

Numerical tables of finite multiple zeta values

By

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Abstract

We review various relations among finite multiple zeta(-star) values and provide numerical tables of the values of weight at most 8.

§ 1. Finite multiple zeta values

An *index* is a finite (possibly empty) sequence of positive integers. For an index $\mathbf{k} = (k_1, \dots, k_n)$ and a prime p , we consider modulo p multiple harmonic sums

$$\zeta_p(\mathbf{k}) = \sum_{p > m_1 > \dots > m_n \geq 1} \frac{1}{m_1^{k_1} \dots m_n^{k_n}} \in \mathbb{Z}/p\mathbb{Z},$$

$$\zeta_p^*(\mathbf{k}) = \sum_{p > m_1 \geq \dots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \dots m_n^{k_n}} \in \mathbb{Z}/p\mathbb{Z};$$

we then look at the sums collectively for all p in the $\mathbb{Q}$ -algebra

$$\mathcal{A} = \left(\prod_p (\mathbb{Z}/p\mathbb{Z}) \right) / \left(\bigoplus_p (\mathbb{Z}/p\mathbb{Z}) \right) \cong \left(\prod_p (\mathbb{Z}/p\mathbb{Z}) \right) \otimes_{\mathbb{Z}} \mathbb{Q},$$

following the idea of Kaneko and Zagier [3]:

Definition 1.1. For an index $\mathbf{k}$, we define the *finite multiple zeta value* (FMZV) $\zeta_{\mathcal{A}}(\mathbf{k})$ and the *finite multiple zeta-star value* (FMZSV) $\zeta_{\mathcal{A}}^*(\mathbf{k})$ by

$$\zeta_{\mathcal{A}}(\mathbf{k}) = (\zeta_p(\mathbf{k}))_p \in \mathcal{A}, \quad \zeta_{\mathcal{A}}^*(\mathbf{k}) = (\zeta_p^*(\mathbf{k}))_p \in \mathcal{A}.$$

Received December 31, 2014. Revised September 1, 2015.

2010 Mathematics Subject Classification(s): 11M32 (MSC2010)

Key Words: Finite multiple zeta value.

This work was supported by JSPS KAKENHI Grant Number 26800018.

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Table 1. Conjectured dimensions of the vector spaces $\mathcal{Z}_{\mathcal{A},k}$.

k	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{Z}_{\mathcal{A},k}$	1	0	0	1	0	1	1	1	2	2

We say that each index $\mathbf{k} = (k_1, \dots, k_n)$ has *weight* $\text{wt } \mathbf{k} = k_1 + \dots + k_n$; the empty index is viewed as having weight 0. For $k \in \mathbb{Z}_{\geq 0}$, the dimension of the $\mathbb{Q}$ -vector subspace

$$\begin{aligned}\mathcal{Z}_{\mathcal{A},k} &= \text{span}\{\zeta_{\mathcal{A}}(\mathbf{k}) \mid \mathbf{k} \text{ is an index of weight } k\} \\ &= \text{span}\{\zeta_{\mathcal{A}}^*(\mathbf{k}) \mid \mathbf{k} \text{ is an index of weight } k\} \subset \mathcal{A}\end{aligned}$$

is conjectured to be given by the following formula (Table 1):

Conjecture 1 (Zagier [3]). If $(d_k)_{k \in \mathbb{Z}_{\geq 0}}$ is the sequence given by $d_0 = 1$, $d_1 = d_2 = 0$, and $d_k = d_{k-2} + d_{k-3}$ for $k \in \mathbb{Z}_{\geq 3}$, then $\dim \mathcal{Z}_{\mathcal{A},k} = d_k$.

§ 2. Relations among finite multiple zeta(-star) values

This section describes various relations among finite multiple zeta(-star) values. Many of them have already been given in [1, 2, 4, 5] in equivalent forms, though some statements and some proofs have not been given explicitly in the literature.

§ 2.1. Zeta and zeta-star values

As in the case of ordinary multiple zeta(-star) values in $\mathbb{R}$, each FMZSV can be written as a $\mathbb{Z}$ -linear combination of FMZVs of the same weight and vice versa, as illustrated by

$$\begin{aligned}\zeta_{\mathcal{A}}^*(k_1, k_2, k_3) &= \zeta_{\mathcal{A}}(k_1, k_2, k_3) + \zeta_{\mathcal{A}}(k_1 + k_2, k_3) + \zeta_{\mathcal{A}}(k_1, k_2 + k_3) + \zeta_{\mathcal{A}}(k_1 + k_2 + k_3), \\ \zeta_{\mathcal{A}}(k_1, k_2, k_3) &= \zeta_{\mathcal{A}}^*(k_1, k_2, k_3) - \zeta_{\mathcal{A}}^*(k_1 + k_2, k_3) - \zeta_{\mathcal{A}}^*(k_1, k_2 + k_3) + \zeta_{\mathcal{A}}^*(k_1 + k_2 + k_3).\end{aligned}$$

§ 2.2. Single zeta values

Lemma 2.1. *If k is an integer and p is a prime, then in $\mathbb{Z}/p\mathbb{Z}$ we have*

$$\sum_{m=1}^{p-1} m^k = \begin{cases} -1 & \text{if } p-1 \text{ divides } k; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $p - 1$ divides k , then Fermat's little theorem implies that $m^k = 1$ in $\mathbb{Z}/p\mathbb{Z}$ for all $m = 1, \dots, p - 1$, so that

$$\sum_{m=1}^{p-1} m^k = \sum_{m=1}^{p-1} 1 = p - 1 = -1$$

in $\mathbb{Z}/p\mathbb{Z}$. Otherwise, taking a primitive root a modulo p , we have $a^k \neq 1$ in $\mathbb{Z}/p\mathbb{Z}$ and so

$$\sum_{m=1}^{p-1} m^k = \sum_{j=0}^{p-2} a^{jk} = \frac{a^{k(p-1)} - 1}{a^k - 1} = 0$$

in $\mathbb{Z}/p\mathbb{Z}$. □

We say that each index $\mathbf{k} = (k_1, \dots, k_n)$ has *depth* $\text{dep } \mathbf{k} = n$; the empty index is viewed as having depth 0.

Proposition 2.2. *If $\mathbf{k} = (k)$ is an index of depth 1, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$.*

Proof. If p is a prime with $p > k + 1$, then $p - 1$ cannot divide k , and so Lemma 2.1 shows that $\sum_{m=1}^{p-1} m^{-k} = 0$ in $\mathbb{Z}/p\mathbb{Z}$. We therefore have $\zeta_{\mathcal{A}}(k) = \zeta_{\mathcal{A}}^*(k) = 0$. □

§ 2.3. Double zeta values and Bernoulli numbers

The *Bernoulli numbers* B_n , where $n \in \mathbb{Z}_{\geq 0}$, are defined by their generating function

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{xe^x}{e^x - 1}.$$

The first few terms are $B_0 = 1$, $B_1 = 1/2$, and $B_2 = 1/6$; we have $B_n = 0$ if n is odd and $n \geq 3$, because $xe^x/(e^x - 1) - x/2$ is an even function.

If $a_p \in \mathbb{Z}/p\mathbb{Z}$ for each prime p , then we write $(a_p)_p \in \mathcal{A}$ simply as a_p . Since finitely many components do not matter in $\mathcal{A}$, it makes sense to consider such an element of $\mathcal{A}$ even when a_p is not defined for finitely many p ; for example B_{p-3} makes perfect sense as an element of $\mathcal{A}$ despite being undefined for $p = 2$.

