

Marcus External Contextual Grammars with Choice of the Languages of Primitive and Generalized Primitive Words. An Alternative Proof

To the honor Professor Masami Ito on his 70-th birthday

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Abstract

In this paper we unify some well-known results describing an alternative proof that all of the languages of primitive, quasi-primitive, and hyper-primitive words are Marcus external contextual languages with choice.

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1 Preliminaries

Marcus contextual grammars were introduced and intensively studied by S. Marcus and his students (see [11, 13]). The word is primitive if it is not a

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power of its proper prefix. The quasi-primitivity and hyper-primitivity are natural extensions of this concept. The relation of the language of primitive words to the Marcus contextual languages was studied first in [4]. On the line of this research, the relation of the language of quasi-primitive and hyper-primitive words and their certain further generalizations was described in [2] and [6]. In this paper we unify some of the results in [4, 2, 6] describing an alternative proof that all of the languages of primitive, quasi-primitive, and hyper-primitive words are Marcus external contextual languages with choice.

All notion and notations not defined here we refer to [3]. A *word* (over Σ) is a finite sequence of elements of some finite non-empty set Σ . We call the set Σ an *alphabet*, the elements of Σ *letters*. If u and v are words over an alphabet Σ , then their *catenation* uv is also a word over Σ . Especially, for every word u over Σ , $u\lambda = \lambda u = u$, where λ denotes the *empty word*. Given a word u , we define $u^0 = \lambda$, $u^n = u^{n-1}u$, $n > 0$, $u^* = \{u^n : n \geq 0\}$ and $u^+ = u^* \setminus \{\lambda\}$.

For every triplet u, v, w of words we say that u is a *prefix*, w is a *suffix*, and v is a *subword* of uvw . If $u (v, w)$ is nonempty then we speak about *proper prefix* (*proper subword*, *proper suffix*). A word z is called *overlapping* or *bordered* if there are $u, v, w \in \Sigma^+$ with $z = uw = vw$.

The *length* $|w|$ of a word w is the number of letters in w , where each letter is counted as many times as it occurs. Thus $|\lambda| = 0$. By the *free monoid* Σ^* *generated by* Σ we mean the set of all words (including the *empty word* λ) having catenation as multiplication. We set $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$, where the subsemigroup Σ^+ of Σ^* is said to be the *free semigroup generated by* Σ . Subsets of Σ^* are referred to as *languages* over Σ .

A *primitive word* (over Σ , or actually over an arbitrary alphabet) is a nonempty word not of the form w^m for any nonempty word w and integer $m \geq 2$. The set of all primitive words over Σ will be denoted by $Q(\Sigma)$, or simply by Q if Σ is understood. Q has received special interest: Q and $\Sigma^+ \setminus Q$ play an important role in the algebraic theory of codes and formal languages (see [7, 8, 9, 14]). If $u \in \Sigma^+$ can not be written into the form $u = v^n v'$, $n \geq 2$ such that $u, v \in \Sigma^+$ and v' is a prefix of u then we say that u is *strongly-primitive*.

We say that a word $u \in \Sigma^+$ is *covered* by the word $v \in \Sigma^+$ if for every $u', u'' \in \Sigma^*$, $a \in \Sigma$ with $u = u'au''$ there are $v_1, v_2, v_3, v_4 \in \Sigma^*$ with $u = v_1v_2av_3v_4$, $v = v_2av_3$, $u' = v_1v_2$, $u'' = v_3v_4$.

A word $u \in \Sigma^+$ is called *hyper-primitive* if it can not be covered by any of its proper subwords.

$u \in \Sigma^+$ is *super strongly primitive* if $u \neq v^n v'$, $n \geq 2$ such that v has a suffix v'' for which $v''v'$ is a prefix of u .

u is called *strongly hyper-primitive* if $u \neq wv'$, where w is covered by v , which is one of its proper subwords, and v' is a prefix of v .

Finally, u is *hyper hyper-primitive* if $u \neq wv'$, where w is covered by v , which is one of its proper subwords, and w has a suffix v'' such that $v''v'$ is a prefix of v .

