ by $\mu \cup j$.

Lemma 7.7. For $\mu = (1^{\beta_1}2^{\beta_2}\dots)$, $\nu = (1^{\gamma_1}2^{\gamma_2}\dots)$, and $x \in \mathbb{Z}_{\geq|\nu|}$, we have

$$f^\mu_\nu(x+1) - f^\mu_\nu(x) = \sum_j f^{\mu\cup j}_\nu(x).$$

Proof. We calculate as:

$$\begin{aligned}
f^\mu_\nu(x+1) - f^\mu_\nu(x) &= \frac{(x-|\nu|)!}{(x-|\nu|-\ell(\nu)+\ell(\mu)+2)!\prod_i(\gamma_i-\beta_i)!} \\
&\quad \times \big\{(x+1-|\nu|)(x-|\mu|+2) - (x-|\nu|-\ell(\nu)+\ell(\mu)+2)(x-|\mu|+1)\big\} \\
&= \frac{(x-|\nu|)!\big\{(\ell(\nu)-\ell(\mu))(x-|\mu|+1)-|\nu|+|\mu|\big\}}{(x-|\nu|-\ell(\nu)+\ell(\mu)+2)!\prod_i(\gamma_i-\beta_i)!} \\
&= \frac{(x-|\nu|)!\big\{\sum_j(\gamma_j-\beta_j)(x-|\mu|-j+1)\big\}}{(x-|\nu|-\ell(\nu)+\ell(\mu)+2)!\prod_i(\gamma_i-\beta_i)!} \\
&= \sum_j \frac{(x-|\nu|)!(x-|\mu\cup j|+1)}{(x-|\nu|-\ell(\nu)+\ell(\mu\cup j)+1)!(\gamma_j-\beta_j-1)!\prod_{i\neq j}(\gamma_i-\beta_i)!} \\
&= \sum_j f^{\mu\cup j}_\nu(x).
\end{aligned}$$

□

Lemma 7.8. For $\mu = (1^{\beta_1}2^{\beta_2}\dots) \preceq \lambda = (1^{\alpha_1}2^{\alpha_2}\dots)$, we have

$$\sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\dots)\\ \mu\preceq\nu\preceq\lambda}} (-1)^{\ell(\nu)+\ell(\lambda)} f^\mu_\nu(k+|\lambda|)\frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|)!}{(|\lambda|-|\nu|)!\prod_i(\alpha_i-\gamma_i)!} = \frac{k!}{(k-\ell(\lambda)+\ell(\mu))!\prod_i(\alpha_i-\beta_i)!}$$

Proof. We prove this formula by induction on $\ell = \ell(\lambda)-\ell(\mu)$. If $\ell = 0$, then the formula is trivial since $\mu = \nu = \lambda$ and $f^\nu_\nu(x) = 1$. Let us assume $\ell > 0$ and the formula holds for smaller ℓ. We set

$$F^\mu_\lambda(k) = \sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\dots)\\ \mu\preceq\nu\preceq\lambda}} (-1)^{\ell(\nu)+\ell(\lambda)} f^\mu_\nu(k+|\lambda|)\frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|)!}{(|\lambda|-|\nu|)!\prod_i(\alpha_i-\gamma_i)!}.$$

By Lemma 7.7 and the induction hypothesis, we have

$$\begin{aligned}
F_\lambda^\mu(k+1) - F_\lambda^\mu(k) &= \sum_j F_\lambda^{\mu\cup j}(k) \\
&= \sum_j \frac{k!}{(k-\ell(\lambda)+\ell(\mu)+1)!(\alpha_j-\beta_j-1)!\prod_{i\neq j}(\alpha_i-\beta_i)!} \\
&= \frac{(k+1)!}{(k+1-\ell(\lambda)+\ell(\mu))!\prod_i(\alpha_i-\beta_i)!} - \frac{k!}{(k-\ell(\lambda)+\ell(\mu))!\prod_i(\alpha_i-\beta_i)!}.
\end{aligned}$$