Proposition 2.3 ([1, 5]). *If $\mathbf{k} = (k_1, k_2)$ is an index of depth 2, then*

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = \frac{(-1)^{k_1}}{k_1 + k_2} \binom{k_1 + k_2}{k_1} B_{p-k_1-k_2}.$$

In particular, we have $\zeta_{\mathcal{A}}(k - 1, 1) = \zeta_{\mathcal{A}}^(k - 1, 1) = B_{p-k}$ for all $k \in \mathbb{Z}_{\geq 2}$.*

Proof. Since $\zeta_{\mathcal{A}}^*(\mathbf{k}) = \zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}(k_1 + k_2)$, we have $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k})$ by Proposition 2.2.

If p is a prime with $p > k_1 + k_2 + 1$, then Fermat's little theorem and Faulhaber's formula show that

$$\begin{aligned} \zeta_p^*(k_1, k_2) &= \sum_{p > m_1 \geq m_2 \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2}} = \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{m_2=1}^{m_1} \frac{1}{m_2^{k_2}} \\ &= \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{m_2=1}^{m_1} m_2^{p-1-k_2} = \frac{1}{p-k_2} \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{j=0}^{p-1-k_2} \binom{p-k_2}{j} B_j m_1^{p-k_2-j} \\ &= \frac{1}{p-k_2} \sum_{j=0}^{p-1-k_2} \binom{p-k_2}{j} B_j \sum_{m_1=1}^{p-1} m_1^{p-k_1-k_2-j} \end{aligned}$$

in $\mathbb{Z}/p\mathbb{Z}$. Since $-p+1 < -k_1+1 \leq p-k_1-k_2-j \leq p-k_1-k_2 < p-1$ for $j = 0, \dots, p-1-k_2$, Lemma 2.1 shows that

$$\begin{aligned} \zeta_p^*(k_1, k_2) &= -\frac{1}{p-k_2} \binom{p-k_2}{p-k_1-k_2} B_{p-k_1-k_2} \\ &= -\frac{(p-k_2-1) \cdots (p-k_1-k_2+1)}{k_1!} B_{p-k_1-k_2} \\ &= \frac{(-1)^{k_1} (k_2+1) \cdots (k_1+k_2-1)}{k_1!} B_{p-k_1-k_2} \\ &= \frac{(-1)^{k_1}}{k_1+k_2} \binom{k_1+k_2}{k_1} B_{p-k_1-k_2} \end{aligned}$$

in $\mathbb{Z}/p\mathbb{Z}$, from which the proposition follows. $\square$

Corollary 2.4. *If $\mathbf{k} = (k_1, k_2)$ is an index of even weight and depth 2, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$.*

Proof. Proposition 2.3 implies the corollary because for all sufficiently large primes p , the number $p - k_1 - k_2$ is an odd integer at least 3, for which the Bernoulli number is 0. $\square$

Corollary 2.5. *If $\mathbf{k} = (k_1, k_2, k_3)$ is an index of even weight and depth 3, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k})$.*

Proof. We have

$$\zeta_{\mathcal{A}}^*(\mathbf{k}) - \zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}(k_1 + k_2, k_3) + \zeta_{\mathcal{A}}(k_1, k_2 + k_3) + \zeta_{\mathcal{A}}(k_1 + k_2 + k_3) = 0$$

by Proposition 2.2 and Corollary 2.4. $\square$

§ 2.4. Reversal

Proposition 2.6 ([1, 5]). *If $\mathbf{k} = (k_1, \dots, k_n)$ is an index and $\mathbf{k}' = (k_n, \dots, k_1)$ is its reversal, then $\zeta_{\mathcal{A}}(\mathbf{k}') = (-1)^{\text{wt } \mathbf{k}} \zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k}') = (-1)^{\text{wt } \mathbf{k}} \zeta_{\mathcal{A}}^*(\mathbf{k})$.*

Proof. For each prime p , we have

$$\begin{aligned} \zeta_p(\mathbf{k}') &= \sum_{p > m_1 > \dots > m_n \geq 1} \frac{1}{m_1^{k_n} \dots m_n^{k_1}} \\ &= (-1)^{\text{wt } \mathbf{k}} \sum_{p > p-m_n > \dots > p-m_1 \geq 1} \frac{1}{(p-m_1)^{k_n} \dots (p-m_n)^{k_1}} \\ &= (-1)^{\text{wt } \mathbf{k}} \zeta_p(\mathbf{k}) \end{aligned}$$

in $\mathbb{Z}/p\mathbb{Z}$, from which it follows that $\zeta_{\mathcal{A}}(\mathbf{k}') = (-1)^{\text{wt } \mathbf{k}} \zeta_{\mathcal{A}}(\mathbf{k})$. We may prove $\zeta_{\mathcal{A}}^*(\mathbf{k}') = (-1)^{\text{wt } \mathbf{k}} \zeta_{\mathcal{A}}^*(\mathbf{k})$ in the same manner. $\square$

§ 2.5. Harmonic product

As in the case of ordinary multiple zeta(-star) values in $\mathbb{R}$, FMZ(S)Vs obey the *harmonic* (also known as *shuffle*) *product* rule, as illustrated by

$$\begin{aligned} \zeta_{\mathcal{A}}(k_1)\zeta(k_2, k_3) &= \zeta_{\mathcal{A}}(k_1, k_2, k_3) + \zeta_{\mathcal{A}}(k_2, k_1, k_3) + \zeta_{\mathcal{A}}(k_2, k_3, k_1) \\ &\quad + \zeta_{\mathcal{A}}(k_1 + k_2, k_3) + \zeta_{\mathcal{A}}(k_2, k_1 + k_3), \\ \zeta_{\mathcal{A}}^*(k_1)\zeta^*(k_2, k_3) &= \zeta_{\mathcal{A}}^*(k_1, k_2, k_3) + \zeta_{\mathcal{A}}^*(k_2, k_1, k_3) + \zeta_{\mathcal{A}}^*(k_2, k_3, k_1) \\ &\quad - \zeta_{\mathcal{A}}^*(k_1 + k_2, k_3) - \zeta_{\mathcal{A}}^*(k_2, k_1 + k_3). \end{aligned}$$

Proposition 2.7 ([1]). *If $\mathbf{k} = (k_1, \dots, k_n)$ is a nonempty index, then*

$$\sum_{\sigma \in \mathfrak{S}_n} \zeta_{\mathcal{A}}(k_{\sigma(1)}, \dots, k_{\sigma(n)}) = \sum_{\sigma \in \mathfrak{S}_n} \zeta_{\mathcal{A}}^*(k_{\sigma(1)}, \dots, k_{\sigma(n)}) = 0,$$

where $\mathfrak{S}_n$ denotes the symmetric group of order n .