Denote, in order, $SQ(\Sigma), HQ(\Sigma), SSQ(\Sigma), SHQ(\Sigma), HHQ(\Sigma)$, or, if Σ is understood, then SQ, HQ, SSQ, SHQ, HHQ the language of all strongly primitive, hyper primitive, super strongly primitive, strongly hyper-primitive, and hyper hyper primitive words (over Σ).

Moreover, denote by $|H|$ the *cardinality* of H for every set H .

A (Marcus) *contextual grammar with choice* is a structure $G = (V, A, C, \varphi)$, where V is an alphabet, A is a finite language over V , C is a finite subset of $V^* \times V^*$, and $\varphi : V^* \rightarrow 2^C$. If $\varphi(x) = C$ holds for every $x \in V^*$ then we say that G is a (Marcus) contextual grammar without choice and then we omit φ sometimes.

We define two relations on V^* as usual: for any $x \in V^*$, we write

$x \Rightarrow_{ex} y$ if and only if $y = uxv$, for a context (u, v) in $\varphi(x)$,

$x \Rightarrow_{int} y$ if and only if $x = x_1 x_2 x_3, y = x_1 u x_2 v x_3$ for any $(u, v) \in \varphi(x_2)$.

Denote $\stackrel{*}{\Rightarrow}_{ex}, \stackrel{*}{\Rightarrow}_{in}$ the reflexive and transitive closure of these relations and let $L_\alpha(G) = \{x \in V^* : w \stackrel{*}{\Rightarrow}_\alpha x, w \in A\}$ for $\alpha \in \{ex, in\}$. Then $L_{ex}(G)$ is the (Marcus) *external contextual language (with or without choice) generated by* G , and similarly, $L_{in}(G)$ is the (Marcus) *internal contextual language (with or without choice) generated by* G . Now let $G = (V, A, \varphi)$, where V is an alphabet, A is a finite language over V , C is a finite subset of $V^* \times V^*$, and $\varphi : V^* \times V^* \times V^* \rightarrow 2^C$.¹

Define the relation $\Rightarrow$ on V^* such that $x \Rightarrow y$ for some $x, y \in V^*$ if and only if $x = x_1 x_2 x_3, y = x_1 u x_2 v x_3, x_1, x_2, x_3 \in V^*, (u, v) \in \varphi(x_1, x_2, x_3)$. Moreover, let $\stackrel{*}{\Rightarrow}$ denote the reflexive and transitive closure of $\Rightarrow$. Thus $L(G)$ is defined to be a (Marcus) *total contextual grammar (with or without choice) generated by* G . If $\varphi(x_1, x_2, x_3) = C$ holds for every $x_1, x_2, x_3 \in V^*$ then we say that G is a (Marcus) total contextual grammar without choice and sometimes we omit φ having this property.

¹Observe that the definition of φ is not the same as before.

The following statement is a unified form of some results in [2, 4, 6]. It has been formulated by [6].

Theorem 1 [2, 4, 6] *The languages Q , SQ , and HQ are external contextual languages with choice. This is not true for the sets SSQ , SHQ , and HHQ , furthermore, none of the sets Q , SQ , HQ , SSQ , SHQ , and HHQ is an external contextual language without choice or an internal contextual language with or without choice.* $\square$ $\square$

We shall use the following results.

Theorem 2 [5] *Let $u, v \in \Sigma^+$, $s, t \geq 1$, with $s \neq t$. If $\sqrt{s}u \neq \sqrt{t}v$ and $uv^s \notin Q$, then $uv^t \in Q$.* $\square$

Theorem 3 [1] *Let $u, v \in Q$, $u^m = v^k w$, $k, m \geq 2$ for some prefix w of v . Then $u = v$ and $w \in \{u, \lambda\}$.* $\square$ ²

Theorem 4 [14]/[BorweinLemma] *Let $u \in \Sigma^+$, $u \notin a^+$, $a \in \Sigma$. Then at least one of ua , u must be primitive.* $\square$

Theorem 5 [10] *If $uv = vq$, $u \in \Sigma^+$, $v, q \in \Sigma^*$, then $u = wz$, $v = (wz)^k w$, $q = zw$ for some $w \in \Sigma^*$, $z \in \Sigma^+$ and $k \geq 0$.* $\square$

We shall use the following two widely known consequences of Theorem 5.