Here, the last equality follows from (7.1). Hence it is enough to prove the case of $k=0$, that is, $F_\lambda^\mu(0)=0$ since we assumed $\ell(\lambda)-\ell(\mu)>0$. We set $n_i=\alpha_i-\beta_i$ and $m_i=\alpha_i-\gamma_i$. Then

$$\begin{aligned}
F_\lambda^\mu(0) &= \sum_{\mu\preceq\nu\preceq\lambda}(-1)^{\ell(\nu)+\ell(\lambda)}\frac{(|\lambda|-|\nu|)!(|\lambda|-|\mu|+1)(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|)!}{(|\lambda|-|\nu|-\ell(\nu)+\ell(\mu)+1)!(|\lambda|-|\nu|)!\prod_i(\gamma_i-\beta_i)!(\alpha_i-\gamma_i)!} \\
&= (|\lambda|-|\mu|+1)\sum_{0\le m_i\le n_i}(-1)^{\sum_i m_i}\frac{(\sum_i(i+1)m_i)!}{(\sum_i(i+1)m_i-\sum_i n_i+1)!\prod_i m_i!(n_i-m_i)!} \\
&= \frac{(|\lambda|-|\mu|+1)(\sum_i n_i-1)!}{\prod_i n_i!}\sum_{0\le m_i\le n_i}(-1)^{\sum_i m_i}\binom{\sum_i(i+1)m_i}{\sum_i n_i-1}\prod_i\binom{n_i}{m_i}
\end{aligned}$$

Let us consider the coefficient of $x^{\sum_i n_i-1}$ in the expansion of $\prod_i((x+1)^{i+1}-1)^{n_i}$. Since we have $\prod_i((x+1)^{i+1}-1)^{n_i}=\prod_i(i+1)^{n_i}x^{\sum_i n_i}+$ (higher order terms), the coefficient of $x^{\sum_i n_i-1}$ is 0. On the other hand, we calculate as:

$$\begin{aligned}
\prod_i((x+1)^{i+1}-1)^{n_i} &= \prod_i\left(\sum_{0\le m_i\le n_i}(-1)^{n_i-m_i}\binom{n_i}{m_i}(x+1)^{(i+1)m_i}\right) \\
&= \sum_{0\le m_i\le n_i}(-1)^{\sum_i(n_i-m_i)}\prod_i\binom{n_i}{m_i}\cdot(x+1)^{\sum_i(i+1)m_i} \\
&= \sum_k\sum_{0\le m_i\le n_i}(-1)^{\sum_i(n_i-m_i)}\binom{\sum_i(i+1)m_i}{k}\prod_i\binom{n_i}{m_i}\cdot x^k.
\end{aligned}$$

Hence we have

$$\sum_{0\le m_i\le n_i}(-1)^{\sum_i m_i}\binom{\sum_i(i+1)m_i}{\sum_i n_i-1}\prod_i\binom{n_i}{m_i}=0.$$

This implies $F^{\mu}_{\lambda}(0)=0$. $\square$

Lemma 7.9. For any $k\ge 0$, $l>0$, and $\lambda=(1^{\alpha_1}2^{\alpha_2}\ldots)$ with $|\lambda|+k-l\ge 0$, we have

$$\begin{aligned}\bar m_{(k,l)(\lambda,0)(0,1)^{|\lambda|+k-l}} = &\sum_{\substack{\mu=(1^{\beta_1}2^{\beta_2}\ldots)\\ \mu\preceq\lambda}}(\beta_{|\lambda|-|\mu|+k}+1)\bar m_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}\\ &\times\left\{\sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\ldots)\\ \mu\preceq\nu\preceq\lambda\\ |\nu|\le|\lambda|+k-l}}(-1)^{\ell(\nu)+\ell(\lambda)+l}f^{\mu}_{\nu}(k+|\lambda|)\frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|+k-l)!}{(|\lambda|-|\nu|+k-l)!\prod_i(\alpha_i-\gamma_i)!}\right\}\end{aligned}$$

Proof. We prove this formula by induction on $|\lambda|+\ell(\lambda)+k-l\ge 0$. Assume $|\lambda|+\ell(\lambda)+k-l=0$. It implies that $\lambda=\emptyset$ and $k=l$. Then the formula reduces to

$$\bar m_{(k,k)}=(-1)^k\bar m_{(k,0)(0,1)^k}. \tag{7.2}$$

This is a special case of Lemma 7.6.