Proof. It suffices to prove that the sum for $\zeta_{\mathcal{A}}$, written as $S(\mathbf{k})$, always vanishes. We proceed by induction on n . If $n = 1$, then the statement immediately follows from Proposition 2.2. If $n = 2$, then Proposition 2.2 and the harmonic product rule show that

$$0 = \zeta_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2) = S(\mathbf{k}) + S(k_1 + k_2)$$

and so we have $S(\mathbf{k}) = 0$ by the inductive hypothesis. If $n = 3$, we similarly have

$$\begin{aligned} 0 &= \zeta_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2)\zeta_{\mathcal{A}}(k_3) \\ &= S(\mathbf{k}) + S(k_1 + k_2, k_3) + S(k_1 + k_3, k_2) + S(k_2 + k_3, k_1) + S(k_1 + k_2 + k_3), \end{aligned}$$

and so we have $S(\mathbf{k}) = 0$ by the inductive hypothesis. The same method works for any n . $\square$

Corollary 2.8 ([5] for the $\zeta_{\mathcal{A}}$ case). *If $\mathbf{k} = (k, \dots, k)$ is a nonempty index whose components are all equal, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$.*

Proof. If n is the depth of $\mathbf{k}$, then Proposition 2.7 shows that $n!\zeta_{\mathcal{A}}(\mathbf{k}) = n!\zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$, from which the corollary follows. $\square$

Proposition 2.9 (e.g. [2]). *If $\mathbf{k} = (k_1, \dots, k_n)$ is a nonempty index, then*

$$\sum_{j=0}^n (-1)^j \zeta_{\mathcal{A}}^*(k_j, \dots, k_1) \zeta_{\mathcal{A}}(k_{j+1}, \dots, k_n) = 0.$$

Proof. Set $S_j = \zeta_{\mathcal{A}}^*(k_j, \dots, k_1) \zeta_{\mathcal{A}}(k_{j+1}, \dots, k_n)$ for $j = 0, \dots, n$. Expanding $\zeta_{\mathcal{A}}^*(k_j, \dots, k_1)$ in terms of FMZVs and using the harmonic product rule, we can write S_j as a sum of FMZVs. For $j = 1, \dots, n-1$, we write $S_j = T_j + U_j$ by letting T_j be the sum of those FMZVs in which the component containing k_j appears strictly left of that containing k_{j+1} , and by letting U_j be the sum of the other FMZVs. If we further set $U_0 = S_0$ and $T_n = S_n$, then we may observe that $U_j = T_{j+1}$ for $j = 0, \dots, n-1$; for example, if $n = 3$, then

$$\begin{aligned} U_0 &= T_1 = \zeta_{\mathcal{A}}(k_1, k_2, k_3), \\ U_1 &= T_2 = \zeta_{\mathcal{A}}(k_2, k_1, k_3) + \zeta_{\mathcal{A}}(k_2, k_3, k_1) + \zeta_{\mathcal{A}}(k_1 + k_2, k_3) + \zeta_{\mathcal{A}}(k_2, k_1 + k_3), \\ U_2 &= T_3 = \zeta_{\mathcal{A}}(k_3, k_2, k_1) + \zeta_{\mathcal{A}}(k_2 + k_3, k_1) + \zeta_{\mathcal{A}}(k_3, k_1 + k_2) + \zeta_{\mathcal{A}}(k_1 + k_2 + k_3). \end{aligned}$$

It follows that

$$\sum_{j=0}^n (-1)^j S_j = \sum_{j=0}^{n-1} (-1)^j (U_j - T_{j+1}) = 0,$$

as desired. $\square$

Corollary 2.10 ([1]). *If $\mathbf{k} = (k_1, k_2, k_3)$ is an index of odd weight k and depth 3, then*

$$\begin{aligned} \zeta_{\mathcal{A}}(\mathbf{k}) &= \left(-(-1)^{k_1} \binom{k}{k_1} + (-1)^{k_3} \binom{k}{k_3} \right) \frac{B_{p-k}}{2k}, \\ \zeta_{\mathcal{A}}^*(\mathbf{k}) &= \left((-1)^{k_1} \binom{k}{k_1} - (-1)^{k_3} \binom{k}{k_3} \right) \frac{B_{p-k}}{2k}. \end{aligned}$$

Proof. Propositions 2.2, 2.6 and 2.9 give

$$\begin{aligned} 0 &= \zeta_{\mathcal{A}}(k_1, k_2, k_3) - \zeta_{\mathcal{A}}^*(k_1) \zeta_{\mathcal{A}}(k_2, k_3) + \zeta_{\mathcal{A}}^*(k_2, k_1) \zeta_{\mathcal{A}}(k_3) - \zeta_{\mathcal{A}}^*(k_3, k_2, k_1) \\ &= \zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}^*(\mathbf{k}). \end{aligned}$$

Also, Propositions 2.2 and 2.3 show that

$$\begin{aligned}\zeta_{\mathcal{A}}^*(\mathbf{k}) &= \zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}(k_1 + k_2, k_3) + \zeta_{\mathcal{A}}(k_1, k_2 + k_3) + \zeta_{\mathcal{A}}(k_1 + k_2 + k_3) \\ &= \zeta_{\mathcal{A}}(\mathbf{k}) + \left(\frac{(-1)^{k_1+k_2}}{k} \binom{k}{k_1+k_2} + \frac{(-1)^{k_1}}{k} \binom{k}{k_1} \right) B_{p-k} \\ &= \zeta_{\mathcal{A}}(\mathbf{k}) + \left((-1)^{k_1} \binom{k}{k_1} - (-1)^{k_3} \binom{k}{k_3} \right) \frac{B_{p-k}}{k}\end{aligned}$$

because $k = k_1 + k_2 + k_3$ is odd, from which the conclusion follows. $\square$

Corollary 2.11. *Let $\mathbf{k} = (k_1, k_2, k_3, k_4)$ be an index of depth 4.*

1. *If $\mathbf{k}$ has odd weight, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k})$.*
2. *If $\mathbf{k}$ has even weight, then $\zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}^*(\mathbf{k}) = \zeta_{\mathcal{A}}(k_1, k_2)\zeta_{\mathcal{A}}(k_3, k_4)$. In particular, $\zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$ unless $k_1 + k_2$ and $k_3 + k_4$ are both odd.*

Proof. Propositions 2.2, 2.6 and 2.9 give

$$\begin{aligned}0 &= \zeta_{\mathcal{A}}(k_1, k_2, k_3, k_4) - \zeta_{\mathcal{A}}^*(k_1)\zeta_{\mathcal{A}}(k_2, k_3, k_4) + \zeta_{\mathcal{A}}^*(k_2, k_1)\zeta_{\mathcal{A}}(k_3, k_4) \\ &\quad - \zeta_{\mathcal{A}}^*(k_3, k_2, k_1)\zeta_{\mathcal{A}}(k_4) + \zeta_{\mathcal{A}}^*(k_4, k_3, k_2, k_1) \\ &= \zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}(k_2, k_1)\zeta_{\mathcal{A}}(k_3, k_4) + (-1)^{\text{wt } \mathbf{k}} \zeta_{\mathcal{A}}^*(\mathbf{k}).\end{aligned}$$

If $\text{wt } \mathbf{k}$ is even, this gives the desired result because $\zeta_{\mathcal{A}}(k_2, k_1) = -\zeta_{\mathcal{A}}(k_1, k_2)$ no matter whether $k_1 + k_2$ is odd or even; if $\text{wt } \mathbf{k}$ is odd, then the middle term vanishes because either $k_1 + k_2$ or $k_3 + k_4$ must be even. $\square$

§ 2.6. Duality

Definition 2.12 ([1]). For each nonempty index $\mathbf{k}$, we define its *dual* index $\mathbf{k}^*$ inductively as follows:

1. $(1)^* = (1)$;
2. if $(k_1, \dots, k_m)^* = (l_1, \dots, l_n)$, then

$$(1, k_1, \dots, k_m)^* = (l_1 + 1, l_2, \dots, l_n), \quad (k_1 + 1, k_2, \dots, k_m)^* = (1, l_1, \dots, l_n).$$

Example 2.13. We have $(1, 2, 2)^* = (2, 2, 1)$ because successive use of item 2 gives $(2)^* = (1, 1)$, $(1, 2)^* = (2, 1)$, $(2, 2)^* = (1, 2, 1)$, and $(1, 2, 2)^* = (2, 2, 1)$, as desired.

Note that induction on weight shows that $\text{wt } \mathbf{k}^* = \text{wt } \mathbf{k}$ and $\text{dep } \mathbf{k}^* = \text{wt } \mathbf{k} - \text{dep } \mathbf{k} + 1$ for all nonempty indices $\mathbf{k}$.