Proposition 6 *For every bordered word $z \in \Sigma^+$ there exists a nonempty word $u \in \Sigma^+$ and a (not necessarily nonempty) word $v \in \Sigma^*$ having $z = uvu$.* $\square$

Theorem 7 [10] *Let $u, v \in \Sigma^+$ with $uv = vu$. There exists $w \in \Sigma^+$ with $u, v \in w^+$.* $\square$

²This statement can also be derived directly from [5].

2 Results

Next we show alternative proofs of some known results.

Theorem 8 [2, 6] *Let V be an alphabet with $|V| \geq 2$. If $awb \in SQ$ where $w \in V^*$ and $a, b \in V$, then $aw \in SQ$ or $wb \in SQ$.³*

Proof: Suppose the contrary. Then $aw, wb \in SPer$, i.e., there are $u, v \in V^+$, $u', v' \in V^*$, positive integers $m, n \geq 2$ such that u' is a prefix of u , v' is a prefix of v , and $u^m u' = aw$, $v^n v' = wb$.

Then $u = aw_1 w_2$ and $u' \in \{\lambda, aw_1\}$ for some $w_1, w_2 \in V^*$. Similarly, $v = w_3 b w_4$ and $v' \in \{\lambda, w_3 b\}$ for an appropriate pair $w_3, w_4 \in V^*$. Thus we can write $w \in \{(w_1 w_2 a)^m w_1, (w_1 w_2 a)^{m-1} w_1 w_2, (w_3 b w_4)^n w_3, (w_3 b w_4)^{n-1} w_3 w_4\}$. Let, say, $w = (w_1 w_2 a)^m w_1 = (w_3 b w_4)^n w_3$. By the symmetricity we may assume $|w_1| \leq |w_3|$. Thus $(w_1 w_2 a)^m = (w_3 b w_4)^n w'$ for some prefix w' of w_3 . Applying Theorem 3, $\sqrt{w_1 w_2 a} = \sqrt{w_3 b w_4}$. Therefore, $w_3 b w_4 = w_3 w'' a$ for some $w'' \in V^*$. Hence $awb = a(w_3 b w'' a)^n w_3 b = (aw_3 b w'')^n aw_3 b \notin SQ$, a contradiction.

We can get the same conclusion if $w = (w_1 w_2 a)^m w_1 = (w_3 w_4 b)^{n-1} w_3 w_4$ and $n > 2$ (or $w = (w_1 w_2 a)^{m-1} w_1 w_2 = (w_3 b w_4)^n w_3$ and $m > 2$). Thus let $w = (w_1 w_2 a)^m w_1 = w_3 w_4 b w_3 w_4$ (with $n = 2$). But then $(w_1 w_2 a)^m w_1 b = (w_3 w_4 b)^2$ with $m \geq 2$. Applying again Theorem 3, $\sqrt{w_1 w_2 a} = \sqrt{w_3 w_4 b}$ with $a = b$. Therefore, $awb = aw_3 w_4 b w_3 w_4 b = (aw_3 w_4)^2 a \notin SQ$, a contradiction.

We can derive the impossibility of $w = w_1 w_2 a w_1 w_2 = (w_3 b w_4)^n w_3$ and $n \geq 2$ in the same way.

The rest of the cases is the equality $w_1 w_2 a w_1 w_2 = w_3 w_4 b w_3 w_4$. But then $|w_1 w_2| = |w_3 w_4|$ which implies $w_1 w_2 a = w_3 w_4 b$, i.e., $a = b$. Then $awb = aw_3 w_4 b w_3 w_4 b = (aw_3 w_4)^2 a \notin SQ$, a contradiction again. $\square$

Lemma 9 *If a word w can be covered by a word va , with $v \in \Sigma^*$, $a \in \Sigma$, then vb is not a subword of w , for any $b \in \Sigma, b \neq a$.*

Proof: Consider a covering of w by va . We will assume that vb can occur in w and show that it leads to a contradiction.