Let us assume $|\lambda|+\ell(\lambda)+k-l>0$. If $k=l$, then by Lemma 7.6 and Lemma 7.8, we have

$$\begin{aligned}\bar m_{(k,k)(\lambda,0)(0,1)^{|\lambda|}} &= \sum_{\mu=(1^{\beta_1}2^{\beta_2}\ldots)}\frac{(-1)^k k!(\beta_{|\lambda|-|\mu|+k}+1)}{(k-\ell(\lambda)+\ell(\mu))!\prod_i(\alpha_i-\beta_i)!}\bar m_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}\\ &= \sum_{\substack{\mu=(1^{\beta_1}2^{\beta_2}\ldots)\\ \mu\preceq\lambda}}(\beta_{|\lambda|-|\mu|+k}+1)\bar m_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}\\ &\quad\times\left\{\sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\ldots)\\ \mu\preceq\nu\preceq\lambda}}(-1)^{\ell(\nu)+\ell(\lambda)+k}f^{\mu}_{\nu}(k+|\lambda|)\frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|)!}{(|\lambda|-|\nu|)!\prod_i(\alpha_i-\gamma_i)!}\right\}.\end{aligned}$$

This implies the formula.

If $k \neq l$, then by Lemma 7.4 and the induction hypothesis, we have

$$\begin{aligned}
\bar{m}_{(k,l)(\lambda,0)(0,1)^{|\lambda|+k-l}} &= -\bar{m}_{(k,l+1)(\lambda,0)(0,1)^{|\lambda|+k-l-1}} - \sum_{j:\alpha_j>0} \bar{m}_{(k+j,l)(\lambda\setminus j,0)(0,1)^{|\lambda|+k-l}} \\
&= \sum_{\mu} (\beta_{|\lambda|-|\mu|+k}+1)\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}} \\
&\quad \times \Bigg\{ \sum_{\substack{\nu \\ \mu\preceq\nu\preceq\lambda \\ |\nu|\leq|\lambda|+k-l-1}} \frac{(-1)^{\ell(\nu)+\ell(\lambda)+l} f_\nu^\mu(k+|\lambda|)(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|+k-l-1)!}{(|\lambda|-|\nu|+k-l-1)!\prod_i(\alpha_i-\gamma_i)!} \\
&\quad + \sum_j \sum_{\substack{\nu \\ \mu\preceq\nu\preceq\lambda\setminus j \\ |\nu|\leq|\lambda|+k-l}} \frac{(-1)^{\ell(\nu)+\ell(\lambda)+l} f_\nu^\mu(k+|\lambda|)(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|+k-l-1)!}{(|\lambda|-|\nu|+k-l)!(\alpha_j-\gamma_j-1)!\prod_{i\neq j}(\alpha_i-\gamma_i)!} \Bigg\} \\
&= \sum_{\mu} (\beta_{|\lambda|-|\mu|+k}+1)\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}} \\
&\quad \times \Bigg\{ \sum_{\substack{\nu \\ \mu\preceq\nu\preceq\lambda \\ |\nu|\leq|\lambda|+k-l}} (-1)^{\ell(\nu)+\ell(\lambda)+l} f_\nu^\mu(k+|\lambda|) \frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|+k-l)!}{(|\lambda|-|\nu|+k-l)!\prod_i(\alpha_i-\gamma_i)!} \Bigg\}.
\end{aligned}$$

Here, the last equality follows from (7.1). This completes the proof of the formula. □

Lemma 7.10. For $k \in \mathbb{Z}_{>0}$ and $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots)$, we have

$$\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} = \sum_{\substack{\mu=(1^{\beta_1}2^{\beta_2}\ldots)\preceq\lambda \\ \ell(\lambda)-\ell(\mu)\leq k+1}} \frac{(-1)^k k!(k+|\lambda|-|\mu|+1)(\beta_{|\lambda|-|\mu|+k}+1)}{(k-\ell(\lambda)+\ell(\mu)+1)!\prod_i(\alpha_i-\beta_i)!} \bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}$$