Theorem 2.14 ([1, Theorem 4.6]). *If $\mathbf{k}$ is a nonempty index, then we have*

$$\zeta_{\mathcal{A}}^*(\mathbf{k}^*) = -\zeta_{\mathcal{A}}^*(\mathbf{k}).$$

For indices $\mathbf{k}$ and $\mathbf{l}$ of the same weight, we write $\mathbf{k} \preceq \mathbf{l}$ if $\mathbf{l}$ is a refinement of $\mathbf{k}$, namely if $\zeta_{\mathcal{A}}(\mathbf{k})$ appears in the expansion of $\zeta_{\mathcal{A}}^*(\mathbf{l})$ in terms of FMZVs.

Corollary 2.15. *If $\mathbf{k}$ is a nonempty index, then*

$$(-1)^{\text{dep } \mathbf{k}} \zeta_{\mathcal{A}}(\mathbf{k}) = \sum_{\mathbf{l} \succeq \mathbf{k}} \zeta_{\mathcal{A}}(\mathbf{l}).$$

Proof. Observe that for indices $\mathbf{k}_1$ and $\mathbf{k}_2$ of the same weight, we have $\mathbf{k}_1 \preceq \mathbf{k}_2$ if and only if $\mathbf{k}_1^* \succeq \mathbf{k}_2^*$. If $\mathbf{k}$ is a nonempty index of weight k and depth n , then Theorem 2.14 shows that

$$\begin{aligned} (-1)^n \zeta_{\mathcal{A}}(\mathbf{k}) &= \sum_{\mathbf{m} \preceq \mathbf{k}} (-1)^{\text{dep } \mathbf{m}} \zeta_{\mathcal{A}}^*(\mathbf{m}) = \sum_{\mathbf{m} \preceq \mathbf{k}} (-1)^{\text{dep } \mathbf{m}+1} \zeta_{\mathcal{A}}^*(\mathbf{m}^*) \\ &= \sum_{\mathbf{m}^* \succeq \mathbf{k}^*} (-1)^{k+\text{dep } \mathbf{m}^*} \zeta_{\mathcal{A}}^*(\mathbf{m}^*) = (-1)^k \sum_{\mathbf{m}^* \succeq \mathbf{k}^*} (-1)^{\text{dep } \mathbf{m}^*} \zeta_{\mathcal{A}}^*(\mathbf{m}^*) \\ &= (-1)^k \sum_{\mathbf{m}^* \succeq \mathbf{k}^*} (-1)^{\text{dep } \mathbf{m}^*} \sum_{\mathbf{l} \preceq \mathbf{m}^*} \zeta_{\mathcal{A}}(\mathbf{l}) = (-1)^k \sum_{\mathbf{l}} \left(\sum_{\substack{\mathbf{m}^* \succeq \mathbf{k}^* \\ \mathbf{m}^* \succeq \mathbf{l}}} (-1)^{\text{dep } \mathbf{m}^*} \right) \zeta_{\mathcal{A}}(\mathbf{l}). \end{aligned}$$

Here for indices $\mathbf{k}_1$ and $\mathbf{k}_2$ of weight k , writing $\mathbf{k}_1 \vee \mathbf{k}_2$ for their coarsest common refinement, we have

$$\sum_{\substack{\mathbf{m} \succeq \mathbf{k}_1 \\ \mathbf{m} \succeq \mathbf{k}_2}} (-1)^{\text{dep } \mathbf{m}} = \sum_{\mathbf{m} \succeq \mathbf{k}_1 \vee \mathbf{k}_2} (-1)^{\text{dep } \mathbf{m}} = \begin{cases} (-1)^k & \text{if } \mathbf{k}_1 \vee \mathbf{k}_2 = (1, \dots, 1); \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{k}^* \vee \mathbf{l} = (1, \dots, 1)$ if and only if $\mathbf{l} \succeq \mathbf{k}$, we obtain

$$(-1)^n \zeta_{\mathcal{A}}(\mathbf{k}) = (-1)^k \sum_{\mathbf{l} \succeq \mathbf{k}} (-1)^k \zeta_{\mathcal{A}}(\mathbf{l}) = \sum_{\mathbf{l} \succeq \mathbf{k}} \zeta_{\mathcal{A}}(\mathbf{l}),$$

as desired. □

Corollary 2.16. *Let $\mathbf{k}$ be an index satisfying $\text{dep } \mathbf{k} = \text{wt } \mathbf{k} - 1$. Such an index can always be written as $\mathbf{k} = (\underbrace{1, \dots, 1}_{i-1}, \underbrace{2, 1, \dots, 1}_{k-i-1})$ where $1 \leq i \leq k-1$, and we have*

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = \frac{(-1)^{i+1}}{k} \binom{k}{i} B_{p-k}.$$

In particular, if $\mathbf{k}$ has even weight, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^(\mathbf{k}) = 0$.*

Proof. If we write $\mathbf{k} = (k_1, \dots, k_n)$, then Proposition 2.9 gives

$$\zeta_{\mathcal{A}}(\mathbf{k}) + (-1)^n \zeta_{\mathcal{A}}^*(k_n, \dots, k_1) = 0$$

because for $j = 1, \dots, n-1$ either $(k_j, \dots, k_1)$ or $(k_{j+1}, \dots, k_n)$ is an all-one sequence. It follows that

$$0 = \zeta_{\mathcal{A}}(\mathbf{k}) + (-1)^n \zeta_{\mathcal{A}}^*(k_n, \dots, k_1) = \zeta_{\mathcal{A}}(\mathbf{k}) + (-1)^{k+n} \zeta_{\mathcal{A}}^*(\mathbf{k}) = \zeta_{\mathcal{A}}(\mathbf{k}) - \zeta_{\mathcal{A}}^*(\mathbf{k}).$$

Now Theorem 2.14 and Proposition 2.3 give

$$\zeta_{\mathcal{A}}^*(\mathbf{k}) = -\zeta_{\mathcal{A}}^*(i, k-i) = \frac{(-1)^{i+1}}{k} \binom{k}{i} B_{p-k}$$

from which the result follows. $\square$

§ 2.7. Result of Pilehrood, Pilehrood, and Tauraso

Pilehrood, Pilehrood, and Tauraso [4] proved relations among FMZ(S)V's of weight 7 and 9 that cannot be deduced from the relations described so far.

Theorem 2.17 ([4, Corollary 4.3]). *We have*

$$\zeta_{\mathcal{A}}^*(2, 2, 2, 1) = \frac{27}{16} B_{p-7}, \quad \zeta_{\mathcal{A}}^*(6, 1, 1, 1) = \frac{1}{54} B_{p-3}^3 + \frac{1899}{648} B_{p-9}.$$

§ 3. Numerical tables

§ 3.1. Weight 0

The only index of weight 0 is the empty index $\emptyset$. Since $\zeta_{\mathcal{A}}(\emptyset) = \zeta_{\mathcal{A}}^*(\emptyset) = 1$ by definition, we have $\dim \mathcal{Z}_{\mathcal{A},0} = 1$.

§ 3.2. Weight 1

The only index of weight 1 is (1). Since $\zeta_{\mathcal{A}}(1) = \zeta_{\mathcal{A}}^*(1) = 0$ by Proposition 2.2, we have $\dim \mathcal{Z}_{\mathcal{A},1} = 0$.