There are two possibilities for vb to occur in w :

Case 1. vb is a proper subword (not only pre- or suffix) of $v'v$, where v' is a prefix of v : in this case vb is neither a prefix nor a suffix of $v'v$ because

³ $a = b$ is possible.

$va \neq vb$. Thus v has two different borders, i.e. by Proposition 6, $v = x_1ux_1$ and $v = x_2y'x_2$. Without loss of generality we can assume $|x_2| < |x_1|$. Then x_1 itself is bordered, hence, applying Proposition 6 again, $x_1 = xyx$, for some x, y . This gives us $v = xyxuxyx$ and because v overlaps twice with itself (by xyx and also by x), $v = xyxuxyx = xuxyxxz$, for some z , but then x is a suffix of z and immediately before it is y , so $xyxuxyx = xuxyxyx$. Simplifying gives us $yxu = uxy$, hence

$$xuxyxyx = xyxuxyx = xyxyxux \text{ with } v = xyxuxyx, \quad (1)$$

taking away the first x , we get $uxyxyx = yxyxux$, so $ux = (yx)^2$. Therefore, by Theorem 7, $ux, (yx)^2 \in z^+$ for some $z \in \Sigma^+$. From here applying (1), $v = xz^k$, where z is a primitive word and $k \geq 3$. Moreover, since x is a suffix of v , we get $x = z'z^j$, with z' a suffix of z and $j < k$, so $z = z''z'$ and $v = z'(z''z')^{j+k}$, with $z''z'$ primitive, therefore $z'(z''z')^{j+k}b$ would have to be a proper subword of either $z'(z''z')^{j+k}az'(z''z')^{j+k}$ or $z'(z''z')^i$, with $i > j+k$. In both cases the first letter of z'' would have to be at the same time a and b , contradiction.

Case 2. vb is a proper subword of vav . In this case v from va overlaps the first v in vav with a part u_1 and the second with u_2 , that is, $v = u_1au_2$ and $v = u_2bu_1$. If $|u_1| = |u_2|$, we instantly get $a = b$, contradiction. Without loss of generality $|u_1| < |u_2|$, and then u_1 is a border of u_2 so, applying Theorem 5, for some $x \in \Sigma^*, y \in \Sigma^+$ we have $u_1 = (xy)^ix = x(yx)^i$ and $u_2 = (xy)^jx = x(yx)^j$, with $1 \leq i < j$. This gives $v = x(yx)^iax(yx)^j = x(yx)^jbx(yx)^i$. Taking away $x(yx)^i$ from both sides we get $ax(yx)^{j-i} = x(yx)^{j-i}b$. By this equality, $x \neq \lambda$ implies $ax = xc$ and $dx = xb$ for some $c, d \in \Sigma$. Hence we could get $x \in a^+ \cap b^+$, a contradiction. Therefore, $x = \lambda$. Then $ay^{j-i} = y^{j-i}b$ with $a \neq b$ and $i < j$. (By $a \neq b$, $i = j$ would be impossible even if we would not suppose before $i < j$.) By this connection, $y \neq \lambda$ implies $ay = yc$ and $dy = yb$ for some $c, d \in \Sigma$. Then $y \in a^+ \cap b^+$, which is impossible unless $a = b$. $\square$

Theorem 10 [6] *For any word w and (not necessarily distinct) letters $a, b \in \Sigma$, if $aw, wb \notin HQ$, then $awb \notin HQ$.*

Proof: If $aw \notin HQ$, then there is some hyper-primitive av which covers aw . Similarly, there is some hyper-primitive ub which covers wb . Without loss

of generality, we can assume $|v| \leq |u|$. Then, u is a suffix of v , therefore wherever there is an occurrence of v in the string, it ends in u . Now, Lemma 9 tells us that if ub covers wb , and $c \neq b$, then uc is not a subword of wb .

There are two cases.