Proof. By using Lemma 7.4 and Lemma 7.9, we calculate as:

$$\begin{aligned}
&\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} = \bar{m}_{(k,k)(\lambda,0)(0,1)^{|\lambda|}} + \bar{m}_{(k,k+1)(\lambda,0)(0,1)^{|\lambda|-1}} + \sum_j \bar{m}_{(k+j,k)(\lambda\setminus j,0)(0,1)^{|\lambda|}} \\
&= \sum_{\mu=(1^{\beta_1}2^{\beta_2}\ldots)\preceq\lambda} (\beta_{|\lambda|-|\mu|+k}+1)\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}} \\
&\quad \times \Bigg\{ \sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\ldots)\\ \mu\preceq\nu\preceq\lambda}} (-1)^{\ell(\nu)+\ell(\lambda)+k} f^{\mu}_{\nu}(k+|\lambda|) \frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|)!}{(|\lambda|-|\nu|)!\prod_i(\alpha_i-\gamma_i)!} \\
&\quad - \sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\ldots)\\ \mu\preceq\nu\preceq\lambda \\ |\nu|\leq|\lambda|-1}} (-1)^{\ell(\nu)+\ell(\lambda)+k} f^{\mu}_{\nu}(k+|\lambda|) \frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|-1)!}{(|\lambda|-|\nu|-1)!\prod_i(\alpha_i-\gamma_i)!} \\
&\quad - \sum_j \sum_{\substack{\nu=(1^{\gamma_1}2^{\gamma_2}\ldots)\\ \mu\preceq\nu\preceq\lambda\setminus j}} (-1)^{\ell(\nu)+\ell(\lambda)+k} f^{\mu}_{\nu}(k+|\lambda|) \frac{(\ell(\lambda)-\ell(\nu)+|\lambda|-|\nu|-1)!}{(|\lambda|-|\nu|)!(\alpha_j-\gamma_j-1)!\prod_{i\neq j}(\alpha_i-\gamma_i)!} \Bigg\} \\
&= \sum_{\mu=(1^{\beta_1}2^{\beta_2}\ldots)\preceq\lambda} \frac{(-1)^k f^{\mu}_{\lambda}(k+|\lambda|)(\beta_{|\lambda|-|\mu|+k}+1)}{\prod_i(\alpha_i-\beta_i)!} \bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}
\end{aligned}$$

Here, the last equality follows from (7.1). □

Corollary 7.11. $\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}$ is contained in the linear span of $\bar{m}_{(\nu,0)(0,1)^{|\nu|}}$ with $\ell(\nu)+|\nu| \geq \ell(\lambda)+|\lambda|$.

Proof. In Lemma 7.10, the coefficient of $\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}$ is zero if $k-\ell(\lambda)+\ell(\mu)+1<0$. Hence if $\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}$ appears in $\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}$, then we have

$$\begin{aligned}
\ell(\mu\cup(|\lambda|-|\mu|+k)) + |\mu\cup(|\lambda|-|\mu|+k)| &= \ell(\mu)+1+|\mu|+|\lambda|-|\mu|+k \\
&\geq \ell(\lambda)+|\lambda|
\end{aligned}$$

This implies the corollary. □

7.3. Proof of Theorem 4.3

Lemma 7.12. The image of $\{m_{(\lambda,0)(0,1)^{|\lambda|}} \mid \ell(\lambda)+|\lambda| \leq n\}$ in $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$ forms a basis of $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$.