§ 3.3. Weight 2

The indices of weight 2 are (2) and (1, 1). Since $\zeta_{\mathcal{A}}(2) = \zeta_{\mathcal{A}}^*(2) = 0$ by Proposition 2.2 and $\zeta_{\mathcal{A}}(1, 1) = \zeta_{\mathcal{A}}^*(1, 1) = 0$ by Corollary 2.4 or Corollary 2.8, we have $\dim \mathcal{Z}_{\mathcal{A},2} = 0$.

§ 3.4. Weight 3

The indices of weight 3 are (3) , $(2, 1)$, $(1, 2)$, and $(1, 1, 1)$. We have $\zeta_{\mathcal{A}}(3) = \zeta_{\mathcal{A}}^*(3) = 0$ by Proposition 2.2 and $\zeta_{\mathcal{A}}(1, 1, 1) = \zeta_{\mathcal{A}}^*(1, 1, 1) = 0$ by Corollary 2.8; we also have $\zeta_{\mathcal{A}}(2, 1) = \zeta_{\mathcal{A}}^*(2, 1) = B_{p-3}$ and $\zeta_{\mathcal{A}}(1, 2) = \zeta_{\mathcal{A}}^*(1, 2) = -B_{p-3}$ by Proposition 2.3. It follows that $\dim \mathcal{Z}_{\mathcal{A},3} \leq 1$; it appears open whether $\dim \mathcal{Z}_{\mathcal{A},3} = 1$, i.e. whether $B_{p-3} \neq 0$ for infinitely many primes p .

§ 3.5. Weight 4

Let $\mathbf{k}$ be an index of weight 4. If $\mathbf{k}$ has depth 1 or 4, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$. If $\mathbf{k}$ has depth 2, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$ by Corollary 2.4. If $\mathbf{k}$ has depth 3, then $\zeta_{\mathcal{A}}^*(\mathbf{k}) = -\zeta_{\mathcal{A}}^*(\mathbf{k}^*) = 0$ by Theorem 2.14 and by the fact that $\mathbf{k}^*$ has depth 2. It follows that $\dim \mathcal{Z}_{\mathcal{A},4} = 0$.

§ 3.6. Weight 5

Proposition 3.1. *Let $\mathbf{k}$ be an index of weight 5. Then the values of $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$ are as given in Table 2 in terms of B_{p-5} .*

Proof. If $\text{dep } \mathbf{k} = 2$, use Proposition 2.3. If $\text{dep } \mathbf{k} = 3$, use Corollary 2.10. If $\text{dep } \mathbf{k} = 4$, use Corollary 2.16. $\square$

Table 2. FMZ(S)Vs of weight 5 in terms of B_{p-5} ; e.g. $\zeta_{\mathcal{A}}(4, 1) = \zeta_{\mathcal{A}}^*(4, 1) = B_{p-5}$, $\zeta_{\mathcal{A}}(3, 1, 1) = B_{p-5}/2$, and $\zeta_{\mathcal{A}}^*(3, 1, 1) = -B_{p-5}/2$. If $\text{wt } \mathbf{k} = 5$ and $\text{dep } \mathbf{k} \in \{1, 5\}$, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$.

$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$
$(4, 1)$	1	$(3, 1, 1)$	$1/2$	$-1/2$	$(2, 1, 1, 1)$	1
$(3, 2)$	-2	$(2, 2, 1)$	$-3/2$	$3/2$	$(1, 2, 1, 1)$	-2
$(2, 3)$	2	$(2, 1, 2)$	0	0	$(1, 1, 2, 1)$	2
$(1, 4)$	-1	$(1, 3, 1)$	0	0	$(1, 1, 1, 2)$	-1
		$(1, 2, 2)$	$3/2$	$-3/2$		
		$(1, 1, 3)$	$-1/2$	$1/2$		

§ 3.7. Weight 6

Proposition 3.2. *Let $\mathbf{k}$ be an index of weight 6. Then the values of $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$ are as given in Table 3 in terms of B_{p-3}^2 .*

Proof. If $\text{dep } \mathbf{k} \in \{1, 2, 5, 6\}$, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$ (use Corollary 2.16 when $\text{dep } \mathbf{k} = 5$).

Note that if $\text{dep } \mathbf{k} = 3$, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k})$ by Corollary 2.5.

We have $\zeta_{\mathcal{A}}(2, 2, 2) = \zeta_{\mathcal{A}}^*(2, 2, 2) = 0$ by Corollary 2.8, and so $\zeta_{\mathcal{A}}^*(1, 2, 2, 1) = -\zeta_{\mathcal{A}}^*(2, 2, 2) = 0$ by Theorem 2.14. It follows that

$$\zeta_{\mathcal{A}}(1, 2, 2, 1) = -\zeta_{\mathcal{A}}^*(1, 2, 2, 1) + \zeta_{\mathcal{A}}(1, 2)\zeta_{\mathcal{A}}(2, 1) = -0 + (-B_{p-3})B_{p-3} = -B_{p-3}^2$$

by Corollary 2.11.

Set $a = \zeta_{\mathcal{A}}(4, 1, 1) = \zeta_{\mathcal{A}}^*(4, 1, 1)$. Then by Proposition 2.6, Corollary 2.11, and Theorem 2.14, we have $\zeta_{\mathcal{A}}(1, 1, 4) = \zeta_{\mathcal{A}}^*(1, 1, 4) = \zeta_{\mathcal{A}}(3, 1, 1, 1) = \zeta_{\mathcal{A}}(1, 1, 1, 3) = a$ and $\zeta_{\mathcal{A}}^*(3, 1, 1, 1) = \zeta_{\mathcal{A}}^*(1, 1, 1, 3) = -a$. Therefore Propositions 2.6 and 2.7 show that $\zeta_{\mathcal{A}}(1, 4, 1) = \zeta_{\mathcal{A}}^*(1, 4, 1) = -2a$, $\zeta_{\mathcal{A}}(1, 1, 3, 1) = \zeta_{\mathcal{A}}(1, 3, 1, 1) = -a$, and $\zeta_{\mathcal{A}}^*(1, 1, 3, 1) = \zeta_{\mathcal{A}}^*(1, 3, 1, 1) = a$. Corollaries 2.15, 2.16, and 2.8 show that

$$-\zeta_{\mathcal{A}}(1, 4, 1) = \zeta_{\mathcal{A}}(1, 4, 1) + \zeta_{\mathcal{A}}(1, 3, 1, 1) + \zeta_{\mathcal{A}}(1, 2, 2, 1) + \zeta_{\mathcal{A}}(1, 1, 3, 1),$$

from which it follows that $2a = -2a - a - B_{p-3}^2 - a$, i.e. $a = -B_{p-3}^2/6$. Then Theorem 2.14 gives $\zeta^*(3, 1, 2)$, $\zeta^*(2, 1, 3)$, and $\zeta_{\mathcal{A}}^*(2, 1, 1, 2)$. Use Corollary 2.11 to obtain $\zeta_{\mathcal{A}}(2, 1, 1, 2)$.

Table 3. FMZ(S)Vs of weight 6 in terms of B_{p-3}^2 ; e.g. $\zeta_{\mathcal{A}}(4, 1, 1) = \zeta_{\mathcal{A}}^*(4, 1, 1) = -B_{p-3}^2/6$, $\zeta_{\mathcal{A}}(3, 1, 1, 1) = -B_{p-3}^2/6$, and $\zeta_{\mathcal{A}}^*(3, 1, 1, 1) = B_{p-3}^2/6$. If $\text{wt } \mathbf{k} = 6$ and $\text{dep } \mathbf{k} \in \{1, 2, 5, 6\}$, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$.