Case 1. $a \neq b$. Whenever v appears in the string wb , it should be followed by b . From here, we get that avb covers awb , so $awb \notin HQ$.

Case 2. $a = b$. Whenever v appears in the string wa , it should be followed by a . From here, we get that ava covers awa , so $awa \notin HQ$. $\square$

Corollary 11 *Let V be an alphabet with $|V| \geq 2$. If $awbc \in XQ$, where $XQ \in \{Q, SQ, HQ\}$, $w \in V^*$ and $a, b, c \in V^4$, then one of aw, awb, wbc is in XQ .*

Proof: If $XQ = Q$ and $awb \notin a^+$, then Theorem 4 implies that one of aw, awb should be in Q . If $XQ = Q$ and $awb \in a^+$, then $awbc \in XQ$ implies $c \neq a$. In this case, $wbc \in a^+c$ with $a \neq c$, for which $wbc \in Q$ obviously holds. If $XQ \in \{SQ\}$ then by Theorem 8, if $XQ \in \{SQ, HQ\}$ then by Theorem 10 we have that one of awb, wbc should be in XQ . $\square$

On the basis of Lemma 11, similarly to Theorem 12 published by [6], next we show an alternative (and unified) proof of the next statement which is a union of three previous results.

Theorem 12 [2, 4, 6] *All of the languages Q, SQ, HQ are Marcus external contextual languages with choice.*

Proof: Let $G = (V, A, C, \varphi)$ be an external Marcus contextual grammar with choice defined by $A = V$, $C = \{(\lambda, \lambda), (\lambda, a), (\lambda, ab), (a, \lambda) : a, b \in V\}$, moreover, let for every $w \in V^*$, $z \in \varphi(w)$ with

$$z = \begin{cases} \{(\lambda, \lambda)\} & \text{if } |V| = 1, \\ \{(a, \lambda)\} & \text{if } a \in V \text{ and } aw \in XQ, \\ \{(\lambda, a)\} & \text{if } a \in V \text{ and } wa \in XQ, \\ \{(\lambda, ab)\} & \text{if } a, b \in V \text{ and } wab \in XQ. \end{cases}^5$$

Moreover, let $XQ \in \{Q, SQ, HQ\}$. Obviously, the proposition holds true for $|V| = 1$. Hence we assume $|V| \geq 2$. By the definition of the grammar G , it is obvious that $L_{ex}(G) \subseteq SQ$. Now we prove that $SQ \subseteq L_{ex}(G)$

⁴ a, b, c are not necessarily distinct

by induction. By definition, $V \cap SQ = V(= A)$ and $V^2 \cap SQ = \{ab \mid a, b \in V, a \neq b\}$. Similarly, $V \cap V^3 = \{abc \mid a, b, c \in V, a \neq b, a \neq c, b \neq c\} \cup \{aab, abb \mid a, b \in V, a \neq b\}$. Moreover, by our construction, $a, b \in V$ and $a \neq b$ imply $a \Rightarrow_{ex} ab$. Thus we have $(V \cup V^2) \cap Q \subseteq L_{ex}(G)$. Similarly, by our construction, $a, b, c \in V$ and $a \neq b, a \neq c, b \neq c$ imply $ab \Rightarrow_{ex} abc$ and $a, b \in V$ and $a \neq b$ imply $ab \Rightarrow_{ex} abb$ and $ab \Rightarrow_{ex} aab$. Now, assume that $(V \cup V^2 \cup \dots \cup V^n) \cap XQ \subseteq L_{ex}(G)$ for some $n \geq 3$. Let $u \in V^{n+1} \cap XQ$ and let $u = awbc \in XQ$ where $a, b, c \in V$. (Note that a, b, c are not necessarily distinct.) Corollary 11 states that, by this condition, one of aw, awb, wbc in XQ . Hence, either $aw \in XQ$ with $aw \Rightarrow_{ex} awbc$ or $awb \in XQ$ with $awb \Rightarrow_{ex} awbc$, or $wbc \in XQ$ with $wbc \Rightarrow_{ex} awbc$. $\square$

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