Proof. By Lemma 7.3, the kernel of the natural surjection $\bar{S} \twoheadrightarrow \mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$ is spanned by $\{\bar{m}_\Lambda \mid \ell(\Lambda) > n\}$. By Lemma 7.5, it suffices to show that each $\bar{m}_\Lambda$ can be written as a linear combination of $\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}$ with $\ell(\lambda)+|\lambda| \geq \ell(\Lambda)$. Set $\deg(\bar{m}_\Lambda) = 2d(\Lambda)$ and set $e(\Lambda)$ to be the number of $(0,1)$ in Λ. We prove this claim by induction $d(\Lambda)-e(\Lambda)$. If $e(\Lambda) = d(\Lambda)$, then there is nothing to prove. If $e(\Lambda) < d(\Lambda)$, then Λ contains (a,b) with $b > 0$ and $(a,b) \neq (0,1)$. Let us write $\Lambda = (a,b)\Lambda'$. By Lemma 7.4, we have

$$\bar{m}_{(a,b-1)}\bar{m}_{(0,1)\Lambda'} = c_0\bar{m}_{(a,b-1)(0,1)\Lambda'} + c_1\bar{m}_\Lambda + \sum_{\substack{\Lambda'' \\ e(\Lambda'')=e(\Lambda) \\ \ell(\Lambda'')=\ell(\Lambda)-1}} c_{\Lambda''}\bar{m}_{(0,1)\Lambda''}$$

for some coefficients c_* with $c_1 \neq 0$. By the induction hypothesis, we have $\bar{m}_{(a,b-1)(0,1)\Lambda'}$, $\bar{m}_{(0,1)\Lambda''} \in \left\langle \bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} \mid \ell(\lambda)+|\lambda| \geq \ell(\Lambda) \right\rangle$. If $b \neq a+1$, then this implies the claim since we have $\bar{m}_{(a,b-1)} = 0$. Let us assume $b = a+1$. Since we have $d((0,1)\Lambda') - e((0,1)\Lambda') = d(\Lambda)-e(\Lambda)-a-1$ and $\ell((0,1)\Lambda') = \ell(\Lambda)$, the induction hypothesis implies that $\bar{m}_{(0,1)\Lambda'} \in \left\langle \bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} \mid \ell(\lambda)+|\lambda| \geq \ell(\Lambda) \right\rangle$. Then Corollary 7.11 implies that $\bar{m}_{(a,a)}\bar{m}_{(0,1)\Lambda'} \in \left\langle \bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} \mid \ell(\lambda)+|\lambda| \geq \ell(\Lambda) \right\rangle$. Therefore, we have $\bar{m}_\Lambda \in \left\langle \bar{m}_{(\lambda,0)(0,1)^{|\lambda|}} \mid \ell(\lambda)+|\lambda| \geq \ell(\Lambda) \right\rangle$ as required.

$\square$

Proof of Theorem 4.3. For a partition $\lambda = (1^{\alpha_1}2^{\alpha_2}\dots)$ with $\ell(\lambda)+|\lambda| \leq n$, we denote by $\hat{\lambda} = (1^{\hat{\alpha}_1}2^{\hat{\alpha}_2}\dots) \vdash n$ the partition given by $\hat{\alpha}_1 = n-\ell(\lambda)-|\lambda|$ and $\hat{\alpha}_i = \alpha_{i-1}$ for $i \geq 2$. Let $\psi : \mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}] \to \operatorname{gr}^F \mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$ be the linear map defined by

$$\psi(\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}) = (-1)^{|\lambda|}\chi_{\hat{\lambda}}.$$

By Lemma 7.12, ψ is well-defined and an isomorphism of graded vector spaces. Since we have $\deg(\sigma) \leq 2(n-1)$ for any $\sigma \in \mathfrak{S}_n$, $\{\bar{m}_{(k,k)} \mid 1 \leq k \leq n-1\}$ generates $\mathbb{C}[(S^n\mathbb{C}^2)^{\mathbb{T}}]$ as a $\mathbb{C}$-algebra by Lemma 7.3. Hence it suffices to prove

$$\psi(\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}) = \psi(\bar{m}_{(k,k)}) \cup \psi(\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}})$$

for $1 \leq k \leq n-1$. By Lemma 7.10, $\psi(\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}})$ is given by

$$\sum_{\substack{\mu=(1^{\beta_1}2^{\beta_2}\ldots)\preceq\lambda \\ \ell(\lambda)-\ell(\mu)\leq k+1}} \left\{ \frac{(-1)^k k!(k+|\lambda|-|\mu|+1)(\beta_{|\lambda|-|\mu|+k}+1)}{(k-\ell(\lambda)+\ell(\mu)+1)!\prod_i(\alpha_i-\beta_i)!}\psi(\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}) \right\}.$$

Since

$$\ell(\mu\cup(|\lambda|-|\mu|+k)) + |\mu\cup(|\lambda|-|\mu|+k)| = |\lambda|+\ell(\mu)+k+1,$$

we have $\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}} = 0$ if $|\lambda|+\ell(\mu)+k+1 > n$. Hence in the above sum, only μ's satisfying $|\lambda|+\ell(\mu)+k+1 \leq n$ contribute.