$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}^*$
(4, 1, 1)	-1/6	(3, 1, 1, 1)	-1/6	1/6
(3, 2, 1)	2/6	(2, 2, 1, 1)	2/6	-2/6
(3, 1, 2)	1/6	(2, 1, 2, 1)	3/6	3/6
(2, 3, 1)	-3/6	(2, 1, 1, 2)	-4/6	-2/6
(2, 2, 2)	0/6	(1, 3, 1, 1)	1/6	-1/6
(2, 1, 3)	1/6	(1, 2, 2, 1)	-6/6	0/6
(1, 4, 1)	2/6	(1, 2, 1, 2)	3/6	3/6
(1, 3, 2)	-3/6	(1, 1, 3, 1)	1/6	-1/6
(1, 2, 3)	2/6	(1, 1, 2, 2)	2/6	-2/6
(1, 1, 4)	-1/6	(1, 1, 1, 3)	-1/6	1/6

In a similar manner, if we set $b = \zeta_{\mathcal{A}}(3, 2, 1) = \zeta_{\mathcal{A}}^*(3, 2, 1)$, then by Proposition 2.6, Proposition 2.7, Corollary 2.11, and Theorem 2.14 we have $\zeta_{\mathcal{A}}(1, 2, 3) = \zeta_{\mathcal{A}}^*(1, 2, 3) = b$, $\zeta_{\mathcal{A}}(1, 3, 2) = \zeta_{\mathcal{A}}^*(1, 3, 2) = \zeta_{\mathcal{A}}(2, 3, 1) = \zeta_{\mathcal{A}}^*(2, 3, 1) = -b - B_{p-3}^2/6$, $\zeta_{\mathcal{A}}(2, 2, 1, 1) = \zeta_{\mathcal{A}}^*(1, 1, 2, 2) = -b$, $\zeta_{\mathcal{A}}(2, 2, 1, 1) = \zeta_{\mathcal{A}}(1, 1, 2, 2) = b$, $\zeta_{\mathcal{A}}(2, 1, 2, 1) = \zeta_{\mathcal{A}}^*(1, 2, 1, 2) = b + B_{p-3}^2/6$, and $\zeta_{\mathcal{A}}(2, 1, 2, 1) = \zeta_{\mathcal{A}}(1, 2, 1, 2) = -b + 5B_{p-3}^2/6$. Corollary 2.15 shows that

$$-\zeta_{\mathcal{A}}(3, 2, 1) = \zeta_{\mathcal{A}}(3, 2, 1) + \zeta_{\mathcal{A}}(2, 1, 2, 1) + \zeta_{\mathcal{A}}(1, 2, 2, 1) + \zeta_{\mathcal{A}}(3, 1, 1, 1),$$

from which it follows that $-b = b + (-b + 5B_{p-3}^2/6) - B_{p-3}^2 - B_{p-3}^2/6$, i.e. $b = B_{p-3}^2/3$. $\square$

§ 3.8. Weight 7

Theorem 3.3. *Let $\mathbf{k}$ be an index of weight 7. Then the values of $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$ are as given in Table 4 in terms of B_{p-7} .*

Proof. If $\text{dep } \mathbf{k} \in \{1, 2, 3, 6, 7\}$, then it is easy to compute $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$.

Apply Proposition 2.9 to $\mathbf{k}$. If we write $\mathbf{k}$ as a concatenation of two nonempty indices, then one of the two has weight in $\{1, 2, 4\}$ and so its $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{A}}^*$ are zero. It follows that $\zeta_{\mathcal{A}}(\mathbf{k}) + (-1)^{\text{dep } \mathbf{k}} \zeta_{\mathcal{A}}^*(\mathbf{k}') = 0$, where $\mathbf{k}'$ is the reversal of $\mathbf{k}$, and therefore we have $\zeta_{\mathcal{A}}(\mathbf{k}) = (-1)^{\text{dep } \mathbf{k}} \zeta_{\mathcal{A}}^*(\mathbf{k})$ by Proposition 2.6 because $\text{wt } \mathbf{k} = 7$ is odd.

Suppose that $\mathbf{k}$ has depth 5. Theorem 2.14 allows us to compute $\zeta_{\mathcal{A}}^*(\mathbf{k})$ because $\mathbf{k}^*$ has depth 3 and so we already know the value of $\zeta_{\mathcal{A}}^*(\mathbf{k}^*)$. We can then use $\zeta_{\mathcal{A}}(\mathbf{k}) = -\zeta_{\mathcal{A}}^*(\mathbf{k})$ deduced above.

In what follows we shall compute $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$ for indices of depth 4. Note that $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k})$ for all such indices $\mathbf{k}$. Since Theorem 2.17 gives the value of $\zeta_{\mathcal{A}}(2, 2, 2, 1)$, it suffices to express the other FMZ(S)V's in terms of $\zeta_{\mathcal{A}}(2, 2, 2, 1)$ and B_{p-7} .

Set $\alpha = \zeta(4, 1, 1, 1)$. Applying the harmonic product rule to $\zeta_{\mathcal{A}}(4, 1, 1)\zeta_{\mathcal{A}}(1)$, we obtain $\zeta_{\mathcal{A}}(1, 4, 1, 1) = -3\alpha + 3B_{p-7}$. Then Proposition 2.6 allows us to find $\zeta_{\mathcal{A}}(1, 1, 4, 1) = 3\alpha - 3B_{p-7}$ and $\zeta_{\mathcal{A}}(1, 1, 1, 4) = -\alpha$. Thus Theorem 2.14 gives $\zeta_{\mathcal{A}}(3, 1, 1, 2) = -3\alpha + 3B_{p-7}$ and $\zeta_{\mathcal{A}}(2, 1, 1, 3) = 3\alpha - 3B_{p-7}$.

Set $\beta = \zeta_{\mathcal{A}}(2, 2, 2, 1)$. We may similarly obtain $\zeta_{\mathcal{A}}(2, 2, 1, 2) = -3\beta + 9B_{p-7}$, $\zeta_{\mathcal{A}}(2, 1, 2, 2) = 3\beta - 9B_{p-7}$, $\zeta_{\mathcal{A}}(1, 2, 2, 2) = -\beta$, $\zeta_{\mathcal{A}}(1, 3, 2, 1) = -3\beta + 9B_{p-7}$, and $\zeta_{\mathcal{A}}(1, 2, 3, 1) = 3\beta - 9B_{p-7}$.

Set $\gamma = \zeta_{\mathcal{A}}(3, 2, 1, 1)$, $\delta = \zeta_{\mathcal{A}}(3, 1, 2, 1)$, and $\varepsilon = \zeta_{\mathcal{A}}(2, 1, 3, 1)$. Then Proposition 2.6 and Theorem 2.14 give all the remaining values. We look at $\zeta_{\mathcal{A}}(3, 2, 1)\zeta_{\mathcal{A}}(1)$, $\zeta_{\mathcal{A}}(3, 1, 2)\zeta_{\mathcal{A}}(1)$, and $\zeta_{\mathcal{A}}(2, 3, 1)\zeta_{\mathcal{A}}(1)$ to obtain $\gamma = -\alpha + 2\beta - 6B_{p-7}$, $\delta = 2\alpha - \beta$, and $\varepsilon = -4\alpha - \beta + 9B_{p-7}$. We then express $\zeta_{\mathcal{A}}^*(2, 2, 1, 1, 1)$ as a sum of FMZVs to get $\alpha = \beta$. $\square$

Table 4. FMZ(S)Vs of weight 7 in terms of B_{p-7} ; e.g. $\zeta_{\mathcal{A}}(6, 1) = \zeta_{\mathcal{A}}^*(6, 1) = B_{p-7}$, $\zeta_{\mathcal{A}}(5, 1, 1) = B_{p-7}$, and $\zeta_{\mathcal{A}}^*(5, 1, 1) = -B_{p-7}$. If $\text{wt } \mathbf{k} = 7$ and $\text{dep } \mathbf{k} \in \{1, 7\}$, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$.

$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}^*$
(6, 1)	1	(5, 1, 1)	1	-1	(3, 1, 1, 1, 1)	1	-1
(5, 2)	-3	(4, 2, 1)	-3	3	(2, 2, 1, 1, 1)	-3	3
(4, 3)	5	(4, 1, 2)	-1	1	(2, 1, 2, 1, 1)	2	-2
(3, 4)	-5	(3, 3, 1)	2	-2	(2, 1, 1, 2, 1)	-2	2
(2, 5)	3	(3, 2, 2)	4	-4	(2, 1, 1, 1, 2)	0	0
(1, 6)	-1	(3, 1, 3)	0	0	(1, 3, 1, 1, 1)	-1	1
		(2, 4, 1)	-2	2	(1, 2, 2, 1, 1)	4	-4
		(2, 3, 2)	0	0	(1, 2, 1, 2, 1)	0	0
		(2, 2, 3)	-4	4	(1, 2, 1, 1, 2)	2	-2
		(2, 1, 4)	1	-1	(1, 1, 3, 1, 1)	0	0
		(1, 5, 1)	0	0	(1, 1, 2, 2, 1)	-4	4
		(1, 4, 2)	2	-2	(1, 1, 2, 1, 2)	-2	2
		(1, 3, 3)	-2	2	(1, 1, 1, 3, 1)	1	-1
		(1, 2, 4)	3	-3	(1, 1, 1, 2, 2)	3	-3
		(1, 1, 5)	-1	1	(1, 1, 1, 1, 3)	-1	1

$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$
(4, 1, 1, 1)	27/16	(1, 4, 1, 1)	-33/16	(2, 1, 1, 1, 1, 1)	1
(3, 2, 1, 1)	-69/16	(1, 3, 2, 1)	63/16	(1, 2, 1, 1, 1, 1)	-3
(3, 1, 2, 1)	27/16	(1, 3, 1, 2)	-9/16	(1, 1, 2, 1, 1, 1)	5
(3, 1, 1, 2)	-33/16	(1, 2, 3, 1)	-63/16	(1, 1, 1, 2, 1, 1)	-5
(2, 3, 1, 1)	27/16	(1, 2, 2, 2)	-27/16	(1, 1, 1, 1, 2, 1)	3
(2, 2, 2, 1)	27/16	(1, 2, 1, 3)	-27/16	(1, 1, 1, 1, 1, 2)	-1
(2, 2, 1, 2)	63/16	(1, 1, 4, 1)	33/16		
(2, 1, 3, 1)	9/16	(1, 1, 3, 2)	-27/16		
(2, 1, 2, 2)	-63/16	(1, 1, 2, 3)	69/16		
(2, 1, 1, 3)	33/16	(1, 1, 1, 4)	-27/16		

§ 3.9. Weight 8

Theorem 3.4. *Let $\mathbf{k}$ be an index of weight 8. Then the values of $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$ are as given in Tables 5–8 in terms of $\zeta_{\mathcal{A}}(6, 1, 1)$ and $B_{p-3}B_{p-5}$.*

Proof. If $\text{dep } \mathbf{k} \in \{1, 2, 7, 8\}$, then $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^*(\mathbf{k}) = 0$. See [1, Theorem 7.4] for how to get the values $\zeta_{\mathcal{A}}^*(\mathbf{k})$ when $\text{dep } \mathbf{k} = 3$. If $\text{dep } \mathbf{k} = 3$, then Corollary 2.5 gives the value of $\zeta_{\mathcal{A}}(\mathbf{k})$. If $\text{dep } \mathbf{k} = 4$, then since we can compute the value of $\zeta_{\mathcal{A}}^*(\mathbf{k}) - \zeta_{\mathcal{A}}(\mathbf{k})$ and Corollary 2.11 gives the value of $\zeta_{\mathcal{A}}(\mathbf{k}) + \zeta_{\mathcal{A}}^*(\mathbf{k})$, we can get the values of $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^*(\mathbf{k})$. If $\text{dep } \mathbf{k} \in \{5, 6\}$, then Theorem 2.14 gives the value of $\zeta_{\mathcal{A}}^*(\mathbf{k})$ and we can get the value of $\zeta_{\mathcal{A}}(\mathbf{k})$ by using Proposition 2.9. $\square$

Table 5. FMZ(S)Vs of weight 8 and depth 3 in terms of $\zeta_{\mathcal{A}}(6, 1, 1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(5, 2, 1) = \zeta_{\mathcal{A}}^*(5, 2, 1) = \zeta_{\mathcal{A}}(6, 1, 1) + B_{p-3}B_{p-5}$.

$\mathbf{k}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$\mathbf{k}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}}^*$
(6, 1, 1)	4/4	0	(2, 4, 2)	−80/4	−16/4
(5, 2, 1)	4/4	4/4	(2, 3, 3)	40/4	8/4
(5, 1, 2)	−14/4	− 2/4	(2, 2, 4)	40/4	8/4
(4, 3, 1)	−50/4	−18/4	(2, 1, 5)	−14/4	− 2/4
(4, 2, 2)	40/4	8/4	(1, 6, 1)	− 8/4	0
(4, 1, 3)	10/4	2/4	(1, 5, 2)	10/4	− 2/4
(3, 4, 1)	40/4	16/4	(1, 4, 3)	40/4	16/4
(3, 3, 2)	40/4	8/4	(1, 3, 4)	−50/4	−18/4
(3, 2, 3)	−80/4	−16/4	(1, 2, 5)	4/4	4/4
(3, 1, 4)	10/4	2/4	(1, 1, 6)	4/4	0
(2, 5, 1)	10/4	− 2/4			

Table 6. FMZ(S)Vs of weight 8 and depth 4 in terms of $\zeta_{\mathcal{A}}(6, 1, 1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(5, 1, 1, 1) = 3\zeta_{\mathcal{A}}(6, 1, 1)/4 - B_{p-3}B_{p-5}/4$ and $\zeta_{\mathcal{A}}^*(5, 1, 1, 1) = -3\zeta_{\mathcal{A}}(6, 1, 1)/4 + B_{p-3}B_{p-5}/4$.

$\mathbf{k}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}}$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}}^*$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}}^*$
(5, 1, 1, 1)	3/4	- 1/4	- 3/4	1/4
(4, 2, 1, 1)	3/4	5/4	- 3/4	- 5/4
(4, 1, 2, 1)	18/4	8/4	-18/4	- 4/4
(4, 1, 1, 2)	-18/4	- 6/4	18/4	2/4
(3, 3, 1, 1)	-42/4	-12/4	42/4	12/4
(3, 2, 2, 1)	18/4	- 6/4	-18/4	- 2/4
(3, 2, 1, 2)	27/4	9/4	-27/4	- 1/4
(3, 1, 3, 1)	0	0	0	0
(3, 1, 2, 2)	-45/4	- 9/4	45/4	9/4
(3, 1, 1, 3)	30/4	6/4	-30/4	- 6/4
(2, 4, 1, 1)	33/4	9/4	-33/4	- 9/4
(2, 3, 2, 1)	-27/4	- 1/4	27/4	9/4
(2, 3, 1, 2)	27/4	1/4	-27/4	- 9/4
(2, 2, 3, 1)	0	6/4	0	- 6/4
(2, 2, 2, 2)	0	0	0	0
(2, 2, 1, 3)	-45/4	- 9/4	45/4	9/4
(2, 1, 4, 1)	-18/4	- 4/4	18/4	8/4
(2, 1, 3, 2)	27/4	1/4	-27/4	- 9/4
(2, 1, 2, 3)	27/4	9/4	-27/4	- 1/4
(2, 1, 1, 4)	-18/4	- 6/4	18/4	2/4
(1, 5, 1, 1)	- 3/4	1/4	3/4	- 1/4
(1, 4, 2, 1)	-18/4	-12/4	18/4	8/4
(1, 4, 1, 2)	-18/4	- 4/4	18/4	8/4
(1, 3, 3, 1)	54/4	18/4	-54/4	-18/4
(1, 3, 2, 2)	0	6/4	0	- 6/4
(1, 3, 1, 3)	0	0	0	0
(1, 2, 4, 1)	-18/4	-12/4	18/4	8/4
(1, 2, 3, 2)	-27/4	- 1/4	27/4	9/4
(1, 2, 2, 3)	18/4	- 6/4	-18/4	- 2/4
(1, 2, 1, 4)	18/4	8/4	-18/4	- 4/4
(1, 1, 5, 1)	- 3/4	1/4	3/4	- 1/4
(1, 1, 4, 2)	33/4	9/4	-33/4	- 9/4
(1, 1, 3, 3)	-42/4	-12/4	42/4	12/4
(1, 1, 2, 4)	3/4	5/4	- 3/4	- 5/4
(1, 1, 1, 5)	3/4	- 1/4	- 3/4	1/4