For a partition $\mu = (1^{\beta_1}2^{\beta_2}\ldots)$ with $\ell(\lambda)-\ell(\mu) \leq k+1$, $\mu \preceq \lambda$, and $|\lambda|+\ell(\mu)+k+1 \leq n$, we associate a partition $\xi(\mu) = (1^{\hat{\gamma}_1}2^{\hat{\gamma}_2}\ldots)$ with $\ell(\xi(\mu)) = k+1$ and by

$$\hat{\gamma}_i = \begin{cases} k+1-\ell(\lambda)+\ell(\mu) & \text{if } i=1 \\ \alpha_{i-1}-\beta_{i-1} & \text{if } i\geq 2. \end{cases}$$

We have $\hat{\gamma}_1 \leq \hat{\alpha}_1$ and hence $\xi(\mu) \preceq \hat{\lambda}$. This ξ gives a bijection between the set of partitions μ with $\ell(\lambda)-\ell(\mu) \leq k+1$, $\mu \preceq \lambda$, and $|\lambda|+\ell(\mu)+k+1 \leq n$ and the set of partitions $\hat{\nu}$ with $\ell(\hat{\nu}) = k+1$ and $\hat{\nu} \preceq \hat{\lambda}$. We have $|\xi(\mu)| = k+1+|\lambda|-|\mu|$ and $\psi(\bar{m}_{(|\lambda|-|\mu|+k,0)(\mu,0)(0,1)^{|\lambda|+k}}) = (-1)^{|\lambda|+k}\chi_{\hat{\lambda}_{\xi(\mu)}}$, where $\hat{\lambda}_{\xi(\mu)} \vdash n$ is defined as in Lemma 7.2. By $\ell(\xi(\mu)) = k+1 > 1$, we have $\hat{\gamma}_{|\xi(\mu)|} = 0$ and hence $\alpha_{|\xi(\mu)|-1} = \beta_{|\xi(\mu)|-1}$. Therefore, by Lemma 7.2, we have

$$\begin{aligned}
\psi(\bar{m}_{(k,k)}\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}) &= \sum_{\substack{\mu=(1^{\beta_1}2^{\beta_2}\dots)\preceq\lambda \\ \ell(\lambda)-\ell(\mu)\leq k+1 \\ |\lambda|+\ell(\mu)+k+1\leq n}} \frac{(-1)^{|\lambda|}k!(k+|\lambda|-|\mu|+1)(\beta_{|\lambda|-|\mu|+k}+1)}{(k-\ell(\lambda)+\ell(\mu)+1)!\prod_i(\alpha_i-\beta_i)!}\chi_{\hat{\lambda}_{\hat{\nu}(\mu)}} \\
&= \sum_{\substack{\hat{\nu}=(1^{\hat{\gamma}_1}2^{\hat{\gamma}_2}\dots)\preceq\hat{\lambda} \\ \ell(\hat{\nu})=k+1}} \frac{(-1)^{|\lambda|}k!|\hat{\nu}|(\hat{\alpha}_{|\hat{\nu}|}+1)}{\prod_i \hat{\gamma}_i!}\chi_{\hat{\lambda}_{\hat{\nu}}} \\
&= (-1)^{|\lambda|}\chi_{(k+1,1^{n-k-1})} \cup \chi_{\hat{\lambda}} \\
&= \psi(\bar{m}_{(k,k)}) \cup \psi(\bar{m}_{(\lambda,0)(0,1)^{|\lambda|}}).
\end{aligned}$$

Here, in the last equality, we used (7.2). This completes the proof of Theorem 4.3.

$\square$

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