Table 7. FMZ(S)Vs of weight 8 and depth 5 in terms of $\zeta_{\mathcal{A}}(6, 1, 1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(4, 1, 1, 1, 1) = \zeta_{\mathcal{A}}^*(4, 1, 1, 1, 1) = 3\zeta_{\mathcal{A}}(6, 1, 1)/4 - B_{p-3}B_{p-5}/4$.

$\mathbf{k}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}}$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}}^*$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}}^*$
(4, 1, 1, 1, 1)	3/4	- 1/4	3/4	- 1/4
(3, 2, 1, 1, 1)	3/4	5/4	3/4	5/4
(3, 1, 2, 1, 1)	-42/4	-12/4	-42/4	-12/4
(3, 1, 1, 2, 1)	33/4	11/4	33/4	9/4
(3, 1, 1, 1, 2)	- 3/4	- 1/4	- 3/4	1/4
(2, 3, 1, 1, 1)	18/4	4/4	18/4	4/4
(2, 2, 2, 1, 1)	18/4	2/4	18/4	2/4
(2, 2, 1, 2, 1)	-27/4	-15/4	-27/4	- 9/4
(2, 2, 1, 1, 2)	-18/4	- 2/4	-18/4	- 8/4
(2, 1, 3, 1, 1)	0	2/4	0	0
(2, 1, 2, 2, 1)	0	0	0	6/4
(2, 1, 2, 1, 2)	54/4	18/4	54/4	18/4
(2, 1, 1, 3, 1)	-18/4	- 8/4	-18/4	- 8/4
(2, 1, 1, 2, 2)	-18/4	- 2/4	-18/4	- 8/4
(2, 1, 1, 1, 3)	- 3/4	- 1/4	- 3/4	1/4
(1, 4, 1, 1, 1)	-18/4	- 2/4	-18/4	- 2/4
(1, 3, 2, 1, 1)	27/4	1/4	27/4	1/4
(1, 3, 1, 2, 1)	27/4	9/4	27/4	9/4
(1, 3, 1, 1, 2)	-18/4	- 8/4	-18/4	- 8/4
(1, 2, 3, 1, 1)	-45/4	-11/4	-45/4	- 9/4
(1, 2, 2, 2, 1)	0	12/4	0	0
(1, 2, 2, 1, 2)	0	0	0	6/4
(1, 2, 1, 3, 1)	27/4	9/4	27/4	9/4
(1, 2, 1, 2, 2)	-27/4	-15/4	-27/4	- 9/4
(1, 2, 1, 1, 3)	33/4	11/4	33/4	9/4
(1, 1, 4, 1, 1)	30/4	6/4	30/4	6/4
(1, 1, 3, 2, 1)	-45/4	-11/4	-45/4	- 9/4
(1, 1, 3, 1, 2)	0	2/4	0	0
(1, 1, 2, 3, 1)	27/4	1/4	27/4	1/4
(1, 1, 2, 2, 2)	18/4	2/4	18/4	2/4
(1, 1, 2, 1, 3)	-42/4	-12/4	-42/4	-12/4
(1, 1, 1, 4, 1)	-18/4	- 2/4	-18/4	- 2/4
(1, 1, 1, 3, 2)	18/4	4/4	18/4	4/4
(1, 1, 1, 2, 3)	3/4	5/4	3/4	5/4
(1, 1, 1, 1, 4)	3/4	- 1/4	3/4	- 1/4

Table 8. FMZ(S)Vs of weight 8 and or 6 in terms of $\zeta_{\mathcal{A}}(6, 1, 1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(2, 2, 1, 1, 1, 1) = \zeta_{\mathcal{A}}(6, 1, 1) + B_{p-3}B_{p-5}$ and $\zeta_{\mathcal{A}}^*(2, 2, 1, 1, 1, 1) = -\zeta_{\mathcal{A}}(6, 1, 1) - B_{p-3}B_{p-5}$.

$\mathbf{k}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}}$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}(6, 1, 1)$ - component of $\zeta_{\mathcal{A}}^*$	$B_{p-3}B_{p-5}$ - component of $\zeta_{\mathcal{A}}^*$
(3, 1, 1, 1, 1, 1)	4/4	0	− 4/4	0
(2, 2, 1, 1, 1, 1)	4/4	4/4	− 4/4	− 4/4
(2, 1, 2, 1, 1, 1)	−50/4	−14/4	50/4	18/4
(2, 1, 1, 2, 1, 1)	40/4	8/4	−40/4	−16/4
(2, 1, 1, 1, 2, 1)	10/4	10/4	−10/4	2/4
(2, 1, 1, 1, 1, 2)	− 8/4	− 8/4	8/4	0
(1, 3, 1, 1, 1, 1)	−14/4	− 2/4	14/4	2/4
(1, 2, 2, 1, 1, 1)	40/4	4/4	−40/4	− 8/4
(1, 2, 1, 2, 1, 1)	40/4	16/4	−40/4	− 8/4
(1, 2, 1, 1, 2, 1)	−80/4	−32/4	80/4	16/4
(1, 2, 1, 1, 1, 2)	10/4	10/4	−10/4	2/4
(1, 1, 3, 1, 1, 1)	10/4	2/4	−10/4	− 2/4
(1, 1, 2, 2, 1, 1)	−80/4	−16/4	80/4	16/4
(1, 1, 2, 1, 2, 1)	40/4	16/4	−40/4	− 8/4
(1, 1, 2, 1, 1, 2)	40/4	8/4	−40/4	−16/4
(1, 1, 1, 3, 1, 1)	10/4	2/4	−10/4	− 2/4
(1, 1, 1, 2, 2, 1)	40/4	4/4	−40/4	− 8/4
(1, 1, 1, 2, 1, 2)	−50/4	−14/4	50/4	18/4
(1, 1, 1, 1, 3, 1)	−14/4	− 2/4	14/4	2/4
(1, 1, 1, 1, 2, 2)	4/4	4/4	− 4/4	− 4/4
(1, 1, 1, 1, 1, 3)	4/4	0	− 4/4	0

Acknowledgements

The author would like to thank the referee for many helpful comments.

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¹The preprint [1] was revised to version 3 on 26 May 2015 and appeared in Kyushu J. Math. **69** (2015), 345–366. The present paper was first submitted on 31 December 2014 and so is independent of the revision of [1].